Block Toeplitz Matrices: Asymptotic Results and Applications

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Abstract

The present monograph studies the asymptotic behaviour of eigenvalues, products and functions of block Toeplitz matrices generated by the Fourier coefficients of a continuous matrix-valued function. This study is based on the concept of asymptotically equivalent sequences of non-square matrices. The asymptotic results on block Toeplitz matrices obtained are applied to vector asymptotically wide sense stationary processes. Therefore, this monograph is a generalization to block Toeplitz matrices of the Gray monograph entitled "Toeplitz and circulant matrices: A review", which was published in the second volume of Foundations and Trends in Communications and Information Theory, and which is the simplest and most famous introduction to the asymptotic theory on Toeplitz matrices.

Contents

1

Introduction

The Gray monograph entitled "Toeplitz and circulant matrices: A review" [8], which was published in the second volume of Foundations and Trends in Communications and Information Theory, is the simplest and most famous introduction to the asymptotic theory on Toeplitz matrices. The secret of the success of that monograph lies in the simplicity of the mathematical tools that Gray used to prove important results on Toeplitz matrices. Specifically, he proved asymptotic results on eigenvalues, products and inverses of Toeplitz matrices by using mainly the concept of asymptotically equivalent sequences of matrices, which he introduced in [7].

The present monograph is a generalization of the Gray monograph to block Toeplitz matrices, which is a type of matrices frequently used in Communications, Information Theory and Signal Processing, because, for instance, matrix representations of discrete-time causal finite impulse response (FIR) multiple-input multiple-output (MIMO) filters and correlation matrices of vector wide sense stationary (WSS) processes are block Toeplitz. Therefore, the present monograph deals with the asymptotic behaviour of eigenvalues, products and inverses of

block Toeplitz matrices, and, as in [8], the concept of asymptotically equivalent sequences of matrices is the key concept in it. However, since the blocks of block Toeplitz matrices are not, in general, square, we extend here the Gray definition of asymptotically equivalent sequences of matrices to sequences of non-square matrices.

Unlike in [8], where powers and inverses of Toeplitz matrices were the only functions of Toeplitz matrices studied, in the present monograph we cover any function of block Toeplitz matrices. Furthermore, while the Toeplitz matrices that Gray considered in [8] were generated by the Fourier coefficients of a function in the Wiener class (which is a special subset of the set of continuous functions) we consider here block Toeplitz matrices generated by the Fourier coefficients of any continuous matrix-valued function. Observe that even for the case of Toeplitz matrices (that is, block Toeplitz matrices with 1×1 blocks) the type of matrices considered here is more general, since they are Toeplitz matrices generated by the Fourier coefficients of continuous functions without imposing the Wiener condition.

Since the continuous functions are Riemann integrable, the integral that we use in the monograph is the Riemann integral. Thus, as in [8], it is not required to use the Lebesgue integral, an integral that is not included in a typical engineering background. We recall here that continuous functions are the most representative type of Riemann integrable functions, since the functions that are Riemann integrable on a closed interval are those which are continuous except on a set of Lebesgue measure zero and which are bounded.

Although there are some new results in the present monograph, most of them were given by the authors in [12] and [13], however, here they are presented in a tutorial manner and proved in detail.

The rest of the monograph is organized as follows. In Section 2 the mathematical preliminaries are given. We review the two matrix norms that are used in Section 3 to define the concept of asymptotically equivalent sequences of matrices: the spectral norm and the Frobenius norm. Furthermore, we review the concept of function of a matrix and we give some examples of functions of matrices, such as, the powers, the exponential or the inverse of a matrix.

Section 3 is devoted to the concept of asymptotically equivalent sequences of non-square matrices. We give its definition, its basic properties and we study its relation to the concept of function of a matrix.

In Section 4 we review the definition of a block Toeplitz matrix and show that matrix representations of discrete-time causal FIR MIMO filters and correlation matrices of vector WSS processes are block Toeplitz. However, most of the section is devoted to the presentation of several non-asymptotic properties of a sequence of block Toeplitz matrices generated by the Fourier coefficients of a continuous matrix-valued function.

In Section 5 we study a special type of block Toeplitz matrices: block circulant matrices. Moreover, we define the sequence of block circulant matrices generated by a continuous matrix-valued function, and we prove several non-asymptotic properties of this sequence.

Section 6 is devoted to the study of the asymptotic behaviour of block Toeplitz matrices. We begin by proving that the sequence of block Toeplitz matrices and the sequence of block circulant matrices generated by the same continuous matrix-valued function are asymptotically equivalent. Based on this result and the results obtained in the previous sections, we analyse the asymptotic behaviour of eigenvalues, products and functions of block Toeplitz matrices generated by the Fourier coefficients of a continuous matrix-valued function. In particular, we prove the most famous asymptotic result on block Toeplitz matrices: the Szegö theorem for block Toeplitz matrices.[1] This theorem deals with the arithmetic mean of the eigenvalues of functions of large Hermitian block Toeplitz matrices, and it has found different applications in Information Theory and Signal Processing (see, e.g., [6, 12, 13, 15, 19, 23]). As a matter of fact, in Section 6 we prove that the Szegö theorem for block Toeplitz matrices is also true if the sequence of Hermitian block Toeplitz matrices is replaced by any other asymptotically equivalent sequence of Hermitian matrices.

Finally, the theory on block Toeplitz matrices developed in this monograph is applied in Section 7 to study some vector non-stationary

[1] The Szegö theorem for block Toeplitz matrices is the generalization to block Toeplitz matrices of the famous result on Toeplitz matrices given by Szegö in [9, p. 64].

processes. Specifically, we define there the concept of vector asymptotically WSS (AWSS) process, which is based on the concept of asymptotically equivalent sequences of matrices, and we give several interesting properties and examples of this kind of process. Moreover, we compute the differential entropy rate of certain vector AWSS processes.

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