Topics in Multi-User Information Theory

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Topics in Multi-User Information Theory

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Abstract

This survey reviews fundamental concepts of multi-user information theory. Starting with typical sequences, the survey builds up knowledge on random coding, binning, superposition coding, and capacity converses by introducing progressively more sophisticated tools for a selection of source and channel models. The problems addressed include: Source Coding; Rate-Distortion and Multiple Descriptions; Capacity-Cost; The Slepian–Wolf Problem; The Wyner-Ziv Problem; The Gelfand-Pinsker Problem; The Broadcast Channel; The Multiaccess Channel; The Relay Channel; The Multiple Relay Channel; and The Multiaccess Channel with Generalized Feedback. The survey also includes a review of basic probability and information theory.

Contents

Notations and Acronyms

We use standard notation for probabilities, random variables, entropy, mutual information, and so forth. Table 1 lists notation developed in the appendices of this survey, and we use this without further explanation in the main body of the survey. We introduce the remaining notation as we go along. The reader is referred to the appendices for a review of the relevant probability and information theory concepts.

Table 1 Probability and information theory notation.

Sequences, Vectors, Matrices	
x^n	the finite sequence $x_1, x_2, \ldots, x_n$
$x^n y^m$	sequence concatenation: $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m$
$\underline{x}$	the vector $[x_1, x_2, \ldots, x_n]$
$\mathbf{H}$	a matrix
$\lvert\mathbf{Q}\rvert$	determinant of the matrix $\mathbf{Q}$

(*Continued*)

Table 1 (*Continued*)

Probability	
$\Pr[\mathcal{A}]$	probability of the event $\mathcal{A}$
$\Pr[\mathcal{A}\|\mathcal{B}]$	probability of event $\mathcal{A}$ conditioned on event $\mathcal{B}$
$P_X(\cdot)$	probability distribution of the random variable X
$P_{X\|Y}(\cdot)$	probability distribution of X conditioned on Y
$\text{supp}(P_X)$	support of P_X
$p_X(\cdot)$	probability density of the random variable X
$p_{X\|Y}(\cdot)$	probability density of X conditioned on Y
$E[X]$	expectation of the real-valued X
$E[X\|\mathcal{A}]$	expectation of X conditioned on event $\mathcal{A}$
$\mathsf{Var}[X]$	variance of X
$\mathbf{Q}_{\underline{X}}$	covariance matrix of $\underline{X}$
Information Theory	
$H(X)$	entropy of the discrete random variable X
$H(X\|Y)$	entropy of X conditioned on Y
$I(X;Y)$	mutual information between X and Y
$I(X;Y\|Z)$	mutual information between X and Y conditioned on Z
$D(P_X\\|P_Y)$	informational divergence between P_X and P_Y
$h(X)$	differential entropy of X
$h(X\|Y)$	differential entropy of X conditioned on Y
$H_2(\cdot)$	binary entropy function

1

Typical Sequences and Source Coding

1.1 Typical Sequences

Shannon introduced the notion of a "typical sequence" in his 1948 paper "A Mathematical Theory of Communication" [55]. To illustrate the idea, consider a discrete memoryless source (DMS), which is a device that emits symbols from a discrete and finite alphabet $\mathcal{X}$ in an independent and identically distributed (i.i.d.) manner (see Figure 1.1). Suppose the source probability distribution is $P_X(\cdot)$ where

$$P_X(0) = 2/3 \quad \text{and} \quad P_X(1) = 1/3. \tag{1.1}$$

Consider the following experiment: we generated a sequence of length 18 by using a random number generator with the distribution (1.1). We write this sequence below along with three other sequences that we generated artificially.

(a) 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
(b) 1, 0, 1, 1, 0, 1, 0, 1, 1, 1, 0, 0, 0, 0, 1, 0, 1, 0
(c) 0, 0, 0, 1, 1, 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 1, 1, 0
(d) 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1.

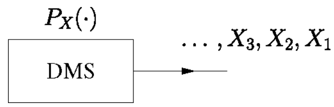

Fig. 1.1 A discrete memoryless source with distribution $P_X(\cdot)$.

If we compute the probabilities that these sequences were emitted by the source (1.1), we have

(a) $(2/3)^{18} \cdot (1/3)^0 \approx 6.77 \cdot 10^{-4}$
(b) $(2/3)^9 \cdot (1/3)^9 \approx 1.32 \cdot 10^{-6}$
(c) $(2/3)^{11} \cdot (1/3)^7 \approx 5.29 \cdot 10^{-6}$
(d) $(2/3)^0 \cdot (1/3)^{18} \approx 2.58 \cdot 10^{-9}$.

Thus, the first sequence is the most probable one by a large margin. However, the reader will likely *not* be surprised to find out that it is sequence (c) that was actually put out by the random number generator. Why is this intuition correct? To explain this, we must define more precisely what one might mean by a "typical" sequence.

1.2 Entropy-Typical Sequences

Let x^n be a finite sequence whose ith entry x_i takes on values in $\mathcal{X}$. We write $\mathcal{X}^n$ for the Cartesian product of the set $\mathcal{X}$ with itself n times, i.e., we have $x^n \in \mathcal{X}^n$. Let $N(a|x^n)$ be the number of positions of x^n having the letter a, where $a \in \mathcal{X}$.

There are several natural definitions for typical sequences. Shannon in [55, Sec. 7] chose a definition based on the entropy of a random variable X. Suppose that X^n is a sequence put out by the DMS $P_X(\cdot)$, which means that $P_{X^n}(x^n) = \prod_{i=1}^n P_X(x_i)$ is the probability that x^n was put out by the DMS $P_X(\cdot)$. More generally, we will use the notation

$$P_X^n(x^n) = \prod_{i=1}^n P_X(x_i). \tag{1.2}$$

We further have

$$P_X^n(x^n) = \begin{cases} \prod_{a \in \text{supp}(P_X)} P_X(a)^{N(a|x^n)} & \text{if } N(a|x^n) = 0 \text{ whenever } P_X(a) = 0 \\ 0 & \text{else} \end{cases} \tag{1.3}$$

and intuitively one might expect that the letter a occurs about $N(a|x^n) \approx nP_X(a)$ times, so that $P_X^n(x^n) \approx \Pi_{a\in \text{supp}(P_X)} P_X(a)^{nP_X(a)}$ or

$$-\frac{1}{n}\log_2 P_X^n(x^n) \approx \sum_{a\in\text{supp}(P_X)} -P_X(a)\log_2 P_X(a).$$

Shannon therefore defined a sequence x^n to be typical with respect to ϵ and $P_X(\cdot)$ if

$$\left|\frac{-\log_2 P_X^n(x^n)}{n} - H(X)\right| < \epsilon \tag{1.4}$$

for some small positive ϵ. The sequences satisfying (1.4) are sometimes called *weakly* typical sequences or *entropy*-typical sequences [19, p. 40]. We can equivalently write (1.4) as

$$2^{-n[H(X)+\epsilon]} < P_X^n(x^n) < 2^{-n[H(X)-\epsilon]}. \tag{1.5}$$

Example 1.1. If $P_X(\cdot)$ is uniform then for any x^n we have

$$P_X^n(x^n) = |\mathcal{X}|^{-n} = 2^{-n\log_2|\mathcal{X}|} = 2^{-nH(X)} \tag{1.6}$$

and *all* sequences in $\mathcal{X}^n$ are entropy-typical.

Example 1.2. The source (1.1) has $H(X) \approx 0.9183$ and the above four sequences are entropy-typical with respect to $P_X(\cdot)$ if

(a) $\epsilon > 1/3$
(b) $\epsilon > 1/6$
(c) $\epsilon > 1/18$
(d) $\epsilon > 2/3$.

Note that sequence (*c*) requires the smallest ϵ.

We remark that *entropy* typicality applies to *continuous* random variables with a density if we replace the probability $P_X^n(x^n)$ in (1.4) with the density value $p_X^n(x^n)$. In contrast, the next definition can be used only for discrete random variables.

1.3 Letter-Typical Sequences

A perhaps more natural definition for *discrete* random variables than (1.4) is the following. For $\epsilon \geq 0$, we say a sequence x^n is ϵ-*letter* typical with respect to $P_X(\cdot)$ if

$$\left|\frac{1}{n}N(a|x^n) - P_X(a)\right| \leq \epsilon \cdot P_X(a) \quad \text{for all } a \in \mathcal{X} \tag{1.7}$$

The set of x^n satisfying (1.7) is called the ϵ-*letter-typical set* $T_\epsilon^n(P_X)$ with respect to $P_X(\cdot)$. The letter typical x^n are thus sequences whose *empirical* probability distribution is close to $P_X(\cdot)$.

Example 1.3. If $P_X(\cdot)$ is uniform then ϵ-letter typical x^n satisfy

$$\frac{(1-\epsilon)n}{|\mathcal{X}|} \leq N(a|x^n) \leq \frac{(1+\epsilon)n}{|\mathcal{X}|} \tag{1.8}$$

and if $\epsilon < |\mathcal{X}| - 1$, as is usually the case, then *not* all x^n are letter-typical. The definition (1.7) is then more restrictive than (1.4) (see Example 1.1).

We will generally rely on letter typicality, since for discrete random variables it is just as easy to use as entropy typicality, but can give stronger results.

We remark that one often finds small variations of the conditions (1.7). For example, for *strongly* typical sequences one replaces the $\epsilon P_X(a)$ on the right-hand side of (1.7) with ϵ or $\epsilon/|\mathcal{X}|$ (see [19, p. 33], and [18, pp. 288, 358]). One further often adds the condition that $N(a|x^n) = 0$ if $P_X(a) = 0$ so that typical sequences cannot have zero-probability letters. Observe, however, that this condition is included in (1.7). We also remark that the letter-typical sequences are simply called "typical sequences" in [44] and "robustly typical sequences" in [46]. In general, by the label "letter-typical" we mean any choice of typicality where one performs a per-alphabet-letter test on the empirical probabilities. We will focus on the definition (1.7).

We next develop the following theorem that describes some of the most important properties of letter-typical sequences and sets.

Let $\mu_X = \min_{x \in \text{supp}(P_X)} P_X(x)$ and define

$$\delta_\epsilon(n) = 2|\mathcal{X}| \cdot e^{-n\epsilon^2 \mu_X}. \tag{1.9}$$

Observe that $\delta_\epsilon(n) \to 0$ for fixed ϵ, $\epsilon > 0$, and $n \to \infty$.

Theorem 1.1. Suppose $0 \le \epsilon \le \mu_X$, $x^n \in T_\epsilon^n(P_X)$, and X^n is emitted by a DMS $P_X(\cdot)$. We have

$$2^{-n(1+\epsilon)H(X)} \le P_X^n(x^n) \le 2^{-n(1-\epsilon)H(X)} \tag{1.10}$$

$$(1 - \delta_\epsilon(n))\, 2^{n(1-\epsilon)H(X)} \le |T_\epsilon^n(P_X)| \le 2^{n(1+\epsilon)H(X)} \tag{1.11}$$

$$1 - \delta_\epsilon(n) \le \Pr[X^n \in T_\epsilon^n(P_X)] \le 1. \tag{1.12}$$

Proof. Consider (1.10). For $x^n \in T_\epsilon^n(P_X)$, we have

$$\begin{aligned} P_X^n(x^n) &= \prod_{a \in \text{supp}(P_X)} P_X(a)^{N(a|x^n)} \\ &\le \prod_{a \in \text{supp}(P_X)} P_X(a)^{nP_X(a)(1-\epsilon)} \\ &= 2^{\sum_{a \in \text{supp}(P_X)} nP_X(a)(1-\epsilon)\log_2 P_X(a)} \\ &= 2^{-n(1-\epsilon)H(X)}, \end{aligned} \tag{1.13}$$

where the inequality follows because, by the definition (1.7), typical x^n satisfy $N(a|x^n)/n \ge P_X(a)(1-\epsilon)$. One can similarly prove the left-hand side of (1.10).

Next, consider (1.12). In the appendix of this section, we prove the following result using the Chernoff bound:

$$\Pr\left[\left|\frac{N(a|X^n)}{n} - P_X(a)\right| > \epsilon P_X(a)\right] \le 2 \cdot e^{-n\epsilon^2 \mu_X}, \tag{1.14}$$

where $0 \le \epsilon \le \mu_X$. We thus have

$$\begin{aligned} \Pr[X^n \notin T_\epsilon^n(P_X)] &= \Pr\left[\bigcup_{a \in \mathcal{X}} \left\{\left|\frac{N(a|X^n)}{n} - P_X(a)\right| > \epsilon P_X(a)\right\}\right] \\ &\le \sum_{a \in \mathcal{X}} \Pr\left[\left|\frac{N(a|X^n)}{n} - P_X(a)\right| > \epsilon P_X(a)\right] \\ &\le 2|\mathcal{X}| \cdot e^{-n\epsilon^2 \mu_X}, \end{aligned} \tag{1.15}$$

where we have used the union bound (see (A.5)) for the second step. This proves the left-hand side of (1.12).

Finally, for (1.11) observe that

$$\begin{aligned}\Pr\left[X^n \in T_\epsilon^n(P_X)\right] &= \sum_{x^n \in T_\epsilon^n(P_X)} P_X^n(x^n) \\ &\le |T_\epsilon^n(P_X)|\, 2^{-n(1-\epsilon)H(X)},\end{aligned} \tag{1.16}$$

where the inequality follows by (1.13). Using (1.15) and (1.16), we thus have

$$|T_\epsilon^n(P_X)| \ge (1 - \delta_\epsilon(n))\, 2^{n(1-\epsilon)H(X)}. \tag{1.17}$$

We similarly derive the right-hand side of (1.11). □

1.4 Source Coding

The source coding problem is depicted in Figure 1.2. A DMS $P_X(\cdot)$ emits a sequence x^n of symbols that are passed to an encoder. The source encoder "compresses" x^n into an index w and sends w to the decoder. The decoder reconstructs x^n from w as $\hat{x}^n(w)$, and is said to be successful if $\hat{x}^n(w) = x^n$.

The source encoding can be done in several ways:

- Fixed-length to fixed-length coding (or block-to-block coding).
- Fixed-length to variable-length coding (block-to-variable-length coding).
- Variable-length to fixed-length coding (variable-length-to-block coding).
- Variable-length to variable-length coding.

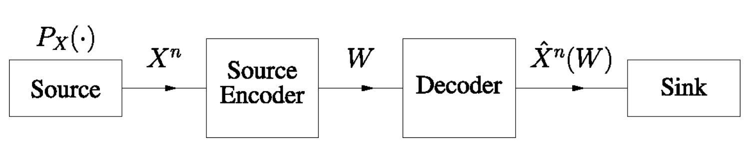

Fig. 1.2 The source coding problem.

We will here consider only the first two approaches. For a block-to-variable-length scheme, the number of bits transmitted by the encoder depends on x^n. We will consider the case where every source sequence is assigned a *unique* index w. Hence, one can reconstruct x^n perfectly. Let $L(x^n)$ be the number of bits transmitted for x^n. The goal is to minimize the *average* rate $R = E\left[L(X^N)\right]/n$.

For a block-to-block encoding scheme, the index w takes on one of 2^{nR} indexes w, $w = 1, 2, \ldots, 2^{nR}$, and we assume that 2^{nR} is a positive integer. The encoder sends exactly nR bits for every source sequence x^n, and the goal is to make R as small as possible. Observe that block-to-block encoding might require the encoder to send the *same* w for two *different* source sequences.

Suppose first that we permit no error in the reconstruction. We use the block-to-variable-length encoder, choose an n and an ϵ, and assign each sequence in $T_\epsilon^n(P_X)$ a unique positive integer w. According to (1.11), these indexes w can be represented by at most $n(1+\epsilon)H(X) + 1$ bits. Next, the encoder collects a sequence x^n. If $x^n \in T_\epsilon^n(P_X)$, then the encoder sends a "0" followed by the $n(1+\epsilon)H(X) + 1$ bits that represent this sequence. If $x^n \notin T_\epsilon^n(P_X)$, then the encoder sends a "1" followed by $n\log_2|\mathcal{X}| + 1$ bits that represent x^n. The average number of bits per source symbol is the compression rate R, and it is upper bounded by

$$\begin{aligned} R &\le \Pr\left[X^n \in T_\epsilon^n(P_X)\right]\left[(1+\epsilon)H(X) + 2/n\right] \\ &\quad + \Pr\left[X^n \notin T_\epsilon^n(P_X)\right](\log_2|\mathcal{X}| + 2/n) \\ &\le (1+\epsilon)H(X) + 2/n + \delta_\epsilon(n)(\log_2|\mathcal{X}| + 2/n). \end{aligned} \tag{1.18}$$

But since $\delta_\epsilon(n) \to 0$ as $n \to \infty$, we can transmit at any rate above $H(X)$ bits per source symbol. For example, if the DMS is binary with $P_X(0) = 1 - P_X(1) = 2/3$, then we can transmit the source outputs in a lossless fashion at any rate above $H(X) \approx 0.9183$ bits per source symbol.

Suppose next that we must use a block-to-block encoder, but that we permit a small error probability in the reconstruction. Based on the above discussion, we can transmit at any rate above $(1+\epsilon)H(X)$ bits per source symbol with an error probability $\delta_\epsilon(n)$. By making n large, we can make $\delta_\epsilon(n)$ as close to zero as desired.

But what about a converse result? Can one compress with a small error probability, or even zero error probability, at rates below $H(X)$? We will prove a converse for block-to-block encoders only, since the block-to-variable-length case requires somewhat more work.

Consider Fano's inequality (see Section A.10) which ensures us that

$$H_2(P_e) + P_e \log_2(|\mathcal{X}|^n - 1) \geq H(X^n|\hat{X}^n), \tag{1.19}$$

where $P_e = \Pr[\hat{X}^n \neq X^n]$. Recall that there are at most 2^{nR} different sequences $\hat{x}^n$, and that $\hat{x}^n$ is a function of x^n. We thus have

$$\begin{aligned} nR &\geq H(\hat{X}^n) \\ &= H(\hat{X}^n) - H(\hat{X}^n|X^n) \\ &= I(X^n;\hat{X}^n) \\ &= H(X^n) - H(X^n|\hat{X}^n) \\ &= nH(X) - H(X^n|\hat{X}^n) \\ &\geq n\left[H(X) - \frac{H_2(P_e)}{n} - P_e\log_2|\mathcal{X}|\right], \end{aligned} \tag{1.20}$$

where the last step follows by (1.19). Since we require that P_e be zero, or approach zero with n, we find that $R \geq H(X)$ for block-to-block encoders with arbitrarily small positive P_e. This is the desired converse.

1.5 Jointly and Conditionally Typical Sequences

Let $N(a,b|x^n,y^n)$ be the number of times the pair (a,b) occurs in the sequence of pairs $(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$. The *jointly* typical set with respect to $P_{XY}(\cdot)$ is simply

$$\begin{aligned} T_\epsilon^n(P_{XY}) = \Bigg\{(x^n,y^n) : &\left|\frac{1}{n}N(a,b|x^n,y^n) - P_{XY}(a,b)\right| \\ &\leq \epsilon \cdot P_{XY}(a,b) \quad \text{for all } (a,b) \in \mathcal{X} \times \mathcal{Y}\Bigg\}. \end{aligned} \tag{1.21}$$

The reader can easily check that $(x^n,y^n) \in T_\epsilon^n(P_{XY})$ implies both $x^n \in T_\epsilon^n(P_X)$ and $y^n \in T_\epsilon^n(P_Y)$.

Consider the conditional distribution $P_{Y|X}(\cdot)$ and define

$$P^n_{Y|X}(y^n|x^n) = \prod_{i=1}^{n} P_{Y|X}(y_i|x_i) \tag{1.22}$$

$$T^n_\epsilon(P_{XY}|x^n) = \{y^n : (x^n,y^n) \in T^n_\epsilon(P_{XY})\}. \tag{1.23}$$

Observe that $T^n_\epsilon(P_{XY}|x^n) = \emptyset$ if x^n is not in $T^n_\epsilon(P_X)$. We shall further need the following counterpart of $\delta_\epsilon(n)$ in (1.9):

$$\delta_{\epsilon_1,\epsilon_2}(n) = 2|\mathcal{X}||\mathcal{Y}| \exp\left(-n \cdot \frac{(\epsilon_2 - \epsilon_1)^2}{1+\epsilon_1} \cdot \mu_{XY}\right), \tag{1.24}$$

where $\mu_{XY} = \min_{(a,b)\in \mathrm{supp}(P_{XY})} P_{XY}(a,b)$ and $0 \le \epsilon_1 < \epsilon_2 \le 1$. Note that $\delta_{\epsilon_1,\epsilon_2}(n) \to 0$ as $n \to \infty$. In the Appendix, we prove the following theorem that generalizes Theorem 1.1 to include conditioning.

Theorem 1.2. Suppose $0 \le \epsilon_1 < \epsilon_2 \le \mu_{XY}$, $(x^n,y^n) \in T^n_{\epsilon_1}(P_{XY})$, and (X^n,Y^n) was emitted by the DMS $P_{XY}(\cdot)$. We have

$$2^{-nH(Y|X)(1+\epsilon_1)} \le P^n_{Y|X}(y^n|x^n) \le 2^{-nH(Y|X)(1-\epsilon_1)} \tag{1.25}$$

$$(1-\delta_{\epsilon_1,\epsilon_2}(n))\, 2^{nH(Y|X)(1-\epsilon_2)} \le |T^n_{\epsilon_2}(P_{XY}|x^n)| \le 2^{nH(Y|X)(1+\epsilon_2)} \tag{1.26}$$

$$1-\delta_{\epsilon_1,\epsilon_2}(n) \le \Pr\left[Y^n \in T^n_{\epsilon_2}(P_{XY}|x^n)\,\middle|\, X^n = x^n\right] \le 1. \tag{1.27}$$

The following result follows easily from Theorem 1.2 and will be extremely useful to us.

Theorem 1.3. Consider a joint distribution $P_{XY}(\cdot)$ and suppose $0 \le \epsilon_1 < \epsilon_2 \le \mu_{XY}$, Y^n is emitted by a DMS $P_Y(\cdot)$, and $x^n \in T^n_{\epsilon_1}(P_X)$. We have

$$\begin{aligned}(1-\delta_{\epsilon_1,\epsilon_2}(n))\; & 2^{-n[I(X;Y)+2\epsilon_2 H(Y)]} \\ & \le \Pr\left[Y^n \in T^n_{\epsilon_2}(P_{XY}|x^n)\right] \le 2^{-n[I(X;Y)-2\epsilon_2 H(Y)]}.\end{aligned} \tag{1.28}$$

Proof. The upper bound follows by (1.25) and (1.26):

$$\begin{aligned}\Pr\left[Y^n \in T^n_{\epsilon_2}(P_{XY}|x^n)\right] &= \sum_{y^n \in T_{\epsilon_2}(P_{XY}|x^n)} P^n_Y(y^n) \\ &\le 2^{nH(Y|X)(1+\epsilon_2)}\, 2^{-nH(Y)(1-\epsilon_2)} \\ &\le 2^{-n[I(X;Y)-2\epsilon_2 H(Y)]}. \end{aligned} \tag{1.29}$$

The lower bound also follows from (1.25) and (1.26). □

For small ϵ_1 and ϵ_2, large n, typical (x^n, y^n), and (X^n, Y^n) emitted by a DMS $P_{XY}(\cdot)$, we thus have

$$P^n_{Y|X}(y^n|x^n) \approx 2^{-nH(Y|X)} \tag{1.30}$$

$$|T^n_{\epsilon_2}(P_{XY}|x^n)| \approx 2^{nH(Y|X)} \tag{1.31}$$

$$\Pr\left[Y^n \in T^n_{\epsilon_2}(P_{XY}|x^n)\,|\,X^n = x^n\right] \approx 1 \tag{1.32}$$

$$\Pr\left[Y^n \in T^n_{\epsilon_2}(P_{XY}|x^n)\right] \approx 2^{-nI(X;Y)}. \tag{1.33}$$

We remark that the probabilities in (1.27) and (1.28) (or (1.32) and (1.33)) differ only in whether or not one conditions on $X^n = x^n$.

Example 1.4. Suppose X and Y are independent, in which case the approximations (1.32) and (1.33) both give

$$\Pr\left[Y^n \in T^n_{\epsilon_2}(P_{XY}|x^n)\right] \approx 1. \tag{1.34}$$

Note, however, that the precise version (1.28) of (1.33) is trivial for large n. This example shows that one must exercise caution when working with the approximations (1.30)–(1.33).

Example 1.5. Suppose that $X = Y$ so that (1.33) gives

$$\Pr\left[Y^n \in T^n_{\epsilon_2}(P_{XY}|x^n)\right] \approx 2^{-nH(X)}. \tag{1.35}$$

This result should not be surprising because $|T^n_{\epsilon_2}(P_X)| \approx 2^{nH(X)}$ and we are computing the probability of the event $X^n = x^n$ for some $x^n \in T^n_{\epsilon_1}(P_{XY})$ (the fact that ϵ_2 is larger than ϵ_1 does not play a role for large n).

1.6 Appendix: Proofs

Proof of Inequality (1.14)

We prove the bound (1.14). Consider first $P_X(a) = 0$ for which we have

$$\Pr\left[\frac{N(a|X^n)}{n} > P_X(a)(1+\epsilon)\right] = 0. \tag{1.36}$$

Next, suppose that $P_X(a) > 0$. Using the Chernoff bound, we have

$$\begin{aligned}
\Pr\left[\frac{N(a|X^n)}{n} > P_X(a)(1+\epsilon)\right] &\le \Pr\left[\frac{N(a|X^n)}{n} \ge P_X(a)(1+\epsilon)\right] \\
&\le E\left[e^{\nu N(a|X^n)/n}\right] e^{-\nu P_X(a)(1+\epsilon)} \\
&= \left[\sum_{m=0}^{n} \Pr\left[N(a|X^n) = m\right] e^{\nu m/n}\right] e^{-\nu P_X(a)(1+\epsilon)} \\
&= \left[\sum_{m=0}^{n} \binom{n}{m} P_X(a)^m (1-P_X(a))^{n-m} e^{\nu m/n}\right] e^{-\nu P_X(a)(1+\epsilon)} \\
&= \left[(1-P_X(a)) + P_X(a)e^{\nu/n}\right]^n e^{-\nu P_X(a)(1+\epsilon)}.
\end{aligned} \tag{1.37}$$

(1.38)

Optimizing (1.38) with respect to ν, we find that

$$\begin{array}{ll} \nu = \infty & \text{if } P_X(a)(1+\epsilon) \ge 1 \\ e^{\nu/n} = \frac{(1-P_X(a))(1+\epsilon)}{1-P_X(a)(1+\epsilon)} & \text{if } P_X(a)(1+\epsilon) < 1. \end{array} \tag{1.39}$$

In fact, the Chernoff bound correctly identifies the probabilities to be 0 and $P_X(a)^n$ for the cases $P_X(a)(1+\epsilon) > 1$ and $P_X(a)(1+\epsilon) = 1$, respectively. More interestingly, for $P_X(a)(1+\epsilon) < 1$ we insert (1.39) into (1.38) and obtain

$$\Pr\left[\frac{N(a|X^n)}{n} \ge P_X(a)(1+\epsilon)\right] \le 2^{-nD(P_B \| P_A)}, \tag{1.40}$$

where A and B are binary random variables with

$$\begin{aligned} P_A(0) &= 1 - P_A(1) = P_X(a) \\ P_B(0) &= 1 - P_B(1) = P_X(a)(1+\epsilon). \end{aligned} \tag{1.41}$$

We can write $P_B(0) = P_A(0)(1+\epsilon)$ and hence

$$D(P_B \| P_A) = P_A(0)(1+\epsilon)\log_2(1+\epsilon) + [1 - P_A(0)(1+\epsilon)]\log_2\left(\frac{1 - P_A(0)(1+\epsilon)}{1 - P_A(0)}\right). \quad (1.42)$$

We wish to further simplify (1.42). The first two derivatives of (1.42) with respect to ϵ are

$$\frac{dD(P_B \| P_A)}{d\epsilon} = P_A(0)\log_2\left(\frac{(1 - P_A(0))(1+\epsilon)}{(1 - P_A(0))(1+\epsilon)}\right) \quad (1.43)$$

$$\frac{d^2 D(P_B \| P_A)}{d\epsilon^2} = \frac{P_A(0)\log_2(e)}{(1+\epsilon)[1 - P_A(0)(1+\epsilon)]}. \quad (1.44)$$

We find that (1.43) is zero for $\epsilon = 0$ and we can lower bound (1.44) by $P_X(a)\log_2(e)$ for $0 \le \epsilon \le \mu_X$. The second derivative of $D(P_B \| P_A)$ with respect to ϵ is thus larger than $P_X(a)\log_2(e)$ and so we have

$$D(P_B \| P_A) \ge \epsilon^2 \cdot P_A(0)\log_2(e) \quad (1.45)$$

for $0 \le \epsilon \le \mu_X$. Combining (1.40) and (1.45) we arrive at

$$\Pr\left[\frac{N(a|X^n)}{n} \ge P_X(a)(1+\epsilon)\right] \le e^{-n\epsilon^2 P_X(a)}. \quad (1.46)$$

One can similarly bound

$$\Pr\left[\frac{N(a|X^n)}{n} \le P_X(a)(1-\epsilon)\right] \le e^{-n\epsilon^2 P_X(a)}. \quad (1.47)$$

Note that (1.46) and (1.47) are valid for all $a \in \mathcal{X}$ including a with $P_X(a) = 0$. However, the event in (1.14) has a strict inequality so we can improve the above bounds for the case $P_X(a) = 0$ (see (1.36)). This observation lets us replace $P_X(a)$ in (1.46) and (1.47) with μ_X and the result is (1.14).

Proof of Theorem 1.2

Suppose that $(x^n, y^n) \in T^n_{\epsilon_1}(P_{XY})$. We prove (1.25) by bounding

$$\begin{aligned} P^n_{Y|X}(y^n|x^n) &= \prod_{(a,b)\in \text{supp}(P_{XY})} P_{Y|X}(b|a)^{N(a,b|x^n,y^n)} \\ &\le \prod_{(a,b)\in \text{supp}(P_{XY})} P_{Y|X}(b|a)^{nP_{XY}(a,b)(1-\epsilon_1)} \\ &= 2^{n(1-\epsilon_1)\sum_{(a,b)\in \text{supp}(P_{XY})} P_{XY}(a,b)\log_2 P_{Y|X}(b|a)} \\ &= 2^{-n(1-\epsilon_1)H(Y|X)}. \end{aligned} \quad (1.48)$$

This gives the lower bound in (1.25) and the upper bound is proved similarly.

Next, suppose that $(x^n, y^n) \in T^n_\epsilon(P_{XY})$ and (X^n, Y^n) was emitted by the DMS $P_{XY}(\cdot)$. We prove (1.27) as follows.

Consider first $P_{XY}(a,b) = 0$ for which we have

$$\Pr\left[\frac{N(a,b|X^n,Y^n)}{n} > P_{XY}(a,b)(1+\epsilon)\right] = 0. \quad (1.49)$$

Now consider $P_{XY}(a,b) > 0$. If $N(a|x^n) = 0$, then $N(a,b|x^n,y^n) = 0$ and

$$\Pr\left[\frac{N(a,b|X^n,Y^n)}{n} > P_{XY}(a,b)(1+\epsilon)\middle| X^n = x^n\right] = 0. \quad (1.50)$$

More interestingly, if $N(a|x^n) > 0$ then the Chernoff bound gives

$$\begin{aligned} &\Pr\left[\frac{N(a,b|X^n,Y^n)}{n} > P_{XY}(a,b)(1+\epsilon)\middle| X^n = x^n\right] \\ &\le \Pr\left[\frac{N(a,b|X^n,Y^n)}{n} \ge P_{XY}(a,b)(1+\epsilon)\middle| X^n = x^n\right] \\ &= \Pr\left[\frac{N(a,b|X^n,Y^n)}{N(a|x^n)} \ge \frac{P_{XY}(a,b)}{N(a|x^n)/n}(1+\epsilon)\middle| X^n = x^n\right] \end{aligned}$$

$$\begin{aligned}
&\le E\left[e^{\nu N(a,b|X^n,Y^n)/N(a|x^n)} \middle| X^n = x^n \right] e^{-\nu \frac{P_{XY}(a,b)(1+\epsilon)}{N(a|x^n)/n}} \\
&= \left[\sum_{m=0}^{N(a|x^n)} \binom{N(a|x^n)}{m} P_{Y|X}(b|a)^m (1 - P_{Y|X}(b|a))^{N(a|x^n)-m} \right. \\
&\qquad \left. e^{\nu m/N(a|x^n)} \right] e^{-\nu \frac{P_{XY}(a,b)(1+\epsilon)}{N(a|x^n)/n}} \\
&= \left[(1 - P_{Y|X}(b|a)) + P_{Y|X}(b|a) e^{\nu/N(a|x^n)} \right]^{N(a|x^n)} e^{-\nu \frac{P_{XY}(a,b)(1+\epsilon)}{N(a|x^n)/n}}.
\end{aligned} \tag{1.51}$$

Minimizing (1.51) with respect to ν, we find that

$$\begin{aligned}
&\nu = \infty && \text{if } P_{XY}(a,b)(1+\epsilon) \ge N(a|x^n)/n \\
&e^{\nu/N(a|x^n)} = \frac{P_X(a)(1-P_{Y|X}(b|a))(1+\epsilon)}{N(a|x^n)/n - P_{XY}(a,b)(1+\epsilon)} && \text{if } P_{XY}(a,b)(1+\epsilon) < N(a|x^n)/n.
\end{aligned} \tag{1.52}$$

Again, the Chernoff bound correctly identifies the probabilities to be 0 and $P_{Y|X}(b|a)^n$ for the cases $P_{XY}(a,b)(1+\epsilon) > N(a|x^n)/n$ and $P_{XY}(a,b)(1+\epsilon) = N(a|x^n)/n$, respectively. More interestingly, for $P_{XY}(a,b)(1+\epsilon) < N(a|x^n)/n$ we insert (1.52) into (1.51) and obtain

$$\Pr\left[\frac{N(a,b|X^n)}{n} \ge P_{XY}(a,b)(1+\epsilon) \middle| X^n = x^n \right] \le 2^{-N(a|x^n)\, D(P_B \| P_A)}, \tag{1.53}$$

where A and B are binary random variables with

$$\begin{aligned}
P_A(0) &= 1 - P_A(1) = P_{Y|X}(b|a) \\
P_B(0) &= 1 - P_B(1) = \frac{P_{XY}(a,b)}{N(a|x^n)/n}(1+\epsilon).
\end{aligned} \tag{1.54}$$

We would like to have the form $P_B(0) = P_A(0)(1+\tilde{\epsilon})$ and compute

$$\tilde{\epsilon} = \frac{P_X(a)}{N(a|x^n)/n}(1+\epsilon) - 1. \tag{1.55}$$

We can now use (1.41)–(1.46) to arrive at

$$\begin{aligned}
&\Pr\left[\frac{N(a,b|X^n,Y^n)}{n} \ge P_{XY}(a,b)(1+\epsilon) \middle| X^n = x^n \right] \\
&\qquad \le e^{-N(a|x^n)\tilde{\epsilon}^2 P_{Y|X}(b|a)}
\end{aligned} \tag{1.56}$$

as long as $\epsilon \leq \min_{b:(a,b)\in \text{supp}(P_{XY})} P_{Y|X}(b|a)$. Now to guarantee that $\tilde{\epsilon}^2$ is positive, we must require that x^n is "more than" ϵ-letter typical, i.e., we must choose $x^n \in T_{\epsilon_1}(P_X)$, where $0 \leq \epsilon_1 < \epsilon$. Inserting $N(a|x^n)/n \geq (1+\epsilon_1)P_X(a)$ into (1.56), we have

$$\begin{aligned}\Pr&\left[\frac{N(a,b|X^n,Y^n)}{n} \geq P_{XY}(a,b)(1+\epsilon)\,\middle|\, X^n = x^n\right]\\ &\leq e^{-n\frac{(\epsilon-\epsilon_1)^2}{1+\epsilon_1}P_{XY}(a,b)} \qquad (1.57)\end{aligned}$$

for $0 \leq \epsilon_1 < \epsilon \leq \mu_{XY}$ (we could allow ϵ to be up to $\min_{b:(a,b)\in \text{supp}(P_{XY})}$ $P_{Y|X}(b|a)$ but we ignore this subtlety). One can similarly bound

$$\begin{aligned}\Pr&\left[\frac{N(a,b|X^n,Y^n)}{n} \leq P_{XY}(a,b)(1-\epsilon)\,\middle|\, X^n = x^n\right]\\ &\leq e^{-n\frac{(\epsilon-\epsilon_1)^2}{1+\epsilon_1}P_{XY}(a,b)}. \qquad (1.58)\end{aligned}$$

As for the unconditioned case, note that (1.57) and (1.58) are valid for all (a,b) including (a,b) with $P_{XY}(a,b) = 0$. However, the event we are interested in has a strict inequality so that we can improve the above bounds for the case $P_{XY}(a,b) = 0$ (see (1.49)). We can thus replace $P_{XY}(a,b)$ in (1.57) and (1.58) with μ_{XY} and the result is

$$\begin{aligned}\Pr&\left[\left|\frac{N(a,b|X^n,Y^n)}{n} - P_{XY}(a,b)\right| > \epsilon P_{XY}(a,b)\,\middle|\, X^n = x^n\right]\\ &\leq 2\cdot e^{-n\frac{(\epsilon-\epsilon_1)^2}{1+\epsilon_1}\mu_{XY}}. \qquad (1.59)\end{aligned}$$

for $0 \leq \epsilon_1 < \epsilon \leq \mu_{XY}$ (we could allow ϵ to be up to $\mu_{Y|X} = \min_{(a,b)\in \text{supp}(P_{XY})} P_{Y|X}(b|a)$ but, again, we ignore this subtlety). We thus have

$$\begin{aligned}&\Pr\left[Y^n \notin T^n_\epsilon(P_{XY}|x^n)\,|\, X^n = x^n\right]\\ &\quad = \Pr\left[\bigcup_{a,b}\left\{\left|\frac{N(a,b|X^n)}{n} - P_{XY}(a,b)\right| > \epsilon P_{XY}(a,b)\right\}\middle|\, X^n = x^n\right]\end{aligned}$$

$$\leq \sum_{a,b} \Pr\left[\left|\frac{N(a,b|X^n,Y^n)}{n} - P_{XY}(a,b)\right| > \epsilon P_{XY}(a,b) \middle| X^n = x^n\right]$$

$$\leq 2|\mathcal{X}||\mathcal{Y}| \cdot e^{-n\frac{(\epsilon-\epsilon_1)^2}{1+\epsilon_1}\mu_{XY}}, \tag{1.60}$$

where we have used the union bound for the last inequality. The result is the left-hand side of (1.27).

Finally, for $x^n \in T^n_{\epsilon_1}(P_X)$ and $0 \leq \epsilon_1 < \epsilon \leq \mu_{XY}$ we have

$$\begin{aligned}\Pr[Y^n \in T^n_\epsilon(P_{XY}|x^n) | X^n = x^n] &= \sum_{y^n \in T^n_\epsilon(P_{XY}|x^n)} P^n_{Y|X}(y^n|x^n) \\ &\leq |T^n_\epsilon(P_{XY}|x^n)| \, 2^{-n(1-\epsilon)H(Y|X)},\end{aligned} \tag{1.61}$$

where the inequality follows by (1.48). We thus have

$$|T^n_\epsilon(P_{XY}|x^n)| \geq (1 - \delta_{\epsilon_1,\epsilon}(n)) \, 2^{n(1-\epsilon)H(Y|X)}. \tag{1.62}$$

We similarly have

$$|T^n_\epsilon(P_{XY}|x^n)| \leq 2^{n(1+\epsilon)H(Y|X)}. \tag{1.63}$$

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