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Power Control for Battery-Limited Energy Harvesting Communications

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Power Control for Battery-Limited Energy Harvesting Communications

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ABSTRACT

Power control is often used to ensure efficient resource utilization in communication systems. Its role becomes even more critical in the emerging paradigm of energy harvesting communications due to the intermittency and randomness of ambient energy sources. This monograph provides a review of the fundamental power control policies and their performance analysis in the basic setting of a discrete-time battery-limited energy harvesting communication system with independent and identically distributed energy arrivals. Three different settings, namely, offline power control, online power control, and power control with lookahead, are considered, corresponding respectively to the cases with non-causal, causal, and partial non-causal knowledge of the energy arrival process. A complete characterization of the optimal offline power control policy is presented. In the online setting, the focus is placed on the greedy policy, which is optimal in the low-battery-capacity regime, and universally near-optimal policies, which include the maximin optimal

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policy, the fixed fraction policy, the two-piece fixed fraction policy, and the locally fixed fraction policy. Finally, power control with lookahead is introduced to bridge offline and online power control, the entire spectrum of optimal policies is characterized for Bernoulli energy arrivals, and the extension beyond the Bernoulli case is also discussed.

Notation

General

$:=$	is defined to be (in a local scope)
$\triangleq$	is defined to be (globally)
$\mathbb{Z}$	set of integers
$\mathbb{Z}_{\geq 0}$	set of non-negative integers
$\mathbb{Z}_{>0}$	set of positive intergers
$\mathbb{R}$	field of real numbers
$\mathbb{R}_{\geq 0}$	set of non-negative real numbers
x_m^n, x^n, x^∞	sequences $(x_i)_{i=m}^n$, $(x_i)_{i=1}^n$, and $(x_i)_{i=1}^\infty$, respectively
X, X^n, X^∞	random variables or sequences denoted by capital letters
e	Euler's number $2.71828\ldots$
$\log x$	$\log_{\mathrm{e}} x$
$\lg x$	$\log_{10} x$
$\lfloor x \rfloor$	the largest integer less than or equal to x
$\lceil x \rceil$	the smallest integer greater than or equal to x
$\langle x \rangle_{a,b}$	clip the number x by the interval $[a, b]$

$\langle x\rangle_{\geq a}$, $\langle x\rangle_{\leq b}$	variants of $\langle x\rangle_{a,+\infty}$ and $\langle x\rangle_{-\infty,b}$, respectively
$f\vert_A$	restriction of the function f to A
$f^{(n)}$	nth iteration of the function f with $f^{(0)}(x) := x$
$1\{S\}$	indicator function that equals 1 if the statement S is true and 0 otherwise
$\mathfrak{B}(S)$	Borel σ-field generated by the topology on a metric space S
$\mathbb{E}X$	expectation of the random variable X
$\mathcal{N}(\mu,\sigma^2)$	normal distribution with mean μ and variance σ^2
δ_x	one-point distribution at x
$\mathrm{B}_{q,a}$	(variant of) Bernoulli distribution supported on $\{0,a\}$ with probablity q at a
E_λ	exponential distribution with the parameter λ
U_b	uniform distribution on $[0,b]$
$\mathrm{epi}(f)$	epigraph of the function f
$\mathrm{hyp}(f)$	hypograph of the function f
$\mathrm{conv}(A)$	convex hull of the set A
$\mathrm{cl}(A)$	closure of the set A
$\underline{{}_{\downarrow}f}_A(x)$	lower semi-continuous envelope of f over A
$\overline{{}^{\uparrow}f}_A(x)$	upper semi-continuous envelope of f over A
$\underline{{}_{\vee}f}_A(x)$	lower convex envelope of f over A
$\overline{{}^{\wedge}f}_A(x)$	upper concave envelope of f over A
$\vee(f)$, $\wedge(f)$	convex and concave points of f, respectively
$f(x)=\mathrm{O}(g(x))$	indicate that $\limsup_{x\to\infty} f(x)/g(x) < +\infty$ (supposing that $f(x)$ and $g(x)$ are both positive) and the point where x tends to, if not infinity, can be specified by, e.g., $\mathrm{O}_{\downarrow 0}(g(x))$
$f(x)=\mathrm{o}(g(x))$	indicate that $\lim_{x\to\infty} f(x)/g(x) = 0$
$f(x)=\Theta(g(x))$	$f(x)=\mathrm{O}(g(x))$ and $g(x)=\mathrm{O}(f(x))$
$f(x)\sim_a g(x)$	indicate that $\lim_{x\to a}(f(x)/g(x)) = 1$

Model

c	battery capacity
$\mathcal{R}_m^n, \mathcal{G}^{(n)}, \mathcal{G}$	finite-horizon total reward, n-horizon average reward, and average reward, respectively
μ, $\bar{\mu}$, $\bar{\varsigma}$	mean, clipped mean, and clipped (standard) deviation of energy arrivals, respectively
MCR	mean-to-capacity ratio
NMCR	nominal mean-to-capacity ratio
NSNR	nominal signal-to-noise ratio
DCR	deviation-to-capacity ratio
$\mathrm{i_v}$	variation index of energy arrivals
$\mathcal{G}^*_{\mathrm{on}}$	maximum online average reward
$\mathcal{G}^*_{\mathrm{lk}[w]}$	maximum w-lookahead average reward
$\overline{\mathrm{G}}_+(\pi)$	nominal additive gap of the policy π
$\overline{\mathrm{G}}_\times(\pi)$	nominal multiplicative gap of the policy π
$\mathrm{G}_+(\pi)$	additive gap of the policy π
$\mathrm{G}_\times(\pi)$	multiplicative gap of the policy π

Policy

$\Pi_\infty^{(n)}$	collection of all (admissible) n-horizon offline (power control) policies
Π_0	collection of all (admissible) online (power control) policies
Π_0^{M}	collection of all Markov online policies
Π_0^{S}	collection of all stationary online policies
Π_0^{SN}	collection of all normal stationary policies
Π_w	collection of all (admissible) w-lookahead (power control) policies
Π_w^{S}	collection of all stationary w-lookahead policies

$\overline{\sigma}$	reserve policy corresponding to the stationary online policy σ
$\mathrm{i_g}(\sigma; s)$	greed index of the stationary online policy σ on $[0, s]$
π^*_{off}	optimal offline policy
π_{st}	save-and-transmit policy
σ_{grd}	greedy policy
$\sigma_{\mathrm{ff}(p)}$	fixed fraction policy with parameter p
$\sigma_{\mathrm{mo}(p)}$	maximin optimal policy with parameter p
$\sigma_{\mathrm{tff}(p)}$	two-piece fixed fraction policy with parameter p
$\sigma_{\mathrm{lff}(p)}$	locally fixed fraction policy with parameter p
$\sigma_{\mathrm{b}(p,w)}$	Bernoulli-optimal w-lookahead policy with parameter p and lookahead window size w

Appendices

A

Technical Details behind Common Reward Functions

A.1 AWGN Reward Function

Suppose that, during a time slot, a transmission is performed over a continuous-time additive white Gaussian noise (AWGN) channel with bandwidth W, which can be represented by a complex baseband AWGN channel (W independent uses per second)

$$Y_i = hX_i + Z_i,$$

where X_i and Y_i are the channel input and output, respectively, h is the complex channel gain, and Z_i is the channel noise that is circular symmetric complex normal with mean zero and variance N_0, denoted by $\mathcal{CN}(0, N_0)$. Thus, the whole system can be formulated as follows.

Suppose that the duration of a time slot is T_s and the energy consumed in time slot t is U_t. Then the maximum amount of data transmitted in time slot t is

$$r(U_t) := WT_s \log\left(1 + \frac{|h|^2 U_t}{N_0 W T_s}\right) \quad \text{nats} \tag{A.1}$$

by the AWGN channel capacity formula (e.g., [118, Eq. (5.26)] with $L = 1$). The n-horizon (expected) throughput and the long-term (expected) throughput can be computed as

$$\mathcal{T}^{(n)}(b) := \frac{1}{nT_s} \mathbb{E}\, \mathcal{R}_1^{n+1}(b) \quad \text{nats/s}$$

and

$$\mathcal{T}(b) := \liminf_{n\to\infty} \mathcal{T}^{(n)}(b) \quad \text{nats/s},$$

respectively. Comparing (A.1) with (1.6), we obtain the conversion law between a real AWGN power control problem and the canonical problem.

Proposition A.1. Let c_{org}, e_{org}, $\mathcal{T}^{(n)}(b_{\text{org}})$, and $\mathcal{T}(b_{\text{org}})$ be the battery capacity, the energy level (consumed, harvested, or available in the battery), the n-horizon throughput, and the long-term throughput, respectively, in a real AWGN power control problem. Then, their relations to the canonical counterparts (c, e, $\mathcal{G}^{(n)}(b)$, and $\mathcal{G}(b)$ for the canonical AWGN reward (1.6), respectively) are given by

$$c = \gamma c_{\text{org}}, \tag{A.2a}$$
$$e = \gamma e_{\text{org}}, \tag{A.2b}$$
$$\mathcal{T}^{(n)}(b_{\text{org}}) = W\,\mathcal{G}^{(n)}(b) = W\,\mathcal{G}^{(n)}(\gamma b_{\text{org}}), \tag{A.2c}$$
$$\mathcal{T}(b_{\text{org}}) = W\,\mathcal{G}(b) = W\,\mathcal{G}(\gamma b_{\text{org}}), \tag{A.2d}$$

where

$$\gamma := \frac{|h|^2}{N_0 W T_s} \tag{A.3}$$

is called *channel coefficient.*

Sketch of Proof. Comparing (1.6) with (A.1) gives (A.2a) and (A.2b). The verification of (A.2c) and (A.2d) is straightforward. □

A.2 MSE Distortion Reward Function

Consider a sensor node collecting data from an i.i.d. Gaussian source with sample rate f_s samples/s. The collected data are then transmitted to another node over a continuous-time AWGN channel with bandwidth

W. Suppose that the duration of a time slot is T_s and the energy consumed in time slot t is U_t. By (A.1), the maximum quantization rate in time slot t is

$$\frac{WT_s \log\left(1 + \dfrac{|h|^2 U_t}{N_0 W T_s}\right)}{f_s T_s} = \frac{\alpha_s}{2}\log(1+\gamma U_t) \quad \text{bits/sample},$$

where $\alpha_s := 2W/f_s$ and γ is the channel coefficient defined by (A.3).

By [30, Thm. 10.3.2], the rate-distortion function for a $\mathcal{N}(0, \sigma_s^2)$ source with squared-error distortion is

$$R(D) = \frac{1}{2}\left\langle \log\frac{\sigma_s^2}{D}\right\rangle_{\geq 0},$$

where $\mathcal{N}(\mu, \sigma_s^2)$ denotes a Gaussian distribution with mean μ and variance σ_s^2. Thus, the minimum MSE distortion of samples in time slot t is

$$c(U_t) = \sigma_s^2 \mathrm{e}^{-\alpha_s \log(1+\gamma U_t)} = \sigma_s^2(1+\gamma U_t)^{-\alpha_s}. \tag{A.4}$$

The n-horizon (expected) average distortion and the long-term (expected) average distortion can be computed as

$$\mathcal{C}^{(n)}(b) := \frac{1}{n}\mathbb{E}\sum_{t=1}^{n} c(U_t)$$

and

$$\mathcal{C}(b) := \liminf_{n\to\infty} \mathcal{C}^{(n)}(b),$$

respectively. Comparing (A.4) with (1.7), we obtain the conversion law between a real MSE distortion power control problem and the canonical problem.

Proposition A.2. Let c_{org}, e_{org}, $\mathcal{C}^{(n)}(b_{\text{org}})$, and $\mathcal{C}(b_{\text{org}})$ be the battery capacity, the energy level (consumed, harvested, or available in the battery), the n-horizon average distortion, and the long-term average distortion, respectively, in a real MSE distortion power control problem. Then, their relations to the canonical counterparts (c, e, $\mathcal{G}^{(n)}(b)$, and $\mathcal{G}(b)$ for the canonical MSE distortion reward (1.7), respectively) are given by

$$c = \gamma c_{\mathrm{org}}, \tag{A.5a}$$

$$e = \gamma e_{\mathrm{org}}, \tag{A.5b}$$

$$\mathcal{C}^{(n)}(b_{\mathrm{org}}) = -\sigma_s^2\,\mathcal{G}^{(n)}(b) = -\sigma_s^2\,\mathcal{G}^{(n)}(\gamma b_{\mathrm{org}}), \tag{A.5c}$$

$$\mathcal{C}(b_{\mathrm{org}}) = -\sigma_s^2\,\mathcal{G}(b) = -\sigma_s^2\,\mathcal{G}(\gamma b_{\mathrm{org}}). \tag{A.5d}$$

Sketch of Proof. Comparing (1.7) with (A.4) gives (A.5a) and (A.5b). The verification of (A.5c) and (A.5d) is straightforward. □

B

Proofs of Some Results in Chapters 3 and 4

B.1 Optimality of Greedy Policy

Proof of Proposition 3.1. For $c \in (\underline{x}, \overline{x})$,

$$\begin{aligned}\check{\psi}(c) &\geq r'(c) - \int_{[0,c)} \overline{{}^\wedge r'}_{[\underline{x},c]}(s) Q(\mathrm{d}s) \\ &\overset{\text{(a)}}{\geq} r'(c) - \rho(c)\overline{{}^\wedge r'}_{[\underline{x},c]}\left(\frac{\int_{[0,c)} s\mathrm{d}Q}{\rho(c)}\right) \\ &\overset{\text{(b)}}{\geq} \check{\psi}_\downarrow(c) := r'(c) - \rho(c)\overline{{}^\wedge r'}_{[\underline{x},c]}(\underline{\xi}(\rho(c))),\end{aligned}$$

where (a) follows from Jensen's inequality, and (b) follows from the strictly decreasing property of $\overline{{}^\wedge r'}_{[\underline{x},c]}$ (Proposition C.4) and the fact that

$$\int_{[0,c)} s\mathrm{d}Q = \mu - \int_{[c,+\infty)} s\mathrm{d}Q \geq \mu - \overline{x}Q([c,+\infty)) = \mu - (1-\rho(c))\overline{x}$$

and

$$\int_{[0,c)} s\mathrm{d}Q \geq \rho(c)\underline{x}.$$

Then the inequality $\check{\psi}_\downarrow(c) \geq 0$ always implies $\check{\psi}(c) \geq 0$, and hence

$$\begin{aligned} c^* &\geq \sup\{c < \overline{x} : \check{\psi}_\downarrow(c) \geq 0\} \qquad \text{(Theorem 3.11 and Remark 3.3)} \\ &\overset{(a)}{=} \sup\{c \in (\underline{x}, \overline{x}) : \check{\psi}_\downarrow(c) \geq 0\} = \underline{c}, \end{aligned}$$

where (a) is due to the fact that $\lim_{c \downarrow \underline{x}} \check{\psi}_\downarrow(c) = r'(\underline{x}) - Q(\underline{x}) r'(\underline{x}) > 0$. □

Proof of Proposition 3.2. For $c \in (\underline{x}, \overline{x})$,

$$\begin{aligned} \check{\psi}(c) &\leq r'(c) - \int_{[0,c)} \underline{\vee r'}_{[\underline{x},c]}(s) Q(\mathrm{d}s) \\ &\overset{(a)}{\leq} r'(c) - \rho(c) \underline{\vee r'}_{[\underline{x},c]} \left(\frac{\int_{[0,c)} s \mathrm{d}Q}{\rho(c)} \right) \\ &\overset{(b)}{\leq} \check{\psi}_\uparrow(c) := r'(c) - \rho(c) \underline{\vee r'}_{[\underline{x},c]} (\langle \overline{\xi}(\rho(c), c) \rangle_{\leq c}), \end{aligned}$$

where (a) follows from Jensen's inequality, and (b) follows from the strictly decreasing property of $\underline{\vee r'}_{[\underline{x},c]}$ (Proposition C.4) and the fact that

$$\int_{[0,c)} s \mathrm{d}Q = \mu - \int_{[c,+\infty)} s \mathrm{d}Q \leq \mu - cQ([c,+\infty)) = \mu - (1-\rho(c))c$$

and

$$\int_{[0,c)} s \mathrm{d}Q \leq \rho(c) c.$$

Then the inequality $\check{\psi}(c) \geq 0$ always implies $\check{\psi}_\uparrow(c) \geq 0$, and hence

$$\begin{aligned} c^* &\leq \sup\{c \in (\underline{x}, \overline{x}) : \check{\psi}_\uparrow(c) \geq 0\} \quad \text{(Theorem 3.11 and Remark 3.3)} \\ &\overset{(a)}{=} \sup\{c \in (\mu, \overline{x}) : \check{\psi}_\uparrow(c) \geq 0\} \\ &\overset{(b)}{=} \overline{c}, \end{aligned}$$

where (a) is due to the fact that

$$\lim_{c \downarrow \mu} \check{\psi}_\uparrow(c) = r'(\mu) - Q([0,\mu]) r'(\mu) > 0,$$

and (b) follows from the inequality $\overline{\xi}(t,c) < c$ for $c > \mu$. □

Proof of Lemma 3.12. By definition, any $c \in [0, \underline{\underline{c}}'(x_1, x_2, \mu_0))$ satisfies

$$r'(c) \geq \overline{\chi}(c, x_1, x_2, \mu_0) \geq \chi(c, Q),$$

or equivalently, $c \leq c^*(Q)$, for all $Q \in \mathcal{Q}_{x_1,x_2,\mu_0}$. This implies that $c \leq \underline{\underline{c}}(x_1, x_2, \mu_0)$ for all $c \in [0, \underline{\underline{c}}'(x_1, x_2, \mu_0))$ and hence $\underline{\underline{c}}'(x_1, x_2, \mu_0) \leq \underline{\underline{c}}(x_1, x_2, \mu_0)$. On the other hand, any $c \in [0, \underline{\underline{c}}(x_1, x_2, \mu_0))$ satisfies $r'(c) \geq \chi(c, Q)$ for all $Q \in \mathcal{Q}_{x_1,x_2,\mu_0}$, or equivalently,

$$r'(c) \geq \overline{\chi}(c, x_1, x_2, \mu_0).$$

This implies that $c \leq \underline{\underline{c}}'(x_1, x_2, \mu_0)$ for all $c \in [0, \underline{\underline{c}}(x_1, x_2, \mu_0))$ and hence $\underline{\underline{c}}(x_1, x_2, \mu_0) \leq \underline{\underline{c}}'(x_1, x_2, \mu_0)$. Therefore, $\underline{\underline{c}}(x_1, x_2, \mu_0) = \underline{\underline{c}}'(x_1, x_2, \mu_0)$.

Similarly, any $c \in (x_1, \overline{\overline{c}}'(x_1, x_2, \mu_0))$ (which is non-empty by Remark 3.3) satisfies

$$r'(c) \geq \underline{\chi}(c, x_1, x_2, \mu_0) = \chi(c, Q_0)$$

for some $Q_0 \in \mathcal{Q}_{x_1,x_2,\mu_0}$ (Lemma 3.13). This implies that $c \leq c^*(Q_0) \leq \overline{\overline{c}}(x_1, x_2, \mu_0)$ for all $c \in (x_1, \overline{\overline{c}}'(x_1, x_2, \mu_0))$ and hence $\overline{\overline{c}}'(x_1, x_2, \mu_0) \leq \overline{\overline{c}}(x_1, x_2, \mu_0)$. On the other hand, for any $c \in [0, \overline{\overline{c}}(x_1, x_2, \mu_0))$, there exists some $Q_0 \in \mathcal{Q}_{x_1,x_2,\mu_0}$ such that $c \leq c^*(Q_0)$, or

$$r'(c) \geq \chi(c, Q_0) \geq \underline{\chi}(c, x_1, x_2, \mu_0).$$

This implies that $c \leq \overline{\overline{c}}'(x_1, x_2, \mu_0)$ for all $c \in [0, \overline{\overline{c}}(x_1, x_2, \mu_0))$ and hence $\overline{\overline{c}}(x_1, x_2, \mu_0) \leq \overline{\overline{c}}'(x_1, x_2, \mu_0)$. Therefore, $\overline{\overline{c}}(x_1, x_2, \mu_0) = \overline{\overline{c}}'(x_1, x_2, \mu_0)$. $\square$

Proof of Proposition 3.3. Observe that

$$\int f(x) Q(\mathrm{d}x) \geq \int \underline{{}_{\vee}f}(x) Q(\mathrm{d}x) \geq \underline{{}_{\vee}f}(\mu_0),$$

where the last inequality follows from Jensen's inequality. Use Proposition C.5 to obtain (3.53) and (3.56).

To prove (3.59), we consider two typical cases, and other cases can be proved in a similar way.

If $f(s) \neq \underline{{}_{\downarrow}f}(s)$, $s < \mu_0$, and $f(t) = \underline{{}_{\downarrow}f}(t)$, then $f(s) > \underline{{}_{\downarrow}f}(s)$, hence the point $p_s := (s, \underline{{}_{\downarrow}f}(s))$ must be a limit point of $\mathrm{epi}(f)$, and therefore there exists a sequence of points (s_n, y_n) converging to p_s and satisfying $s_n \in [a, \mu_0)$ and $y_n \geq f(s_n)$. Since $\liminf_{x\to s} f(x) \geq \underline{{}_\downarrow f}(s)$ (by the lower semi-continuity of $\underline{{}_\downarrow f}(x)$), the sequence $(s_n, f(s_n))_{n=1}^\infty$ also converges to p_s. In other words,

$$a \leq s_n < \mu_0, \quad \lim_{n\to\infty} s_n = s, \quad \lim_{n\to\infty} f(s_n) = \underline{{}_\downarrow f}(s).$$

It is easy to verify that

$$Q_n := \hat{\rho}(s_n, t, \mu_0)\delta_{s_n} + (1 - \hat{\rho}(s_n, t, \mu_0))\delta_t$$

satisfies (3.59) as well as $Q_n([a,b]) = 1$ and $\mu(Q_n) = \mu_0$.

If $f(s) \neq \underline{{}_\downarrow f}(s)$ but $s = t = \mu_0 \in (a,b)$ (regardless of whether $f(t) = \underline{{}_\downarrow f}(t)$), then there exists a sequence $(s_n)_{n=1}^\infty$ satisfying

$$|s_n - \mu_0| < \epsilon, \quad \lim_{n\to\infty} s_n = \mu_0, \quad \lim_{n\to\infty} f(s_n) = \underline{{}_\downarrow f}(\mu_0),$$

where $\epsilon := (\min\{\mu_0 - a, b - \mu_0, 1\})^2$. Let

$$t_n := \mu_0 + |s_n - \mu_0|^{1/2}(1\{s_n < \mu_0\} - 1\{s_n > \mu_0\}).$$

It is clear that $s_n, t_n \in (a,b)$, $\lim_{n\to\infty} t_n = \mu_0$, and

$$\lim_{n\to\infty} \hat{\rho}(s_n, t_n, \mu_0) = \lim_{n\to\infty} \frac{t_n - \mu_0}{t_n - s_n} = \lim_{n\to\infty} \frac{|s_n - \mu_0|^{1/2}}{|s_n - \mu_0| + |s_n - \mu_0|^{1/2}} = 1.$$

It is then easy to verify that

$$Q_n := \hat{\rho}(s_n, t_n, \mu_0)\delta_{s_n} + (1 - \hat{\rho}(s_n, t_n, \mu_0))\delta_{t_n}$$

satisfies (3.59) as well as $Q_n([a,b]) = 1$ and $\mu(Q_n) = \mu_0$. □

Proof of Lemma 3.13. Let

$$f(x) := r'(x)1\{0 \leq x < c\}.$$

For $c \in (x_1, x_2)$, it is clear that $\underline{{}_\downarrow f}_{[x_1,x_2]}(x) = f(x)$ and $\overline{{}^\uparrow f}_{[x_1,x_2]}(x) = r'(x)1\{0 \leq x \leq c\}$.

By Proposition 3.3,

$$\begin{aligned}
\underline{\chi}(c, x_1, x_2, \mu_0) &= \sup_{Q\in\mathcal{Q}_{x_1,x_2,\mu_0}} \chi(c, Q) \\
&= \overline{{}^\wedge f}_{[x_1,x_2]}(\mu_0) \\
&= t\,\overline{{}^\uparrow f}_{[x_1,x_2]}(s_1) + (1-t)\overline{{}^\uparrow f}_{[x_1,x_2]}(s_2),
\end{aligned}$$

where $s_1 \leq \mu_0 \leq s_2$, $ts_1 + (1-t)s_2 = \mu_0$, and both s_1 and s_2 are adjacent or equal concave points of $f|_{[x_1,x_2]}$. Note that the points x_1 and x_2 are the concave points of $f|_{[x_1,x_2]}$. Candidates of other concave points of $f|_{[x_1,x_2]}$ are the concave points of $r'|_{[x_1,c]}$. If $s_2 = \mu_0$, then

$$x_1 < s_1 = s_2 = \mu_0 \leq c$$

and

$$\overline{{}^{\wedge}f}_{[x_1,x_2]}(\mu_0) = r'(\mu_0).$$

This case can be regarded as a degenerate case of the next case with μ_0 being a concave point of $r'|_{[x_1,c]}$. If $\mu_0 < s_2 < x_2$, then

$$x_1 \leq s_1 < \mu_0 < s_2 \leq c < x_2$$

and

$$\overline{{}^{\wedge}f}_{[x_1,x_2]}(\mu_0) = tr'(s_1) + (1-t)r'(s_2) = \overline{{}^{\wedge}r'}_{[x_1,c]}(\mu_0).$$

If $s_2 = x_2$, then

$$x_1 \leq s_1 \leq c < s_2 = x_2, \quad x_1 \leq s_1 < \mu_0 < s_2 = x_2,$$

and

$$\overline{{}^{\wedge}f}_{[x_1,x_2]}(\mu_0) = \hat{\rho}(s_1, x_2, \mu_0) r'(s_1),$$

where $\hat{\rho}(s,t,v)$ is defined by (3.55). In summary, we have

$$\overline{\chi}(c, x_1, x_2, \mu_0) = \max_{s \in \overline{A}} \hat{\rho}(s, x_2, \mu_0) \overline{r'}_{[x_1,c]}(s),$$

where $\overline{A}$ is defined by (3.63).

In the same vein, by Proposition 3.3 with $\underline{{}_{\downarrow}f}_{[x_1,x_2]} = f(x)$,

$$\begin{aligned} \underline{\chi}(c, x_1, x_2, \mu_0) &= \min_{Q \in \mathcal{Q}_{x_1,x_2,\mu_0}} \chi(c, Q) \\ &= tf(s_1) + (1-t)f(s_2), \end{aligned}$$

where $s_1 \leq \mu_0 \leq s_2$, $ts_1 + (1-t)s_2 = \mu_0$, and both s_1 and s_2 are adjacent or equal convex points of $f|_{[x_1,x_2]}$. Note that the points x_1, c, and x_2 are the convex points of $f|_{[x_1,x_2]}$. Candidates of other convex points are the convex points of $r'|_{[x_1,c)}$.

For $c \in (x_1, \mu_0]$, we have $s_1 = c$ and $s_2 = x_2$ (or c if $c = \mu_0$), and hence

$$\underline{\chi}(c, x_1, x_2, \mu_0) = \underline{{}_{\vee}f}_{[x_1,x_2]}(\mu_0) = \chi(c, \delta_{\mu_0}) = 0.$$

For $c \in (\mu_0, x_2)$, according to the value of s_2, there are three cases to be considered. Note that $s_2 \neq x_2$, because in this case, s_1 and s_2 are not adjacent convex points of $f|_{[x_1,x_2]}$. If $s_2 = \mu_0$, then

$$x_1 < s_1 = s_2 = \mu_0 < c$$

and

$$\underline{\vee f}_{[x_1,x_2]}(\mu_0) = r'(\mu_0).$$

This case can be regarded as a degenerate case of the next case with μ_0 being a concave point of $r'|_{[x_1,c)}$. If $\mu_0 < s_2 < c$, then

$$x_1 \leq s_1 < \mu_0 < s_2 < c < x_2$$

and

$$\underline{\vee f}_{[x_1,x_2]}(\mu_0) = tr'(s_1) + (1-t)r'(s_2) = \underline{\vee r'}_{[x_1,c)}(\mu_0).$$

If $s_2 = c$, then

$$x_1 \leq s_1 < \mu_0 < s_2 = c < x_2$$

and

$$\underline{\vee f}_{[x_1,x_2]}(\mu_0) = \hat{\rho}(s_1, c, \mu_0) r'(s_1).$$

In all the three cases, we have

$$\chi(c, Q_0) = \underline{\vee f}_{[x_1,x_2]}(\mu_0)$$

with $Q_0 := t\delta_{s_1} + (1-t)\delta_{s_2}$. In summary, we have

$$\begin{aligned}\underline{\chi}(c, x_1, x_2, \mu_0) &= \min_{s \in \underline{A}'} \hat{\rho}(s, c, \mu_0) \underline{\vee r'}_{[x_1,c)}(s) = \min_{s \in \underline{A}} \hat{\rho}(s, c, \mu_0) \underline{\vee r'}_{[x_1,c]}(s) \\ &= \chi(c, Q_0),\end{aligned}$$

where $\underline{A}' := \langle \vee (r'|_{[x_1,c)}) \rangle_{\leq \mu_0} \cup \{\mu_0\}$ and $\underline{A}$ is defined by (3.64). $\square$

Proof of Proposition 3.4. When $r = r_{\mathrm{awgn}}$,

$$\overline{\wedge r'}_{[\underline{x},c]}(x) = \frac{1 + \underline{x} + c - x}{(1+\underline{x})(1+c)} \quad \text{for } x \in [\underline{x}, c] \tag{B.1}$$

and

$$\underline{\vee r'}_{[\underline{x},c]}(x) = \frac{1}{1+x} \quad \text{for } x \in [\underline{x}, c], \tag{B.2}$$

from which (3.69) and (3.70) follow immediately (Propositions 3.1 and 3.2).

By Theorem 3.14 and (B.1),

$$\overline{\chi}(c,\underline{x},\overline{x},\mu) = \max\left\{\frac{\hat{\rho}(\underline{x},\overline{x},\mu)}{1+\underline{x}}, \frac{\hat{\rho}(\langle c\rangle_{\le\mu},\overline{x},\mu)(1+\underline{x}+c-\langle c\rangle_{\le\mu})}{(1+\underline{x})(1+c)}\right\}$$

for $c \in (\underline{x},\overline{x})$, and the semi-universal lower bound $\underline{c}(\underline{x},\overline{x},\mu)$ is the supremum of the intersection of the solution sets

$$\underline{C}_1 := \left\{c \in (\underline{x},\overline{x}) : r'(c) \ge \frac{\hat{\rho}(\underline{x},\overline{x},\mu)}{1+\underline{x}}\right\}$$

and

$$\underline{C}_2 := \left\{c \in (\underline{x},\overline{x}) : r'(c) \ge \frac{\hat{\rho}(\langle c\rangle_{\le\mu},\overline{x},\mu)(1+\underline{x}+c-\langle c\rangle_{\le\mu})}{(1+\underline{x})(1+c)}\right\}.$$

It is clear that

$$\underline{C}_1 = \left(\underline{x}, \frac{(1+\underline{x})(\overline{x}-\underline{x})}{\overline{x}-\mu} - 1\right]$$

and $\underline{C}_2 = (\underline{x},\mu]$, from which (3.71) and (3.72) follow.

By Theorem 3.14 and (B.2), for $c \in (\mu,\overline{x})$,

$$\begin{aligned}
\underline{\chi}(c,\underline{x},\overline{x},\mu) &= \min_{s\in[\underline{x},\mu]} \frac{c-\mu}{(c-s)(1+s)} \\
&= \min_{s\in[\underline{x},\mu]} \frac{c-\mu}{-\left(s-\frac{c-1}{2}\right)^2 + \frac{(c+1)^2}{4}} \\
&= \begin{cases} \dfrac{c-\mu}{(c-\underline{x})(1+\underline{x})}, & c < \iota(\underline{x}), \\ \dfrac{4(c-\mu)}{(c+1)^2}, & \iota(\underline{x}) \le c \le \iota(\mu), \\ \dfrac{1}{1+\mu}, & c > \iota(\mu), \end{cases}
\end{aligned}$$

and the semi-universal upper bound $\overline{c}(\underline{x},\overline{x},\mu)$ is the supremum of the union of the solution sets

$$\overline{C}_1 := \left\{c \in (\mu,\overline{x}) \cap [0,\iota(\underline{x})) : r'(x) \ge \frac{c-\mu}{(c-\underline{x})(1+\underline{x})}\right\},$$

$$\overline{C}_2 := \left\{c \in (\mu,\overline{x}) \cap [\iota(\underline{x}),\iota(\mu)] : r'(x) \ge \frac{4(c-\mu)}{(c+1)^2}\right\},$$

$$\overline{C}_3 := \left\{c \in (\mu,\overline{x}) \cap (\iota(\mu),+\infty) : r'(x) \ge \frac{1}{1+\mu}\right\},$$

where

$$\iota(x) := 2x + 1.$$

Some algebraic manipulations yield

$$\begin{aligned}
\overline{C}_1 &= (\mu, \overline{x}) \cap [0, \iota(\underline{x})) \cap (\mu, c_1], \\
\overline{C}_2 &= (\mu, \overline{x}) \cap [\iota(\underline{x}), \iota(\mu)] \cap (\mu, c_2], \\
\overline{C}_3 &= (\mu, \overline{x}) \cap (\iota(\mu), +\infty) \cap [0, \mu] = \varnothing.
\end{aligned}$$

Thus,

$$\begin{aligned}
\overline{\overline{c}}(\underline{x}, \overline{x}, \mu) &= \begin{cases} \sup((\mu, c_1] \cap (\mu, \overline{x})), & \mu \leq \frac{3}{2}\underline{x} + \frac{1}{2}, \\ \sup((\mu, c_2] \cap (\mu, \overline{x})), & \mu > \frac{3}{2}\underline{x} + \frac{1}{2}, \end{cases} \\
&= \begin{cases} \langle c_1 \rangle_{\leq \overline{x}}, & \mu \leq \frac{3}{2}\underline{x} + \frac{1}{2}, \\ \langle c_2 \rangle_{\leq \overline{x}}, & \mu > \frac{3}{2}\underline{x} + \frac{1}{2}, \end{cases}
\end{aligned}$$

where c_1 and c_2 are defined by (3.77) and (3.78), respectively, because

$$c_1 \lesseqqgtr \iota(\underline{x}) \quad \text{for } \mu \lesseqqgtr \frac{3}{2}\underline{x} + \frac{1}{2}$$

and

$$c_2 \lesseqqgtr \iota(\underline{x}) \quad \text{for } \mu \lesseqqgtr \frac{3}{2}\underline{x} + \frac{1}{2}.$$

This proves (3.73). □

B.2 Universally Near-Optimal Online Policies

Proof of Theorem 3.22. By Lemma 3.18 and Theorem 3.19,

$$\begin{aligned}
&\sup_{c>0} \overline{\mathrm{G}}_+(\sigma_{\mathrm{ff}}; \mathcal{Q}_{c,p}) \\
&\quad = \lim_{c \to +\infty} \left(r(pc) - \sum_{i=0}^{\infty} p(1-p)^i r(\sigma_{\mathrm{ff}(p)}(\overline{\sigma_{\mathrm{ff}(p)}}^{(i)}(c))) \right) \\
&\quad = \lim_{c \to +\infty} \sum_{i=0}^{\infty} p(1-p)^i (\log(1+pc) - \log(1+p(1-p)^i c)) \\
&\quad \overset{(a)}{=} \sum_{i=0}^{\infty} p(1-p)^i \lim_{c \to +\infty} \log \frac{1+pc}{1+p(1-p)^i c} \\
&\quad = \sum_{i=1}^{\infty} ip(1-p)^i \log \frac{1}{1-p}
\end{aligned}$$

$$= \frac{1-p}{p} \log \frac{1}{1-p},$$

where (a) follows from the dominated convergence theorem. Since

$$\left(\frac{1-p}{p} \log \frac{1}{1-p} \right)' = \frac{\log(1-p)}{p^2} + \frac{1}{p} \leq -\frac{p}{p^2} + \frac{1}{p} = 0,$$

we have

$$\begin{aligned} \mathcal{M}\,\overline{\mathrm{G}}_+(\sigma_{\mathrm{ff}}) &= \sup_{p \in (0,1)} \frac{1-p}{p} \log \frac{1}{1-p} \\ &= \lim_{p \to 0} \frac{1-p}{p} \log \frac{1}{1-p} = 1. \end{aligned}$$

□

Proof of Theorem 3.23. It is easy to verify that for any $u > 0$,

$$\sigma_{\mathrm{ff}(p)}(\overline{\sigma_{\mathrm{ff}(p)}}^{(i)}(\sigma^{-1}_{\mathrm{ff}(p)}(u))) = (1-p)^i u$$

and

$$p\sigma^{-1}_{\mathrm{ff}(p)}(u) = u. \tag{B.3}$$

Let $h_{u,p}(t) := \sigma_{\mathrm{ff}(p)}(\overline{\sigma_{\mathrm{ff}(p)}}^{(\lfloor t/p \rfloor)}(\sigma^{-1}_{\mathrm{ff}(p)}(u)))$. Then we have $h_{u,p}(t) \leq u\mathrm{e}^{1-t}$ and

$$\lim_{p \to 0} h_{u,p}(t) = u\mathrm{e}^{-t} \quad \text{(Proposition C.11)}. \tag{B.4}$$

Hence,

$$r_0 := \int_0^{+\infty} \mathrm{e}^{-t} r(u\mathrm{e}^{-t}) \mathrm{d}t = \log(1+u) - 1 + \frac{\log(1+u)}{u}$$

and

$$\begin{aligned} \lim_{p \to 0} \left. \frac{\overline{g}_{\sigma_{\mathrm{ff}(p)}}(c,p)}{|r(pc)|} \right|_{c = \sigma^{-1}_{\mathrm{ff}(p)}(u)} &= \frac{\log(1+u) - r_0}{\log(1+u)} \quad \text{(Theorem 3.20)} \\ &= \frac{1}{\log(1+u)} - \frac{1}{u}, \end{aligned}$$

which concludes (3.130). Furthermore, we have

$$
\begin{aligned}
\mathcal{M}\,\overline{\mathrm{G}}_{\times}(\sigma_{\mathrm{ff}}) &= \sup_{\substack{c>0,\\ p\in(0,1)}} \frac{\overline{g}_{\sigma_{\mathrm{ff}}}(c,p)}{r(pc)} \quad \text{(Lemma 3.18)}\\
&= \sup_{\substack{u>0,\\ p\in(0,1)}} \left.\frac{\overline{g}_{\sigma_{\mathrm{ff}}}(c,p)}{r(pc)}\right|_{c=\sigma^{-1}_{\mathrm{ff}(p)}(u)}\\
&\geq \sup_{u>0} \lim_{p\to 0} \left.\frac{\overline{g}_{\sigma_{\mathrm{ff}}}(c,p)}{r(pc)}\right|_{c=\sigma^{-1}_{\mathrm{ff}(p)}(u)}\\
&= \sup_{u>0} \left(\frac{1}{\log(1+u)} - \frac{1}{u}\right)\\
&\overset{\text{(a)}}{=} \lim_{u\to 0} \left(\frac{1}{\log(1+u)} - \frac{1}{u}\right) = \frac{1}{2},
\end{aligned} \tag{B.5}
$$

where (a) is due to the strictly decreasing property of $1/\log(1+u) - 1/u$ for $u > 0$ because

$$
\left(\frac{1}{\log(1+u)} - \frac{1}{u}\right)' = -\frac{1}{(1+u)(\log(1+u))^2} + \frac{1}{u^2} < 0 \quad \text{([26, p. 184]).}
$$

On the other hand, since for all $c > 0$ and $p \in (0,1)$,

$$
\begin{aligned}
\mathcal{G}_{\mathrm{B}}(c,p;\sigma_{\mathrm{ff}(p)}) &= \sum_{i=0}^{\infty} p(1-p)^i r((1-p)^i pc + [1-(1-p)^i]0)\\
&\geq \sum_{i=0}^{\infty} p(1-p)^{2i} r(pc) \quad \text{(Jensen's inequality)}\\
&= \frac{p}{1-(1-p)^2} r(pc) = \frac{1}{2-p} r(pc),
\end{aligned}
$$

we have

$$
\mathcal{M}\,\overline{\mathrm{G}}_{\times}(\sigma_{\mathrm{ff}}) \leq \sup_{p\in(0,1)} \left(1 - \frac{1}{2-p}\right) \leq \frac{1}{2},
$$

which together with (B.5) concludes (3.131). □

Proof of Theorem 3.24. By Lemma 3.18 and Theorem 3.19,

$$
\mathcal{M}\,\overline{\mathrm{G}}_{+}(\sigma_{\mathrm{grd}}) = \sup_{p\in(0,1)} \lim_{c\to+\infty} \overline{g}_{\sigma_{\mathrm{grd}}}(c,p)
$$

and

$$\mathcal{M}\overline{\mathrm{G}}_{\times}(\sigma_{\mathrm{grd}}) = \sup_{\substack{c>0,\\ p\in(0,1)}} \frac{\overline{g}_{\sigma_{\mathrm{grd}}}(c,p)}{r(pc)},$$

where $\overline{g}_{\sigma_{\mathrm{grd}}}(c,p) = r(pc) - pr(c)$. Furthermore, we have

$$\begin{aligned}
\sup_{p\in(0,1)} \lim_{c\to+\infty} \overline{g}_{\sigma_{\mathrm{grd}}}(c,p) &= \sup_{p\in(0,1)} \lim_{c\to+\infty} (\log(1+pc) - p\log(1+c)) \\
&= \sup_{p\in(0,1)} \lim_{c\to+\infty} \log \frac{1+pc}{(1+c)^p} \\
&= \sup_{p\in(0,1)} \lim_{c\to+\infty} \log \frac{c^{1-p}(1/c+p)}{(1/c+1)^p} = +\infty
\end{aligned}$$

and

$$\begin{aligned}
\sup_{\substack{c>0,\\ p\in(0,1)}} \frac{\overline{g}_{\sigma_{\mathrm{grd}}}(c,p)}{r(pc)} &\geq \sup_{c>0} \lim_{p\to 0} \frac{r(pc)-pr(c)}{r(pc)} \\
&\overset{(a)}{=} \sup_{c>0} \lim_{p\to 0} \frac{cr'(pc)-r(c)}{cr'(pc)} \\
&= \sup_{c>0} \frac{c-r(c)}{c} = 1,
\end{aligned}$$

where (a) follows from L'Hospital's rule. The proof is complete by noting that the nominal multiplicative gap cannot be larger than one. □

Proof of Corollary 3.28. Let $y = \sigma_{\mathrm{mo}(p)}(x)$. We have

$$\kappa_s(x) = \frac{1+x}{s} - 1, \tag{B.6}$$

so

$$\begin{aligned}
x &= \eta_{1/(1-p)}(y) && \text{(Theorem 3.27)} \\
&= \sum_{i=0}^{\tilde{M}-1} [(1+y)(1-p)^i - 1] && \text{(Proposition 3.8)} \\
&= (1+y)\frac{1-(1-p)^{\tilde{M}}}{p} - \tilde{M}, && \text{(B.7)}
\end{aligned}$$

where

$$-\frac{\log(1+y)}{\log(1-p)} < \tilde{M} = \left\lfloor -\frac{\log(1+y)}{\log(1-p)} \right\rfloor + 1 \le -\frac{\log(1+y)}{\log(1-p)} + 1.$$

Solving (B.7) for y then gives

$$y = \frac{p(x+\tilde{M})}{1-(1-p)^{\tilde{M}}} - 1.$$

The proof is complete by noting that

$$\begin{aligned} & 1-p \le (1+y)(1-p)^{\tilde{M}} < 1 \\ \Rightarrow \quad & (1-p)[1-(1-p)^{\tilde{M}}] \le p(x+\tilde{M})(1-p)^{\tilde{M}} < 1-(1-p)^{\tilde{M}} \\ \Rightarrow\, & [1+p(x+\tilde{M})](1-p)^{\tilde{M}} < 1 \le [1+p(x+\tilde{M}-1)](1-p)^{\tilde{M}-1}. \ \square \end{aligned}$$

Proof of Theorem 3.29. By (B.6), $q_\infty(\kappa_{1/(1-p)}) = 1-p$, and hence $\bar{\rho} = 1 - q_\infty(\kappa_{1/(1-p)}) = p$. By Theorem 3.27, we have

$$\sigma_{\mathrm{mo}(p)}(x) \ge \bar{\rho}x = \sigma_{\mathrm{ff}(p)}(x)$$

and

$$\begin{aligned} \lim_{x\to+\infty}(r(\sigma_{\mathrm{mo}(p)}(x)) - r(\sigma_{\mathrm{ff}(p)}(x))) &= \log \lim_{x\to+\infty} \frac{1+\sigma_{\mathrm{mo}(p)}(x)}{1+px} \\ &= \log \frac{q_\infty(\sigma_{\mathrm{mo}(p)})}{p} = \log\frac{\bar{\rho}}{p} = 0. \end{aligned}$$

Then,

$$\limsup_{c\to+\infty}(\mathcal{G}_{\mathrm{B}}(c,p;\sigma_{\mathrm{mo}(p)}) - \mathcal{G}_{\mathrm{B}}(c,p;\sigma_{\mathrm{ff}(p)})) \le 0 \quad \text{(Theorem 3.21)},$$

hence

$$\lim_{c\to+\infty}(\mathcal{G}_{\mathrm{B}}(c,p;\sigma_{\mathrm{mo}(p)}) - \mathcal{G}_{\mathrm{B}}(c,p;\sigma_{\mathrm{ff}(p)})) = 0 \tag{B.8}$$

by the optimality of $\sigma_{\mathrm{mo}(p)}$ for B_p, and therefore

$$\begin{aligned} \sup_{c>0} \overline{\mathrm{G}}_+(\sigma_{\mathrm{mo}}; \mathcal{Q}_{c,p}) &= \lim_{c\to+\infty} \overline{g}_{\sigma_{\mathrm{mo}}}(c,p) \quad && \text{(Theorems 3.19 and 3.27)} \\ &= \lim_{c\to+\infty} \overline{g}_{\sigma_{\mathrm{ff}}}(c,p) \\ &= \sup_{c>0} \overline{\mathrm{G}}_+(\sigma_{\mathrm{ff}}; \mathcal{Q}_{c,p}) && \text{(Theorem 3.19)}. \end{aligned}$$

This together with Theorem 3.22 establishes the theorem. $\square$

Proof of Theorem 3.30. By Theorem 3.27, it is easy to see that

$$\sigma_{\mathrm{mo}(p)}(\overline{\sigma_{\mathrm{mo}(p)}}^{(i)}(\sigma_{\mathrm{mo}(p)}^{-1}(u))) = \begin{cases} \kappa_{1/(1-p)^i}(u), & \text{if } i < M(u), \\ 0, & \text{otherwise,} \end{cases}$$

and

$$p\sigma_{\mathrm{mo}(p)}^{-1}(u) = p \sum_{i=0}^{M(u)-1} \sigma_{\mathrm{mo}(p)}(\overline{\sigma_{\mathrm{mo}(p)}}^{(i)}(\sigma_{\mathrm{mo}(p)}^{-1}(u))),$$

where

$$\kappa_{1/(1-p)^i}(u) = (1-p)^i(1+u) - 1$$

and

$$M(u) = \left\lceil -\frac{\log(1+u)}{\log(1-p)} \right\rceil.$$

Let $h_{u,p}(t) := \sigma_{\mathrm{mo}(p)}(\overline{\sigma_{\mathrm{mo}(p)}}^{(\lfloor t/p \rfloor)}(\sigma_{\mathrm{mo}(p)}^{-1}(u)))$. Observe that

$$pM(u) < -p\frac{\log(1+u)}{\log(1-p)} + p \le \log(1+u) + 1$$

and

$$\lim_{p\to 0} pM(u) = \log(1+u).$$

Then by Proposition C.11,

$$\begin{aligned} h_{u,p}(t) &\le [\mathrm{e}^{1-t}(1+u) - 1]1\{0 \le t < pM(u)\} \\ &\le [\mathrm{e}^{1-t}(1+u) - 1]1\{0 \le t < \log(1+u) + 1\} \end{aligned}$$

and

$$\begin{aligned} \lim_{p\to 0} h_{u,p}(t) &= \lim_{p\to 0}[(1-p)^{\lfloor t/p \rfloor}(1+u) - 1]1\{0 \le t < pM(u)\} \\ &= [\mathrm{e}^{-t}(1+u) - 1]1\{0 \le t < \log(1+u)\}. \end{aligned}$$

Applying Theorem 3.20, we have

$$\begin{aligned} r_0 &:= \int_0^{\log(1+u)} \mathrm{e}^{-t} r(\mathrm{e}^{-t}(1+u) - 1)\mathrm{d}t \\ &= \int_0^{\log(1+u)} \mathrm{e}^{-t}(\log(1+u) - t)\mathrm{d}t = \frac{1}{1+u} + \log(1+u) - 1, \\ \bar{\mu}_0 &:= \int_0^{\log(1+u)} [\mathrm{e}^{-t}(1+u) - 1]\mathrm{d}t = u - \log(1+u), \end{aligned}$$

and

$$\lim_{p\to 0} \frac{\overline{g}_{\sigma_{\mathrm{mo}}}(c,p)}{r(pc)}\bigg|_{c=\sigma^{-1}_{\mathrm{mo}(p)}(u)}$$
$$= \frac{\log(1-\log(1+u)/(1+u)) - 1/(1+u) + 1}{\log(1+u-\log(1+u))}$$
$$= \overline{g}_0^{\times}(1+u).$$

Finally,

$$\begin{aligned}
\mathcal{M}\,\overline{\mathrm{G}}_{\times}(\sigma_{\mathrm{mo}}) &= \sup_{\substack{c>0,\\ p\in(0,1)}} \frac{\overline{g}_{\sigma_{\mathrm{mo}}}(c,p)}{r(pc)} \quad \text{(Lemma 3.18)}\\
&\geq \sup_{u>0} \lim_{p\to 0} \frac{\overline{g}_{\sigma_{\mathrm{mo}}}(c,p)}{r(pc)}\bigg|_{c=\sigma^{-1}_{\mathrm{mo}(p)}(u)}\\
&= \sup_{t>1} \overline{g}_0^{\times}(t).
\end{aligned}$$ □

Proof of Theorem 3.31. By Theorem 3.23 and Lemma 3.15, we have $\mathcal{M}\,\mathrm{G}_{\times}(\sigma_{\mathrm{ff}}) \leq \frac{1}{2}$, so it suffices to show that $\mathcal{M}\,\mathrm{G}_{\times}(\sigma_{\mathrm{ff}}) \geq \frac{1}{2}$. For this purpose, we will show that $\mathrm{G}_{\times}(\sigma_{\mathrm{ff}};\mathrm{B}_p)$ converges to $1/2$ as c and p go to zero. We have

$$\begin{aligned}
\lim_{p\to 0}\lim_{c\to 0} \mathrm{G}_{\times}(\sigma_{\mathrm{ff}};\mathrm{B}_p) &= 1 - \lim_{p\to 0}\lim_{c\to 0} \frac{\mathcal{G}_{\mathrm{B}}(c,p;\sigma_{\mathrm{ff}(p)})}{\mathcal{G}_{\mathrm{B}}(c,p;\sigma_{\mathrm{mo}(p)})}\\
&\overset{\text{(a)}}{=} 1 - \lim_{p\to 0}\sum_{i=0}^{\infty} p(1-p)^i \lim_{c\to 0} \frac{r(p(1-p)^i c)}{\mathcal{G}_{\mathrm{B}}(c,p;\sigma_{\mathrm{mo}(p)})}\\
&\overset{\text{(b)}}{=} 1 - \lim_{p\to 0}\sum_{i=0}^{\infty} p(1-p)^i \lim_{c\to 0} \frac{p(1-p)^i r'(p(1-p)^i c)}{p r'(\sigma_{\mathrm{mo}(p)}(0))}\\
&\overset{\text{(c)}}{=} 1 - \lim_{p\to 0}\sum_{i=0}^{\infty} p(1-p)^{2i}\\
&= 1 - \lim_{p\to 0} \frac{1}{2-p} = \frac{1}{2},
\end{aligned}$$

where (a) follows from Theorem 3.17 and the dominated convergence theorem with

$$\frac{r(p(1-p)^i c)}{\mathcal{G}_{\mathrm{B}}(c,p;\sigma_{\mathrm{mo}(p)})} \leq \frac{\mathcal{G}_{\mathrm{B}}(c,p;\sigma_{\mathrm{ff}(p)})}{\mathcal{G}_{\mathrm{B}}(c,p;\sigma_{\mathrm{mo}(p)})} \leq 1,$$

(b) from L'Hospital's rule and Theorem 3.32, and (c) follows from Corollary 3.28. □

Proof of Theorem 3.32. For almost every $c \ge 0$,

$$\begin{aligned}
&\frac{\partial \mathcal{G}_{\mathrm{B}}(c, p; \sigma_{\mathrm{mo}(p)})}{\partial c} \\
&\quad = \sum_{i=0}^{M(\sigma_{\mathrm{mo}(p)}(c))-1} p(1-p)^i r'(\sigma_{\mathrm{mo}(p)}(\overline{\sigma_{\mathrm{mo}(p)}}^i(c))) \frac{\partial \sigma_{\mathrm{mo}(p)}(\overline{\sigma_{\mathrm{mo}(p)}}^i(c))}{\partial c} \\
&\quad \overset{\text{(a)}}{=} \sum_{i=0}^{M(\sigma_{\mathrm{mo}(p)}(c))-1} pr'(\sigma_{\mathrm{mo}(p)}(c)) \frac{\partial \sigma_{\mathrm{mo}(p)}(\overline{\sigma_{\mathrm{mo}(p)}}^i(c))}{\partial c} \\
&\quad = pr'(\sigma_{\mathrm{mo}(p)}(c)) \frac{\partial \sum_{i=0}^{M(\sigma_{\mathrm{mo}(p)}(c))-1} \sigma_{\mathrm{mo}(p)}(\overline{\sigma_{\mathrm{mo}(p)}}^i(c))}{\partial c} \\
&\quad \overset{\text{(b)}}{=} pr'(\sigma_{\mathrm{mo}(p)}(c)) \frac{\partial \eta_{1/(1-p)}(\sigma_{\mathrm{mo}(p)}(c))}{\partial c} \\
&\quad = pr'(\sigma_{\mathrm{mo}(p)}(c)),
\end{aligned}$$

where (a) and (b) both follow from Theorem 3.27. □

Proof of Corollary 3.35. At first, we have $\tau_{1/(1-p)} = p/(1-p)$ and $\bar{\rho} = p$ (see (B.6)). (3.177) and the first piece of (3.178) then follow easily. It remains to prove the second piece of (3.178), which can be obtained by solving the equation

$$x = \frac{y^2}{y - \kappa_{1/(1-p)}(y)}$$

for $y \ge p/(1-p)$, which also implies that $x \ge p/(1-p)$. □

Proof of Theorem 3.36. Since

$$\sigma_{\mathrm{ff}(p)}(x) \le \sigma_{\mathrm{tff}(p)}(x) \le \sigma_{\mathrm{lff}(p)}(x) \le \sigma_{\mathrm{mo}(p)}(x) \quad \text{(Theorem 3.34)}$$

and

$$\lim_{x \to +\infty} (r(\sigma_{\mathrm{mo}(p)}(x)) - r(\sigma_{\mathrm{ff}(p)}(x))) = 0 \quad \text{(B.8)},$$

we have

$$\begin{aligned}&\lim_{c\to+\infty}(\mathcal{G}_{\mathrm{B}}(c,p;\sigma_{\mathrm{mo}(p)})-\mathcal{G}_{\mathrm{B}}(c,p;\sigma_{\mathrm{tff}(p)}))\\&\quad=\lim_{c\to+\infty}(\mathcal{G}_{\mathrm{B}}(c,p;\sigma_{\mathrm{mo}(p)})-\mathcal{G}_{\mathrm{B}}(c,p;\sigma_{\mathrm{lff}(p)}))=0.\quad\text{(Theorem 3.21)}\end{aligned}$$

With the help of Theorem 3.34 again, the theorem is established by a similar argument to the proof of Theorem 3.29. □

Proof of Theorem 3.37. It is easy to verify that for any $u>0$,

$$\begin{aligned}\sigma_{\mathrm{tff}(p)}(\overline{\sigma_{\mathrm{tff}(p)}}^{(i)}(\sigma^{-1}_{\mathrm{tff}(p)}(u)))&=\begin{cases}(1-p)^i u, & 0\le i<i_0,\\ \dfrac{(1-p)^i}{p}u-1, & i=i_0,\\ 0, & i>i_0,\end{cases}\\ p\sigma^{-1}_{\mathrm{tff}(p)}(u)&=\begin{cases}pu, & 0\le u<\dfrac{p}{1-p},\\ u-p, & u\ge\dfrac{p}{1-p},\end{cases}\\ i_0&:=\left\lfloor\frac{\log(p/u)}{\log(1-p)}\right\rfloor.\end{aligned}$$

Let $h_{u,p}(t):=\sigma_{\mathrm{tff}(p)}(\overline{\sigma_{\mathrm{tff}(p)}}^{(\lfloor t/p\rfloor)}(\sigma^{-1}_{\mathrm{tff}(p)}(u)))$. Then we have

$$\lim_{p\to0}h_{u,p}(t)=u\mathrm{e}^{-t},\qquad\lim_{p\to0}p\sigma^{-1}_{\mathrm{tff}(p)}(u)=u,\tag{B.9}$$

and for $p\in(0,1/2)$,

$$\begin{aligned}h_{u,p}(t)&\overset{\text{(a)}}{\le}(1-p)^{\lfloor t/p\rfloor}u1\{0\le t<pi_0\}+\frac{p}{1-p}1\{pi_0\le t<p(i_0+1)\}\\&\overset{\text{(b)}}{\le}u\mathrm{e}^{1-t}+2u\mathrm{e}^{1-t}=3u\mathrm{e}^{1-t},\end{aligned}$$

where (a) is due to

$$\frac{(1-p)^{i_0}}{p}u-1<\frac{(1-p)^{\log(p/u)/\log(1-p)-1}}{p}u-1=\frac{p}{1-p}$$

and (b) follows from

$$2u\mathrm{e}^{1-t}\ge2u\mathrm{e}^{1-p(i_0+1)}\ge\frac{p}{1-p}\quad\text{for }p\in(0,1/2)$$

with

$$pi_0 \le \frac{p\log(u/p)}{-\log(1-p)} \le \log(u/p).$$

Comparing (B.9) with the limit (B.4) and the identity (B.3) in the proof of Theorem 3.23, we conclude that

$$\lim_{p\to 0} \frac{\overline{g}_{\sigma_{\mathrm{tff}}}(c,p)}{|r(pc)|}\bigg|_{c=\sigma^{-1}_{\mathrm{tff}(p)}(u)} = \lim_{p\to 0} \frac{\overline{g}_{\sigma_{\mathrm{ff}}}(c,p)}{|r(pc)|}\bigg|_{c=\sigma^{-1}_{\mathrm{ff}(p)}(u)} \quad \text{(Theorem 3.20)},$$

which further gives (3.181) and (3.182) (Theorem 3.23). □

B.3 Optimal w-lookahead Policy for Bernoulli Energy Arrivals

Proof of Proposition 4.1. The function $g(\xi_0^\infty)$ can be rewritten as

$$\begin{aligned} g(\xi_0^\infty) &= \sum_{k=1}^{w} p^2(1-p)^{k-1} kr\left(\frac{c}{k}\right) \\ &\quad + \sum_{i=w}^{\infty} p^2(1-p)^i \left(\sum_{j=0}^{i-w} r(\xi_j) + wr\left(\frac{c-\sum_{j=0}^{i-w}\xi_j}{w}\right)\right) \end{aligned}$$

by the step (b) of (4.12). Observing that for $i \ge n+w$,

$$\frac{1}{i-n+1}\left(\sum_{j=n}^{i-w} r(\xi_j) + wr\left(\frac{c-\sum_{j=0}^{i-w}\xi_j}{w}\right)\right) \le r\left(\frac{c-\sum_{j=0}^{n-1}\xi_j}{i-n+1}\right)$$

by Jensen's inequality, we further have

$$\begin{aligned} g(\xi_0^\infty) &\le \sum_{k=1}^{w} p^2(1-p)^{k-1} kr\left(\frac{c}{k}\right) \\ &\quad + \sum_{i=w}^{n+w-1} p^2(1-p)^i \left(\sum_{j=0}^{i-w} r(\xi_j) + wr\left(\frac{c-\sum_{j=0}^{i-w}\xi_j}{w}\right)\right) \\ &\quad + \sum_{i=n+w}^{\infty} p^2(1-p)^i \left(\sum_{j=0}^{n-1} r(\xi_j) + (i-n+1)r\left(\frac{c-\sum_{j=0}^{n-1}\xi_j}{i-n+1}\right)\right) \\ &= \sum_{k=1}^{w} p^2(1-p)^{k-1} kr\left(\frac{c}{k}\right) + \sum_{j=0}^{n-1} p^2 r(\xi_j) \sum_{i=j+w}^{\infty} (1-p)^i \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{k=0}^{n-1} p^2(1-p)^{k+w} wr\left(\frac{c-\sum_{i=0}^{k}\xi_i}{w}\right) \\
&+ \sum_{k=w+1}^{\infty} p^2(1-p)^{k+n-1} kr\left(\frac{c-\sum_{i=0}^{n-1}\xi_i}{k}\right) \\
= &\sum_{k=1}^{w} p^2(1-p)^{k-1} kr\left(\frac{c}{k}\right) + \sum_{k=0}^{n-1} p(1-p)^{k+w} r(\xi_k) \\
&+ \sum_{k=0}^{n-2} p^2(1-p)^{k+w} wr\left(\frac{c-\sum_{i=0}^{k}\xi_i}{w}\right) \\
&+ \sum_{k=w}^{\infty} p^2(1-p)^{k+n-1} kr\left(\frac{c-\sum_{i=0}^{n-1}\xi_i}{k}\right),
\end{aligned}
$$

which concludes (4.17).

The identity (4.18) follows easily from (4.16) with $\xi_i = 0$ for $i \geq n$ and the observation that

$$
\begin{aligned}
&\sum_{k=n-1}^{\infty} p^2(1-p)^{k+w} wr\left(\frac{c-\sum_{i=0}^{k}\xi_i}{w}\right) \\
&\qquad = p^2(1-p)^{n+w-1} wr\left(\frac{c-\sum_{i=0}^{n-1}\xi_i}{w}\right)\sum_{i=0}^{\infty}(1-p)^i \\
&\qquad = p(1-p)^{n+w-1} wr\left(\frac{c-\sum_{i=0}^{n-1}\xi_i}{w}\right).
\end{aligned}
$$

By (4.17) and (4.18), the gap $g(\xi_0^\infty)$ is

$$
\begin{aligned}
&\sum_{k=w}^{\infty} p^2(1-p)^{k+n-1} kr\left(\frac{\Delta(\xi_0^{n-1})}{k}\right) - p(1-p)^{n+w-1} wr\left(\frac{\Delta(\xi_0^{n-1})}{w}\right) \\
&\quad = \sum_{k=w+1}^{\infty} p^2(1-p)^{k+n-1}\left(kr\left(\frac{\Delta(\xi_0^{n-1})}{k}\right) - wr\left(\frac{\Delta(\xi_0^{n-1})}{w}\right)\right) \\
&\quad \overset{\text{(a)}}{\leq} \sum_{k=w+1}^{\infty} p^2(1-p)^{k+n-1}\left(kr(0) + r'(0)\Delta(\xi_0^{n-1}) - wr\left(\frac{\Delta(\xi_0^{n-1})}{w}\right)\right) \\
&\quad = p(1-p)^{n+w}\left[r(0)(w+\varpi) + r'(0)\Delta(\xi_0^{n-1}) - wr\left(\frac{\Delta(\xi_0^{n-1})}{w}\right)\right]
\end{aligned}
$$

which concludes (4.19b), where (a) follows from the second inequality of

$$r(0) + r'(x)x \leq r(x) \leq r(0) + r'(0)x \quad \text{for } x \geq 0, \tag{B.10}$$

which is a consequence of the mean value theorem and the concavity of $r(x)$. By the first inequality of (B.10), we further have

$$\begin{aligned} g(\xi_0^\infty) &\leq p(1-p)^{n+w}\left[r(0)\varpi + \left(r'(0) - r'\left(\frac{\Delta(\xi_0^{n-1})}{w}\right)\right)\Delta(\xi_0^{n-1})\right] \\ &\overset{\text{(a)}}{\leq} p(1-p)^{n+w}\left[r(0)\varpi + \frac{M_r}{w}(\Delta(\xi_0^{n-1}))^2\right], \end{aligned}$$

where (a) follows from the mean value theorem with $r''(x) \geq -M_r$. □

Proof of Theorem 4.4. The inequality (4.25) as well as the uniqueness of the three maximizers is an easy consequence of Proposition 4.1 and the strict concavity of $r(x)$.

It is easy to see that $\underline{\mathcal{G}}_n$ is bounded and non-decreasing in n, so $\underline{\mathcal{G}}_n$ converges as $n \to \infty$. Observe that

$$\overline{\mathcal{G}}_n - \underline{\mathcal{G}}_n \leq \overline{g}_n((\overline{\xi}^{*(n)})_0^{n-1}) - \underline{g}_n((\overline{\xi}^{*(n)})_0^{n-1}) \leq \Delta g_n((\overline{\xi}^{*(n)})_0^{n-1}),$$

which, combined with Proposition 4.1, gives (4.26) and (4.27).

If for every $i \geq 0$, $\underline{\xi}_i^{*(n)}$ converges as $n \to \infty$, then we will show that the limit $(\underline{\xi}_i^{*(\infty)} := \lim_{n\to\infty} \underline{\xi}_i^{*(n)})_{i=0}^\infty$ is a maximizer of Problem 4.2, and hence the uniqueness of the maximizer implies that $\underline{\xi}_i^{*(\infty)} = \xi_i^*$ for all $i \geq 0$. To this end, it suffices to show that $g((\underline{\xi}^{*(\infty)})_0^\infty) \geq g((\xi^*)_0^\infty)$. It is easy to verify that $\sum_{i=0}^\infty \underline{\xi}_i^{*(\infty)} \leq c$. Then we have

$$g((\underline{\xi}^{*(\infty)})_0^\infty) \overset{\text{(a)}}{=} \lim_{n\to\infty} g((\underline{\xi}^{*(n)})_0^\infty) = \lim_{n\to\infty} \underline{g}_n((\underline{\xi}^{*(n)})_0^{n-1}) \overset{\text{(b)}}{=} \mathcal{G}^*,$$

where (a) follows from the continuity of $r(x)$ and the dominated convergence theorem with

$$r(\underline{\xi}_i^{*(n)}) \leq r(c) \quad \text{and} \quad wr\left(\frac{c - \sum_{j=0}^i \underline{\xi}_j^{*(n)}}{w}\right) \leq r'(0)c, \tag{B.11}$$

and (b) follows from (4.26).

Similarly, if for every $i \geq 0$, $\overline{\xi}_i^{*(n)}$ converges as $n \to \infty$, then we will show that the limit $(\overline{\xi}_i^{*(\infty)} := \lim_{n\to\infty} \overline{\xi}_i^{*(n)})_{i=0}^{\infty}$ is a maximizer of Problem 4.2, and hence the uniqueness of the maximizer implies that $\overline{\xi}_i^{*(\infty)} = \xi_i^*$ for all $i \geq 0$. It is easy to verify that $\sum_{i=0}^{\infty} \overline{\xi}_i^{*(\infty)} \leq c$, and then

$$
\begin{aligned}
g((\overline{\xi}^{*(\infty)})_0^{\infty}) &\overset{(a)}{=} \lim_{n\to\infty} g((\overline{\xi}^{*(n)})_0^{\infty}) \\
&= \lim_{n\to\infty} \underline{g}_n((\overline{\xi}^{*(n)})_0^{n-1}) \\
&\overset{(b)}{=} \lim_{n\to\infty} (\overline{\mathcal{G}}_n - \Delta g_n((\overline{\xi}^{*(n)})_0^{n-1})) \\
&\geq \limsup_{n\to\infty} \overline{g}_n((\overline{\xi}^{*(n)})_0^{n-1}) - \limsup_{n\to\infty} \Delta g_n((\overline{\xi}^{*(n)})_0^{n-1}) \\
&\overset{(c)}{\geq} \mathcal{G}^* - \lim_{n\to\infty} p(1-p)^{n+w}[r(0)(w+\varpi) + r'(0)c] \\
&= \mathcal{G}^*,
\end{aligned}
$$

where (a) follows from the continuity of $r(x)$ and the dominated convergence theorem with (B.11), (b) from (4.27a), and (c) from (4.26) and (4.27b) with $\Delta((\overline{\xi}^{*(n)})) \leq c$. □

Proof of Theorem 4.5. Define the Lagrangian function

$$
\underline{L}(\xi_0^{n-1}, \gamma_0^{n-1}, \delta) := \underline{g}_n(\xi_0^{n-1}) + \sum_{i=0}^{n-1} \gamma_i \xi_i - \delta \left(\sum_{i=0}^{n-1} \xi_i - c \right)
$$

for Problem 4.3, where $\xi_0^{n-1}, \gamma_0^{n-1} \in \mathbb{R}_{\geq 0}^n$ and $\delta \geq 0$. The corresponding KKT conditions for the maximizer $(\underline{\xi}^{*(n)})_0^{n-1}$ are

$$
\begin{aligned}
p(1-p)^{i+w} r'(\underline{\xi}_i^{*(n)}) - \sum_{k=i}^{n-2} p^2 (1-p)^{k+w} r' \left(\frac{c - \sum_{j=0}^{k} \underline{\xi}_j^{*(n)}}{w} \right) \\
- p(1-p)^{n+w-1} r' \left(\frac{c - \sum_{j=0}^{n-1} \underline{\xi}_j^{*(n)}}{w} \right) + \gamma_i - \delta = 0, \\
i = 0, \ldots, n-1, \\
\gamma_i \geq 0, \quad i = 0, \ldots, n-1, \\
\delta \geq 0,
\end{aligned}
$$

$$\gamma_i \underline{\xi}_i^{*(n)} = 0, \quad i = 0, \ldots, n-1,$$
$$\delta\left(\sum_{i=0}^{n-1} \underline{\xi}_i^{*(n)} - c\right) = 0.$$

Note that the uniqueness of the maximizer (Theorem 4.4) implies the uniqueness of the solution to the KKT conditions. Substituting γ_i and δ with $p(1-p)^{i+w}\gamma_i$ and $p(1-p)^{n+w-1}\delta$, respectively, we further have

$$r'(\underline{\xi}_i^{*(n)}) + \gamma_i = pr'\left(\frac{c - \sum_{j=0}^{i} \underline{\xi}_j^{*(n)}}{w}\right) + (1-p)(r'(\underline{\xi}_{i+1}^{*(n)}) + \gamma_{i+1}),$$
$$i = 0, \ldots, n-2, \quad \text{(B.12a)}$$
$$r'(\underline{\xi}_{n-1}^{*(n)}) + \gamma_{n-1} = r'\left(\frac{c - \sum_{j=0}^{n-1} \underline{\xi}_j^{*(n)}}{w}\right) + \delta, \quad \text{(B.12b)}$$
$$\gamma_i \geq 0, \quad i = 0, \ldots, n-1, \quad \text{(B.12c)}$$
$$\delta \geq 0, \quad \text{(B.12d)}$$
$$\gamma_i \underline{\xi}_i^{*(n)} = 0, \quad i = 0, \ldots, n-1, \quad \text{(B.12e)}$$
$$\delta\left(\sum_{i=0}^{n-1} \underline{\xi}_i^{*(n)} - c\right) = 0. \quad \text{(B.12f)}$$

We first show that $\sum_{j=0}^{n-1} \underline{\xi}_j^{*(n)} < c$. If $\sum_{j=0}^{n-1} \underline{\xi}_j^{*(n)} = c$, then

$$r'(\underline{\xi}_{n-1}^{*(n)}) + \gamma_{n-1} \geq r'(0) \quad \text{(B.12b)},$$

which together with (B.12e) implies that $\underline{\xi}_{n-1}^{*(n)} = 0$, whether γ_{n-1} is positive or not, because $r'(x)$ is strictly decreasing. This further implies that $\sum_{j=0}^{n-2} \underline{\xi}_j^{*(n)} = c$, so

$$r'(\underline{\xi}_{n-2}^{*(n)}) + \gamma_{n-2} \geq r'(0) \quad \text{(B.12a)},$$

which in the same vein implies that $\underline{\xi}_{n-2}^{*(n)} = 0$. Repeating this backward induction thus leads to the conclusion $\underline{\xi}_0^{*(n)} = c = 0$, which is absurd. Hence $\sum_{j=0}^{n-1} \underline{\xi}_j^{*(n)} < c$, and therefore $\delta = 0$ (B.12f).

From this conclusion and (B.12b), it follows that

$$r'(\underline{\xi}_{n-1}^{*(n)}) \leq r'(\underline{\xi}_{n-1}^{*(n)}) + \gamma_{n-1} < r'(0),$$

which implies that $\underline{\xi}_{n-1}^{*(n)} > 0$ and $\gamma_{n-1} = 0$ (B.12e). Hence,

$$r'(\underline{\xi}_{n-1}^{*(n)}) = r'\left(\frac{c - \sum_{j=0}^{n-1} \underline{\xi}_j^{*(n)}}{w}\right) \quad \text{(B.12b)},$$

which concludes (4.31b). Continuing this kind of arguments, we further have

$$r'(\underline{\xi}_{n-2}^{*(n)}) \leq r'(\underline{\xi}_{n-2}^{*(n)}) + \gamma_{n-2} < r'(0) \quad \text{(B.12a)},$$

which implies that $\underline{\xi}_{n-2}^{*(n)} > 0$ and $\gamma_{n-2} = 0$ (B.12e). Consequently,

$$r'(\underline{\xi}_{n-2}^{*(n)}) = pr'\left(\frac{c - \sum_{j=0}^{n-2} \underline{\xi}_j^{*(n)}}{w}\right) + (1-p)r'(\underline{\xi}_{n-1}^{*(n)}) \quad \text{(B.12a)},$$

which concludes (4.31a) for $i = n-2$. Since

$$0 < \underline{\xi}_{n-1}^{*(n)} = \frac{c - \sum_{j=0}^{n-1} \underline{\xi}_j^{*(n)}}{w} < \frac{c - \sum_{j=0}^{n-2} \underline{\xi}_j^{*(n)}}{w},$$

we immediately have

$$\underline{\xi}_{n-1}^{*(n)} < \underline{\xi}_{n-2}^{*(n)} < \frac{c - \sum_{j=0}^{n-2} \underline{\xi}_j^{*(n)}}{w}$$

because of the monotonicity of $r'(x)$. Repeating such a backward induction finally establishes the theorem. □

Proof of Theorem 4.6. Define the Lagrangian function

$$\overline{L}(\xi_0^{n-1}, \gamma_0^{n-1}, \delta) := \overline{g}_n(\xi_0^{n-1}) + \sum_{i=0}^{n-1} \gamma_i \xi_i - \delta\left(\sum_{i=0}^{n-1} \xi_i - c\right)$$

for Problem 4.4, where $\xi_0^{n-1}, \gamma_0^{n-1} \in \mathbb{R}_{\geq 0}^n$ and $\delta \geq 0$. The corresponding KKT conditions for the maximizer $(\overline{\xi}^{*(n)})_0^{n-1}$ are

$$
\begin{aligned}
p(1-p)^{i+w} r'(\overline{\xi}_i^{*(n)}) - \sum_{k=i}^{n-2} p^2(1-p)^{k+w} r'\left(\frac{c-\sum_{j=0}^{k}\overline{\xi}_j^{*(n)}}{w}\right)& \\
- \sum_{k=w}^{\infty} p^2(1-p)^{k+n-1} r'\left(\frac{c-\sum_{j=0}^{n-1}\overline{\xi}_j^{*(n)}}{k}\right) + \gamma_i - \delta &= 0, \\
i &= 0,\ldots,n-1,
\end{aligned}
$$

$$
\begin{aligned}
\gamma_i &\geq 0, \quad i = 0,\ldots,n-1, \\
\delta &\geq 0, \\
\gamma_i \overline{\xi}_i^{*(n)} &= 0, \quad i = 0,\ldots,n-1, \\
\delta\left(\sum_{i=0}^{n-1}\overline{\xi}_i^{*(n)} - c\right) &= 0.
\end{aligned}
$$

Note that the uniqueness of the maximizer (Theorem 4.4) implies the uniqueness of the solution to the KKT conditions. Substituting γ_i and δ with $p(1-p)^{i+w}\gamma_i$ and $p(1-p)^{n+w-1}\delta$, respectively, we further have

$$
r'(\overline{\xi}_i^{*(n)}) + \gamma_i = pr'\left(\frac{c-\sum_{j=0}^{i}\overline{\xi}_j^{*(n)}}{w}\right) + (1-p)(r'(\overline{\xi}_{i+1}^{*(n)}) + \gamma_{i+1}), \quad i = 0,\ldots,n-2, \tag{B.13a}
$$

$$
r'(\overline{\xi}_{n-1}^{*(n)}) + \gamma_{n-1} = \sum_{k=w}^{\infty} p(1-p)^{k-w} r'\left(\frac{c-\sum_{j=0}^{n-1}\overline{\xi}_j^{*(n)}}{k}\right) + \delta, \tag{B.13b}
$$

$$
\gamma_i \geq 0, \quad i = 0,\ldots,n-1, \tag{B.13c}
$$

$$
\delta \geq 0, \tag{B.13d}
$$

$$
\gamma_i \overline{\xi}_i^{*(n)} = 0, \quad i = 0,\ldots,n-1, \tag{B.13e}
$$

$$
\delta\left(\sum_{i=0}^{n-1}\overline{\xi}_i^{*(n)} - c\right) = 0. \tag{B.13f}
$$

We first show that $\sum_{j=0}^{n-1}\overline{\xi}_j^{*(n)} < c$. If $\sum_{j=0}^{n-1}\overline{\xi}_j^{*(n)} = c$, then

$$
r'(\overline{\xi}_{n-1}^{*(n)}) + \gamma_{n-1} \geq r'(0) \quad \text{(B.13b)},
$$

which together with (B.13e) implies that $\overline{\xi}_{n-1}^{*(n)} = 0$, whether γ_{n-1} is positive or not, because $r'(x)$ is strictly decreasing. This further implies that $\sum_{j=0}^{n-2} \overline{\xi}_j^{*(n)} = c$, so

$$r'(\overline{\xi}_{n-2}^{*(n)}) + \gamma_{n-2} \geq r'(0) \quad \text{(B.13a)},$$

which in the same vein implies that $\overline{\xi}_{n-2}^{*(n)} = 0$. Repeating this backward induction thus leads to the conclusion $\overline{\xi}_0^{*(n)} = c = 0$, which is absurd. Hence $\sum_{j=0}^{n-1} \overline{\xi}_j^{*(n)} < c$, and therefore $\delta = 0$ (B.13f).

From this conclusion and (B.13b), it follows that

$$r'(\overline{\xi}_{n-1}^{*(n)}) \leq r'(\overline{\xi}_{n-1}^{*(n)}) + \gamma_{n-1} < r'(0),$$

which implies that $\overline{\xi}_{n-1}^{*(n)} > 0$ and $\gamma_{n-1} = 0$ (B.13e). Hence,

$$\begin{aligned} r'(\overline{\xi}_{n-1}^{*(n)}) &= \sum_{k=w}^{\infty} p(1-p)^{k-w} r'\left(\frac{c - \sum_{j=0}^{n-1} \underline{\xi}_j^{*(n)}}{k}\right) \quad \text{(B.13b)} \\ &> r'\left(\frac{c - \sum_{j=0}^{n-1} \underline{\xi}_j^{*(n)}}{w}\right), \end{aligned}$$

which concludes (4.33b) as well as (4.34) for $i = n-1$. Continuing this kind of arguments, we further have

$$r'(\overline{\xi}_{n-2}^{*(n)}) \leq r'(\overline{\xi}_{n-2}^{*(n)}) + \gamma_{n-2} < r'(0) \quad \text{(B.13a)},$$

which implies that $\overline{\xi}_{n-2}^{*(n)} > 0$ and $\gamma_{n-2} = 0$ (B.12e). Consequently,

$$r'(\overline{\xi}_{n-2}^{*(n)}) = pr'\left(\frac{c - \sum_{j=0}^{n-2} \overline{\xi}_j^{*(n)}}{w}\right) + (1-p) r'(\overline{\xi}_{n-1}^{*(n)}) \quad \text{(B.13a)},$$

which concludes (4.33a) for $i = n-2$. Since

$$0 < \overline{\xi}_{n-1}^{*(n)} < \frac{c - \sum_{j=0}^{n-1} \overline{\xi}_j^{*(n)}}{w} < \frac{c - \sum_{j=0}^{n-2} \overline{\xi}_j^{*(n)}}{w},$$

we immediately have

$$\overline{\xi}_{n-1}^{*(n)} < \overline{\xi}_{n-2}^{*(n)} < \frac{c - \sum_{j=0}^{n-2} \overline{\xi}_j^{*(n)}}{w}$$

because of the monotonicity of $r'(x)$. Repeating such a backward induction finally establishes the theorem. $\square$

C

Auxiliary Results

C.1 Optimality of Greedy Policy

Proposition C.1. Let

$$\phi(u;b) := r(u) + \int r(\langle b-u+s\rangle_{\leq c})Q(\mathrm{d}s) \tag{C.1}$$

and

$$\psi(u;b) := r'(u) - \int_{[0,c-b+u)} r'(b-u+s)Q(\mathrm{d}s), \tag{C.2}$$

where Q is the energy arrival distribution, $b \in [0,c]$, and $x \in [0,b]$. For $0 \leq u_1 < u_2 \leq b$,

$$\phi(u_2;b) - \phi(u_1;b) = \int_{(u_1,u_2)} \psi(t;b)\mathrm{d}t.$$

Proof. By definition,

$$\begin{aligned}&\phi(u_2;b) - \phi(u_1;b)\\ &\quad = r(u_2) - r(u_1) - \int (r(\langle b-u_1+s\rangle_{\leq c}) - r(\langle b-u_2+s\rangle_{\leq c}))Q(\mathrm{d}s).\end{aligned}$$

Since

$$r(u_2) - r(u_1) = \int_{(u_1,u_2)} r'(t)\mathrm{d}t$$

and

$$
\begin{aligned}
&\int (r(\langle b - u_1 + s\rangle_{\le c}) - r(\langle b - u_2 + s\rangle_{\le c}))Q(\mathrm{d}s) \\
&\qquad = \int Q(\mathrm{d}s) \int r'(b-t+s)1\{b-t+s<c\}1\{u_1<t<u_2\}\mathrm{d}t \\
&\qquad \overset{(a)}{=} \int 1\{u_1<t<u_2\}\mathrm{d}t \int r'(b-t+s)1\{b-t+s<c\}Q(\mathrm{d}s) \\
&\qquad = \int_{(u_1,u_2)} \mathrm{d}t \int_{[0,c-b+t)} r'(b-t+s)Q(\mathrm{d}s),
\end{aligned}
$$

where (a) follows from Fubini's theorem, we have

$$
\phi(u_2;b) - \phi(u_1;b) = \int_{(u_1,u_2)} \psi(t;b)\mathrm{d}t. \qquad \square
$$

Proposition C.2. The function $\psi(u;b)$ is strictly decreasing in $u \in [0,b]$ for fixed $b \in [0,c]$, and is increasing in $b \in [u,c]$ for fixed $u \in [0,c]$. Let

$$
\underline{\psi}(b) := \psi(b;b). \tag{C.3}
$$

The function $\underline{\psi}(b)$ is strictly decreasing on $[0,c]$.

Sketch of Proof. Observe that the function $r'(u)$ is positive and strictly decreasing (Assumption 1.1) and that the interval $[0, c-b+u)$ is increasing in u. Applying these two properties to (C.2) establishes the proposition. $\square$

Proposition C.3. The function $\psi(u;b)$ is left continuous in $u \in [0,b]$ for fixed $b \in [0,c]$. The function $\underline{\psi}(b)$ is continuous on $[0,c]$.

Proof. The continuity of $\underline{\psi}(b)$ is an easy consequence of the continuity of $r'(x)$ (Assumption 1.1). As for $\psi(u;b)$, note that for $\epsilon > 0$,

$$
\begin{aligned}
&\int_{[0,c-b+u)} r'(b-u+s)Q(\mathrm{d}s) - \int_{[0,c-b+(u-\epsilon))} r'(b-(u-\epsilon)+s)Q(\mathrm{d}s) \\
&\qquad = \int_{[c-b+u-\epsilon,c-b+u)} r'(b-u+s)Q(\mathrm{d}s) \\
&\qquad\quad + \int_{[0,c-b+u-\epsilon)} (r'(b-u+s) - r'(b-u+s+\epsilon))Q(\mathrm{d}s) \\
&\qquad \le r'(0)Q([c-b+u-\epsilon, c-b+u))
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_{[0,c-b+u)} (r'(b-u+s) - r'(b-u+s+\epsilon))Q(\mathrm{d}s) \\
&\to r'(0)Q(\varnothing) + \int_{[0,c-b+u)} 0Q(\mathrm{d}s) = 0
\end{aligned}
$$

as $\epsilon \to 0$ by the continuity of Q and the dominated convergence theorem. Therefore, $\psi(u;b)$ is left continuous in u. □

Proposition C.4. Let $\overline{{}^\wedge f}_{[a,b]}(x)$ and $\underline{{}_\vee f}_{[a,b]}(x)$ be the upper concave envelope and lower convex envelope of f over $[a,b]$, respectively. If f is strictly decreasing (resp., non-increasing), then both $\overline{{}^\wedge f}_{[a,b]}(x)$ and $\underline{{}_\vee f}_{[a,b]}(x)$ are strictly decreasing (resp., non-increasing).

Proof. It is clear that $f(x) < f(a)$ for $x \in (a,b]$. Then for any $a \le x_1 < x_2 \le b$, $\overline{{}^\wedge f}_{[a,b]}(x_1) \le f(a)$ and $\overline{{}^\wedge f}_{[a,b]}(x_2) < f(a)$. Note that the number x_1 can be rewritten as

$$x_1 = \lambda a + (1-\lambda)x_2,$$

where

$$\lambda := \frac{x_2 - x_1}{x_2 - a} \in (0,1].$$

By the concavity of $\overline{{}^\wedge f}_{[a,b]}(x)$,

$$
\begin{aligned}
\overline{{}^\wedge f}_{[a,b]}(x_2) &< \lambda f(a) + (1-\lambda)\overline{{}^\wedge f}_{[a,b]}(x_2) \\
&= \lambda \overline{{}^\wedge f}_{[a,b]}(a) + (1-\lambda)\overline{{}^\wedge f}_{[a,b]}(x_2) \\
&\le \overline{{}^\wedge f}_{[a,b]}(\lambda a + (1-\lambda)x_2) \\
&= \overline{{}^\wedge f}_{[a,b]}(x_1).
\end{aligned}
$$

Analogously, it can be shown that $\underline{f}_{[a,b]}(x)$ is strictly decreasing. □

Proposition C.5. Let $f(x)$ be a bounded real-valued function on $[a,b]$. Every point on the graph of the lower convex (resp., upper concave) envelope of f (over $[a,b]$) lies on a line segment between two points on the graph of the lower (resp., upper) semi-continuous envelope of f, corresponding to two adjacent or equal convex (resp., concave) points of f.

Proof. Since $f(x)$ is bounded,

$$\begin{aligned}\underline{\vee f}_{[a,b]}(x) &= \min\{y : (x,y) \in \mathrm{cl}(\mathrm{conv}(\mathrm{epi}(\langle f\rangle_{\le M})))\} \\ &\overset{(a)}{=} \min\{y : (x,y) \in \mathrm{conv}(C')\}\end{aligned}$$

for some sufficiently large $M > 0$, where

$$\begin{aligned}C' &:= \mathrm{cl}(\mathrm{epi}(\langle f\rangle_{\le M})), \\ \langle f\rangle_{\le M}(x) &:= \langle f(x)\rangle_{\le M},\end{aligned}$$

and (a) follows from the compactness of C'. Note that $\mathrm{conv}(C')$ is also compact.

By the Krein-Milman theorem [41, Thm 5.5] with the compactness of $\mathrm{conv}(C')$, every point $p_x := (x, \underline{\vee f}_{[a,b]}(x))$ is a convex combination of extreme points of $\mathrm{conv}(C')$. In particular, these extreme points are on the lower boundary of C', or equivalently, on the lower boundary of $C := \mathrm{cl}(\mathrm{epi}(f))$, the graph of $\underline{\downarrow f}_{[a,b]}$. Since p_x is on the lower boundary of $\mathrm{conv}\, C'$, we consider a support line L (which exists by [41, Thm 4.2]) of $\mathrm{conv}\, C'$ at p_x. Let $L' := L \cap \mathrm{conv}\, C'$. It is easy to see that L' is a closed line segment, whose two endpoints (or one point in the degenerate case) are the extreme points of $\mathrm{conv}(C')$ on the graph of $\underline{\downarrow f}_{[a,b]}$. Let $(\alpha, \underline{\downarrow f}_{[a,b]}(\alpha))$ and $(\beta, \underline{\downarrow f}_{[a,b]}(\beta))$ denote the two (possibly equal) endpoints. It is clear that α and β are adjacent or equal convex points of f.

The "convex" part is thus established, and the "concave" part is an easy consequence of the "convex" part with f replaced with $-f$. □

Proposition C.6. Let

$$\hat{\rho}(s,t,v) := \frac{t-v}{t-s} \quad \text{for } s < v < t. \tag{C.4}$$

Then $\hat{\rho}(s,t,v)$ is strictly increasing in $t \in (s, +\infty)$ for fixed s and v, and $\hat{\rho}(s,t,v)$ is strictly increasing in $s \in [0,t)$ for fixed t and v.

Proof. Observe that

$$\hat{\rho}(s,t,v) = 1 + \frac{s-v}{t-s},$$

so $\hat{\rho}(s,t,v)$ is strictly increasing in $t \in (s, \overline{x})$ for fixed s. The remaining part is straightforward by definition. □

C.2 Universally Near-Optimal Online Policies

Proposition C.7. A non-negative convex function $f(x)$ on $[0,c]$ with $f(0)=0$ must be non-decreasing.

Proof. For $0 \le x < y \le c$,

$$f(x) = f\left(\frac{x}{y}y + \left(1-\frac{x}{y}\right)0\right) \le \frac{x}{y}f(y) + \left(1-\frac{x}{y}\right)f(0) \le f(y). \quad \square$$

Proposition C.8 (Generalization of [137, Lemma 1])**.** Let $x_1 \in [0,c]$. For a normal policy $\sigma \in \Pi_0^{\mathrm{SN}}$ and a real-valued function g on $[0,c]$, if σ is affine on $[x_1,c]$, g is non-decreasing, Lipschitz, and concave, and $g'(\overline{\sigma}(x)) \le r'(\sigma(x))$ a.e. on $[0,x_1]$, then

$$h(x) := r(\sigma(x)) + g(\overline{\sigma}(x)) \tag{C.5}$$

is non-decreasing, Lipschitz, and concave, and $h'(x) \le r'(\sigma(x))$ a.e. on $[0,x_1]$.

Proof. By the conditions and Proposition 3.5, h is non-decreasing and Lipschitz. Then, all Lipschitz functions, h, r, g, σ, and $\overline{\sigma}$, are differentiable a.e. [48, Lemma 6.1.3 and Cor. 6.1.5]. Moreover, the derivative $h'(x)$ can be computed by the chain rule [48, Thm. 6.5.2] as follows:

$$\begin{aligned} h'(x) &= r'(\sigma(x))\sigma'(x) + g'(\overline{\sigma}(x))\overline{\sigma}'(x) \\ &= r'(\sigma(x)) + \overline{\sigma}'(x)(g'(\overline{\sigma}(x)) - r'(\sigma(x))) \quad \text{a.e.,} \end{aligned} \tag{C.6}$$

which implies $h'(x) \le r'(\sigma(x))$ a.e. on $[0,x_1]$ because $\overline{\sigma}'$ is non-negative a.e. (Proposition 3.5).

Let

$$\begin{aligned} A := \{&x \in [0,c] : \text{(C.6) holds}\} \\ &\cap (\{x \in [0,x_1] : g'(\overline{\sigma}(x)) \le r'(\sigma(x))\} \cup [x_1,c]). \end{aligned}$$

It is clear that A is measurable and its Lebesgue measure is c. For any $x,y \in A$, we have

$$\begin{aligned} h'(y) - h'(x) = (&r'(\sigma(y)) - r'(\sigma(x)))\sigma'(y) \\ &+ (g'(\overline{\sigma}(y)) - g'(\overline{\sigma}(x)))\overline{\sigma}'(y) \\ &+ (\sigma'(y) - \sigma'(x))(r'(\sigma(x)) - g'(\overline{\sigma}(x))). \end{aligned}$$

Note that r', g', and σ' are all non-increasing on A ([48, Thm. 6.6.7]). Then for $x < y$ (both in A),

$$h'(y) - h'(x) \overset{\text{(a)}}{\leq} (\sigma'(y) - \sigma'(x))(r'(\sigma(x)) - g'(\overline{\sigma}(x))) \overset{\text{(b)}}{\leq} 0,$$

where (a) follows from the non-increasing property of $r'(\sigma(x))$ and $g'(\overline{\sigma}(x))$ and the non-negativity of σ' and $\overline{\sigma}'$ (Proposition 3.5), and (b) from

$$\sigma'(y) = \sigma'(x) \quad \text{for } x \geq x_1$$

or

$$\begin{aligned} \sigma'(y) &\leq \sigma'(x), \\ g'(\overline{\sigma}(x)) &\leq r'(\sigma(x)) \quad \text{for } x \leq x_1. \end{aligned}$$

Therefore, h' is non-increasing on A and hence h is concave on $[0, c]$ (Proposition C.9). □

Proposition C.9. If $f : [a, b] \to \mathbb{R}$ is absolutely continuous and its derivative f' is non-decreasing a.e. on the set where it exists, then f is convex.

Proof. Since f is absolutely continuous, it is differentiable a.e. and

$$f(x) - f(a) = \int_a^x f'(s)\mathrm{d}s \quad \text{for all } x \in [a, b] \text{ ([48, Thm. 6.4.2])}.$$

Let A be the set where f' exists and is non-decreasing. The Lebesgue measure of A is $b - a$. Then, for any $a \leq x < y \leq b$ and $t \in (0, 1)$,

$$\begin{aligned} &(1-t)f(x) + tf(y) - f(z) \\ &\quad = (1-t)(f(x) - f(z)) + t(f(y) - f(z)) \\ &\quad = -(1-t)\int_x^z f'(s)\mathrm{d}s + t\int_z^y f'(s)\mathrm{d}s \\ &\quad = -(1-t)\int_{[x,z]\cap A} f'(s)\mathrm{d}s + t\int_{[z,y]\cap A} f'(s)\mathrm{d}s \\ &\quad \geq -(1-t)\int_{[x,z]\cap A} g_z\mathrm{d}s + t\int_{[z,y]\cap A} g_z\mathrm{d}s \\ &\quad = g_z[-(1-t)(z-x) + t(y-z)] = 0, \end{aligned}$$

where $z := (1-t)x + ty$, $g_z := \sup_{s\in[x,z]\cap A} f'(s)$. Consequently, f is a convex function. □

Proposition C.10. Let $\sigma \in \Pi_0^{\mathrm{SN}}$ be a normal policy. If $\sigma(x) > 0$ for $x > 0$, then

$$\sum_{i=0}^{\infty} \sigma(\overline{\sigma}^{(i)}(x)) = x \tag{C.7}$$

for all $x > 0$.

Proof. Since $\sigma(x) > 0$ for $x > 0$, $\overline{\sigma}(x) < x$ for $x > 0$, and hence $x_i := \overline{\sigma}^{(i)}(x) < x$ is strictly decreasing in i. Let $x_\infty := \lim_{n\to\infty} \overline{\sigma}^{(n)}(x)$. It is clear that

$$\begin{aligned}\overline{\sigma}(x_\infty) &= \overline{\sigma}(\lim_{n\to\infty} \overline{\sigma}^{(n)}(x)) = \lim_{n\to\infty} \overline{\sigma}(\overline{\sigma}^{(n)}(x)) \\ &= \lim_{n\to\infty} \overline{\sigma}^{(n+1)}(x) = x_\infty,\end{aligned}$$

which must be zero, the unique solution of $\overline{\sigma}(x) = x$. Observing that

$$\sum_{i=0}^{n-1} \sigma(\overline{\sigma}^{(i)}(x)) = \sum_{i=0}^{n-1} (\overline{\sigma}^{(i)}(x) - \overline{\sigma}^{(i+1)}(x)) = x - \overline{\sigma}^{(n)}(x),$$

we finally have $\sum_{i=0}^{\infty} \sigma(\overline{\sigma}^{(i)}(x)) = x$. □

Proposition C.11. For $t \geq 0$ and $p \in (0, 1)$,

$$\mathrm{e}^{-t/(1-p)} \leq (1-p)^{\lfloor t/p \rfloor} \leq \mathrm{e}^{p-t}, \tag{C.8a}$$

$$\lim_{p\to 0} (1-p)^{\lfloor t/p \rfloor} = \mathrm{e}^{-t}. \tag{C.8b}$$

Proof. Recall the inequalities $\mathrm{e}^{x/(1+x)} \leq 1 + x \leq \mathrm{e}^x$ for $x > -1$ and $x - 1 < \lfloor x \rfloor \leq x$. We have

$$(1-p)^{\lfloor t/p \rfloor} \leq \mathrm{e}^{-p(t/p-1)} = \mathrm{e}^{p-t}$$

and

$$(1-p)^{\lfloor t/p \rfloor} \geq \mathrm{e}^{-t/(1-p)}.$$

Letting $p \to 0$, we have $\mathrm{e}^{-t} \leq \lim_{p\to 0}(1-p)^{\lfloor t/p \rfloor} \leq \mathrm{e}^{-t}$, and hence $\lim_{p\to 0}(1-p)^{\lfloor t/p \rfloor} = \mathrm{e}^{-t}$. □

Proposition C.12. Let $f(x)$ be a non-negative function on $\mathbb{R}_{\geq 0}$. Let $q(x) := f(x)/x$. If $f(x)$ is convex (resp., concave), then $q(x)$ is non-decreasing (resp., non-increasing).

Sketch of Proof. Use [48, Lemma 6.6.3]). □

Proposition C.13. Let $f(x)$ be a positive and strictly decreasing function on $\mathbb{R}_{\geq 0}$. If $g(x) := f^{-1}(sf(x))$ is convex (on its domain of definition) for some positive $s \neq 1$, then $\lim_{x\to+\infty} f(x) = 0$ and $\lim_{x\to+\infty} g(x) = +\infty$.

Proof. Choose a point q from the domain of $g(x)$. It is clear that $g(2q) > g(q) \geq 0$, and furthermore,

$$\frac{g(x)}{x} \geq \frac{g(2q)}{2q} > 0 \quad \text{for all } x \geq 2q \text{ (Proposition C.12)},$$

which implies that $\lim_{x\to+\infty} g(x) = +\infty$. Therefore,

$$\begin{aligned}\lim_{x\to+\infty} f(x) &= \lim_{x'\to+\infty} f(g(x')) \\ &= \lim_{x'\to+\infty} f(f^{-1}(sf(x'))) = s \lim_{x'\to+\infty} f(x'),\end{aligned}$$

which implies that $\lim_{x\to+\infty} f(x) = 0$. □

Proposition C.14. Let f be a strictly increasing function from A to B, where both A and B are convex subsets of $\mathbb{R}$. Then f is convex if and only if f^{-1} is concave.

Proof. Since f is strictly increasing, the epigraph $\mathrm{epi}(f) = \{(x,y) : y \geq f(x)\}$ is exactly the hypograph $\mathrm{hyp}(f^{-1}) = \{(x,y) : x \leq f^{-1}(y)\}$. Therefore, f is convex if and only if f^{-1} is concave. □

Proposition C.15. Let

$$f(x) := \frac{x^2}{x - g(x)}, \tag{C.9}$$

where $g(x)$ is a twice-differentiable, convex function satisfying $g(x) < x$. Then $f(x)$ is convex (on its domain of definition).

Sketch of Proof. It is straightforward to verify that

$$f''(x) = \frac{2(g(x) - xg'(x))^2 + x^2 g''(x)(x - g(x))}{(x - g(x))^3} \geq 0. \qquad \square$$

Proposition C.16. Let f and g be two bijections from A to A, where A is a subset of $\mathbb{R}$. If f and g are both strictly increasing and $f(x) \leq g(x)$ for all $x \in A$, then $f^{-1}(x) \geq g^{-1}(x)$ for all $x \in A$.

Proof. Let $y_1 := f^{-1}(x)$ and $y_2 := g^{-1}(x)$. Then we have $x = f(y_1) \leq g(y_1)$, and hence $y_2 = g^{-1}(x) \leq g^{-1}(g(y_1)) = y_1$. □

C.3 Optimal w-lookahead Policy for Bernoulli Energy Arrivals

Proposition C.17. Let $f(x)$ be a positive, strictly decreasing function on $\mathbb{R}_{\geq 0}$. Let a_w^∞ be a sequence of positive numbers satisfying $\sum_{k=w}^\infty a_k = 1$. Let

$$b_k(x,y) := \frac{f((x+y)/(k+1)) - f(x/k)}{f(y) - f(x/k)}, \tag{C.10}$$

where $x, y > 0$ and $k \in \mathbb{Z}_{>0}$ but $k \neq x/y$. For a given $x > 0$, if

$$\min_{w \leq k < x/\bar{x}} b_k(x,\bar{x}) \geq \max_{k > x/\bar{x}} b_k(x,\bar{x}), \tag{C.11}$$

then

$$\sum_{k=w}^\infty a_k f\left(\frac{x}{k}\right) < \sum_{k=w}^\infty a_k f\left(\frac{x+\bar{x}}{k+1}\right), \tag{C.12}$$

where

$$\bar{x} := f^{-1}\left(\sum_{k=w}^\infty a_k f\left(\frac{x}{k}\right)\right). \tag{C.13}$$

Proof. At first, note that $\lim_{k\to\infty} b_k(x,\bar{x}) = 0$, so the supremum of $b_k(x,\bar{x})$ over all $k > x/\bar{x}$ is always a maximum attained at finite number of points.

Let $d(x,\bar{x}) := \min_{w \leq k < x/\bar{x}} b_k(x,\bar{x})$, which is positive. Then,

$$f\left(\frac{x+\bar{x}}{k+1}\right) - f\left(\frac{x}{k}\right) \geq d(x,\bar{x})\left(f(\bar{x}) - f\left(\frac{x}{k}\right)\right) \quad \text{for } w \leq k < x/\bar{x}$$

and

$$f\left(\frac{x+\bar{x}}{k+1}\right) - f\left(\frac{x}{k}\right) \geq d(x,\bar{x})\left(f(\bar{x}) - f\left(\frac{x}{k}\right)\right) \quad \text{for } k > x/\bar{x},$$

but with equality only for finite number of k's. Therefore,

$$\begin{aligned}\sum_{k=w}^{\infty} a_k f\left(\frac{x+\bar{x}}{k+1}\right) - \sum_{k=w}^{\infty} a_k f\left(\frac{x}{k}\right) &= \sum_{k=w}^{\infty} a_k \left(f\left(\frac{x+\bar{x}}{k+1}\right) - f\left(\frac{x}{k}\right)\right) \\ &> d(x,\bar{x}) \sum_{k=w}^{\infty} a_k \left(f(\bar{x}) - f\left(\frac{x}{k}\right)\right) \\ &= 0. \end{aligned}$$

□

Proposition C.18. Let $f(x)$ be a positive, strictly decreasing, and continuously differentiable function on $\mathbb{R}_{\geq 0}$ with $f'(0) \in (-\infty, 0)$. Let

$$\bar{x} := f^{-1}\left(\sum_{k=w}^{\infty} p(1-p)^{k-w} f\left(\frac{x}{k}\right)\right), \tag{C.14}$$

where $w \in \mathbb{Z}_{>0}$ and $p \in (0,1)$. Then there exists some $x_0 > 0$ such that for all $x \in (0, x_0]$,

$$\frac{x}{2\left(w + \left\lceil \frac{\log(1/4)}{\log(1-p)} \right\rceil\right)} < \bar{x}. \tag{C.15}$$

Proof. Since $f(x)$ is continuously differentiable and $f'(0) \in (-\infty, 0)$, we can choose $x_0 > 0$ such that

$$\frac{4f'(0)}{3} < f'(\chi) < \frac{2f'(0)}{3} \quad \text{for all } \chi \in [0, x_0]. \tag{C.16}$$

Choose

$$k_1 := w + \left\lceil \frac{\log(1/4)}{\log(1-p)} \right\rceil \geq w + 1$$

so that

$$(1-p)^{k_1 - w} \leq \frac{1}{4}.$$

Then for $x \in (0, x_0]$,

$$\begin{aligned}\sum_{k=w}^{\infty} p(1-p)^{k-w} f\left(\frac{x}{k}\right) &< \sum_{k=w}^{k_1-1} p(1-p)^{k-w} f\left(\frac{x}{k_1}\right) \\ &\quad + \sum_{k=k_1}^{\infty} p(1-p)^{k-w} f(0) \\ &= (1-(1-p)^{k_1-w}) f\left(\frac{x}{k_1}\right) + (1-p)^{k_1-w} f(0)\end{aligned}$$

$$
\begin{aligned}
&= f\left(\frac{x}{k_1}\right) + (1-p)^{k_1-w}\left(f(0) - f\left(\frac{x}{k_1}\right)\right) \\
&\overset{\text{(a)}}{=} f\left(\frac{x}{k_1}\right) - (1-p)^{k_1-w} f'(\chi_1)\frac{x}{k_1} \\
&\overset{\text{(b)}}{=} f\left(\frac{x}{k_1}\left[1 - (1-p)^{k_1-w}\frac{f'(\chi_1)}{f'(\chi_2)}\right]\right) \\
&\overset{\text{(c)}}{<} f\left(\frac{x}{k_1}\left[1 - 2(1-p)^{k_1-w}\right]\right) \\
&\overset{\text{(d)}}{\leq} f\left(\frac{x}{k_1[1+4(1-p)^{k_1-w}]}\right) \\
&\leq f\left(\frac{x}{2k_1}\right),
\end{aligned}
$$

where $\chi_1, \chi_2 \in (0, x_0)$, (a) follows from the mean value theorem, (b) from the identity

$$
f\left(a + \frac{b}{f'(\chi_2)}\right) - f(a) = b
$$

for some $\chi_2 \in (0, a)$ by the mean value theorem again, (c) from (C.16), and (d) from the inequality

$$
1 - t \geq \frac{1}{1+2t}, \quad t \in [0, \tfrac{1}{2}].
$$

By the strictly decreasing property of $f(x)$, we have $\bar{x} > x/(2k_1)$. $\square$

Proposition C.19. Let $f(x)$ be a positive, strictly decreasing, and continuously differentiable function on $\mathbb{R}_{\geq 0}$ with $f'(0) \in (-\infty, 0)$. Then there exists some $x_0 > 0$ such that for all $x \in (0, x_0]$,

$$
\min_{w \leq k < x/\bar{x}} b_k(x, \bar{x}) \geq \max_{k > x/\bar{x}} b_k(x, \bar{x}), \tag{C.17}
$$

where $b_k(x, y)$ and $\bar{x}$ are defined by (C.10) and (C.14), respectively.

Proof. Let

$$
k_0 := 2\left(w + \left\lceil \frac{\log(1/4)}{\log(1-p)} \right\rceil\right).
$$

Choose $x_1 > 0$ such that

$$
\frac{f'(\chi_1)}{f'(\chi_2)} < \sqrt{1 + \frac{1}{k_0}} \quad \text{for all } \chi_1, \chi_2 \in [0, x_1].
$$

By Proposition C.18, there exists some $x_2 > 0$ such that for all $x \in (0, x_2]$, $\bar{x} > x/k_0$. Let $x_0 := \min\{x_1, x_2\}$.

Then, for any $x \in (0, x_0]$ and any $k_1 < x/\bar{x} < k_2$,

$$\begin{aligned}
\frac{b_{k_1}(x,\bar{x})}{b_{k_2}(x,\bar{x})} &\overset{(a)}{=} \frac{f'(\chi_{11})\left(\dfrac{x+\bar{x}}{k_1+1} - \dfrac{x}{k_1}\right)}{f'(\chi_{12})\left(\bar{x} - \dfrac{x}{k_1}\right)} \cdot \frac{f'(\chi_{22})\left(\bar{x} - \dfrac{x}{k_2}\right)}{f'(\chi_{21})\left(\dfrac{x+\bar{x}}{k_2+1} - \dfrac{x}{k_2}\right)} \\
&= \frac{f'(\chi_{11})f'(\chi_{22})(k_2+1)}{f'(\chi_{12})f'(\chi_{21})(k_1+1)} \\
&> \frac{k_0(\lfloor x/\bar{x} \rfloor + 2)}{(k_0+1)\lceil x/\bar{x} \rceil} \\
&\geq \frac{k_0(1 + 1/\lceil x/\bar{x} \rceil)}{(k_0+1)} \\
&\geq \frac{k_0(1+1/k_0)}{(k_0+1)} = 1,
\end{aligned}$$

where (a) follows from the mean value theorem and $\chi_{11}, \chi_{12}, \chi_{21}, \chi_{22} \in (0, x_0)$. This thus establishes the proposition. □

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