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Quantile Methods for Stochastic Frontier Analysis

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Contents

Quantile Methods for Stochastic Frontier Analysis

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ABSTRACT

Quantile regression has become one of the standard tools of econometrics. We examine its compatibility with the special goals of stochastic frontier analysis. We document several conflicts between quantile regression and stochastic frontier analysis. From there we review what has been done up to now, we propose ways to overcome the conflicts that exist, and we develop new tools to do applied efficiency analysis using quantile methods in the context of stochastic frontier models. The work includes an empirical illustration to reify the issues and methods discussed, and catalogs the many open issues and topics for future research.

Keywords: Conditional quantile function; quantile regression; deterministic frontier; inefficiency; stochastic noise; heteroskedasticity; non-linear quantile estimator

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1

Introduction

This monograph seeks to merge two seemingly disparate econometric fields, quantile estimation and stochastic frontier analysis (SFA). Why might these two fields be viewed as disparate? Quantiles exist on a continuum of the distribution, the frontier is a fixed object of it. As will be seen, these two approaches can, when used properly, be merged to provide a unified approach to studying a stochastic boundary.

The use of distribution quantiles for estimation purposes is an old story. The most well known case is the use of the sample median to estimate location parameters instead of the sample mean. More generally, sample quantiles are more robust than sample means against outliers, and this advantage has always been championed, in the face of contaminated samples that did not conform with the idealized conditions needed for sample means to fully perform as theory tells us that they should.

In econometrics, "quantile regression", introduced by Koenker and Bassett (1978), has become the most popular method to use quantile methods in estimation. With hindsight, it was a package with two interconnected but distinct offerings. The first offering was a new estimator focused on linear regression analysis that required no distributional assumptions, was more robust than least-squares methods with respect to outliers in the data, and, when the regressors were independent from the error term, allowed one to estimate consistently the slope regression coefficients. We will call it the "Q-estimator" henceforth, and "quantile regression" will also refer to their approach. We will use the term "quantile estimation" to refer more generally to approaches that pursue their inference goals by estimating quantiles.

The second offering was based on the fact that the Q-estimator could estimate the effects of regressors at different quantiles of the conditional distribution of the dependent variable, simply by choosing the probability associated with each quantile. In the case where the error term of the regression was independent from the regressors, this multiple-quantiles view gave rise to changes only in the value of the constant term of the regression. But when some form of dependence is present between regressors and the error term (like heteroskedasticity), the marginal effects of the regressors differ at different quantiles of the dependent variable, and the Q-estimator was able to provide this much richer information, making the least-squares estimator to suddenly look like a poor relative.

This is the "quantile approach" to statistical analysis and econometric inference proper, where a statistical aspect of our data, the conditional quantiles of the dependent variable, are mapped to an important structural aspect. To provide a prototypical example, if the dependent variable is earnings, and the regressor is a government subsidy program for professional education, "different effects at different quantiles" would tell us whether it was the low-earners or the high-earners that tended to benefit most from the policy. This multiple-quantiles estimation can be achieved by the Q-estimator and it is no wonder that it spread and saw intense use in the treatment effects literature. It remains today a methodology of choice when one wants to drill down on marginal effects and policy evaluation.[1]

In the process, the quantile regression toolkit expanded to include non-linear setups and quantile regression models that accounted for

[1]We take here the opportunity to note that our work will be focused solely on conditional quantile methods. The "unconditional quantiles" approach made widely known by Firpo *et al.* (2009) is beyond the scope of this work.

endogeneity, while its basic incarnation has become a standard feature of most econometric software, that makes it an off-the-shelf choice even for beginners ... which does not always lead to valid outcomes.

SFA, on the other hand, which begun with Aigner *et al.* (1977) and Meeusen and van den Broeck (1977), attempts to discern a stochastic boundary of firm performance, usually cost or output. These methods are not interested in average behavior, but in both an idealized level of performance and any deviations from it. This puts the model, and its corresponding estimators, at odds with traditional regression. However, it has connections to conditional quantiles as the frontier exists somewhere in the output space.

The application of quantile methods in SFA has in the past been rather sparse, but a recent flurry of interest in combining the two was what motivated the present work. We have four main goals: the first is to examine whether, and to what degree, the popular and easily available Q-estimator and quantile regression align well, or not, with the special properties and goals of SFA. We warn the reader that the conclusions here will be partly negative: there exist certain fundamental incompatibilities that do not allow the Q-estimator to provide in stochastic frontier models (SFMs) what it provides in treatment effects studies. Our second goal is to offer an overview of how the quantile approach has been used up to now in stochastic frontier analysis, using quantile regression or likelihood-based analysis. This gives an acute picture of the aforementioned incompatibilities. Our third goal is to provide new and ready-to-implement tools that allow the valid use of quantile regression for efficiency analysis, including estimation of the frontier but also quantile-dependent (in)efficiency measures. Our fourth goal is to sketch avenues for future research and *name* the many prints that are not yet blue, which is done partly throughout the text but also compiled in the penultimate Section of this work.

Sections 2 to 6 present the current state of affairs. We start in Section 2 by detailing the very close link between the regression function and the conditional quantile function, in order to show that the quantile relation is not some disconnected statistical aspect that lives independently of our regression specification. This section also shows what the quantile approach and the Q-estimator actually do, and we contrast this with what SFA models *want* to do, using also a simulated example. Already we encounter the first points of tension between the quantile regression approach and the SFA view.

The reader may be disappointed that we will not provide a detailed treatment here on the use of quantile approaches in the sister field of data envelopment analysis (DEA). This is intentional. The presence of stochastic noise makes the treatment of SFA and DEA distinct, and there are subtle, nontrivial complications that warrant a more thorough discussion of SFA methods due to the presence of both noise *and* inefficiency. We provide a heuristic discussion of this difference at the end of Section 2 to illustrate the main intuitive distinction of the stochastic and deterministic frontier models when quantile methods are applied.

In Section 3 we present the main characteristics and properties of the linear Q-estimator when the error term is independent of the regressors, as a necessary preparation to move to Section 4, where we show how some of these properties are fundamentally incompatible with the goals and purposes of SFA. Essentially, the area of friction is the quantile probability of the deterministic component (DF) of the stochastic frontier (SF).[2] To make the point forcefully, we include in this section a review of applied SFA studies that have used quantile regression, and we show how this incompatibility undermines the reliability and usefulness of their results. Section 5 begins the healing process: we discuss recent advances that properly construct the deterministic frontier.

Section 6 is where we move away from quantile regression, and we present likelihood-based approaches that use density functions that include as one of their parameters the probability of the zero-quantile of their distributions. We focus on a specific incarnation of the Asymmetric Laplace distribution for the noise term in a composite error SFA specification. We examine both frequentist and Bayesian lines of research. The Bayesian approach appears to achieve the *desideratum* of obtaining different estimates of regression coefficients and of inefficiency

[2]The word "deterministic" is used to describe the component of the dependent variable that depends on variables traditionally treated as *decision* variables of the firm.

per quantile, but we clarify that this only reflects the interconnected estimation uncertainty that is inherent in Bayesian econometrics.

Sections 7 to 10 present a new estimator, but also metrics and insights that allow to fruitfully use the quantile approach in SFA. In Section 7 we show how one can use the Q-estimator together with additional assumptions in order to provide conceptually valid and useful estimation and inference results in SFMs. In Section 8 we present quantile-dependent measures of efficiency both at the sample level, and at the individual level, but also how the conditional quantiles of the distribution of inefficiency can be used to offer a picture of how individual efficiency scores are distributed around a chosen quantile of the efficiency distribution.

In Section 9 we prove a fundamental result: that positive and high values of the composite error term of production SFA models, are expected to co-exist with low inefficiency, in a concrete probabilistic sense. An analogous result holds for cost models: when the value of the composite error term is negative and large in absolute value, cost inefficiency is expected to also be low. We decided to separate and stress this result because, with minimum assumptions and certainly for the distributions that are the mainstays of stochastic frontier analysis, it allows us to gauge *individual* inefficiency based on estimated quantities like the residuals of the model.

Section 10 examines the case of dependence between the error term and the regressors or other covariates. We first discuss the issues that generally arise when “traditional” heteroskedasticity co-exists with a skewed and non zero-mean error term, and how we can obtain consistent estimation in such a case. We then examine the particular SFA setting of “determinants of inefficiency” situation, and we develop a non-linear quantile regression model for this setup.

In Section 11 we provide an empirical illustration that showcases the approach of the four previous Sections, and functions as a guide for detailed applied studies.

Section 12 includes a list of the various open issues as well as ideas and directions for future research, while Section 13 offers a short summary and a few parting thoughts.

Part III

For the Road

12

Challenges Ahead

At various places in this study we have mentioned or hinted at unfinished business. Here we collect and present open issues that relate to asymptotic theory, statistical testing, accounting for panel data and nonparametric estimation, as interesting topics for future theoretical breakthroughs.

Inference for the Corrected Q-Estimator

In order to strengthen the reliability of the Corrected Q-estimator a worthy endeavor would be to determine its asymptotic distribution. This should start with the asymptotic distribution of the centered quantile residuals, and proceed to determine the distribution of the individual coefficient estimators, *a la* Olson *et al.* (1980), while Coelli (1995) could inspire adjusted significance tests.

Testing the Distributional Assumption Using Quantiles

We provide here the preliminaries on distributional specification tests based on quantiles.

Using Conditional Quantiles of the Error Components

The Production Frontier. Here the composite error term is $\varepsilon_i = v_i - u_i$. We consider the conditional quantile relation, for some τ, $q_{v\,|\,\varepsilon}(\tau\,|\,\varepsilon_i) \equiv G^{-1}_{v\,|\,\varepsilon}(\tau)$. For the same τ, we have

$$
\begin{aligned}
&\Pr(u_i \leq q_{u\,|\,\varepsilon}(\tau\,|\,\varepsilon_i)\,|\,\varepsilon_i) = \tau \\
&\quad\Longrightarrow \Pr(v_i - \varepsilon_i \leq q_{u\,|\,\varepsilon}(\tau\,|\,\varepsilon_i)\,|\,\varepsilon_i) = \tau \\
&\quad\Longrightarrow \Pr(v_i \leq \varepsilon_i + q_{u\,|\,\varepsilon}(\tau\,|\,\varepsilon_i)\,|\,\varepsilon_i) = \tau \\
&\quad\Longrightarrow G_{v\,|\,\varepsilon}\left(\varepsilon_i + q_{u\,|\,\varepsilon}(\tau\,|\,\varepsilon_i)\right) = \tau.
\end{aligned}
$$

Applying $G^{-1}_{v\,|\,\varepsilon}(\cdot)$ to both sides of the expression and rearranging, we obtain

$$\varepsilon_i = q_{v\,|\,\varepsilon}(\tau\,|\,\varepsilon_i)\; - \; q_{u\,|\,\varepsilon}(\tau\,|\,\varepsilon_i).$$

Essentially the same principle that we saw in Section 2 applies here.[1]

Consider then for the left-hand side the consistent predictor $\widehat{\varepsilon}_i(\widehat{\tau}_{DF})$ and assume distributions for the right-hand side. Under the null hypothesis of correct specification, it will be the case that

$$|\widehat{\varepsilon}_i(\widehat{\tau}_{DF}) - (\hat{q}_{v\,|\,\varepsilon}(\tau\,|\,\varepsilon_i)\; - \;\hat{q}_{u\,|\,\varepsilon}(\tau\,|\,\varepsilon_i))| \underset{\mathrm{H}_0}{\longrightarrow}_p 0, \quad \forall\, i, \quad \forall\, \tau.$$

Notice that this should hold for every i and for every τ, and so it can form the basis for a formal statistical test.

The Cost Frontier. The expression for a cost frontier changes, because we have to transform a tail probability to a cumulative probability. Here the composite error term is $\varepsilon_i = v_i + u_i$, and we consider the conditional quantile relation, $q_{v\,|\,\varepsilon}(1-\tau\,|\,\varepsilon_i) \equiv G^{-1}_{v\,|\,\varepsilon}(1-\tau)$. For the same τ, we have

$$
\begin{aligned}
&\Pr(u_i \leq q_{u\,|\,\varepsilon}(\tau\,|\,\varepsilon_i)\,|\,\varepsilon_i) = \tau \\
&\quad\Longrightarrow \Pr(\varepsilon_i - v_i \leq q_{u\,|\,\varepsilon}(\tau\,|\,\varepsilon_i)\,|\,\varepsilon_i) = \tau \\
&\quad\Longrightarrow 1 - \Pr(v_i \leq \varepsilon_i - q_{u\,|\,\varepsilon}(\tau\,|\,\varepsilon_i)\,|\,\varepsilon_i) = \tau \\
&\quad\Longrightarrow G_{v\,|\,\varepsilon}\left(\varepsilon_i - q_{u\,|\,\varepsilon}(\tau\,|\,\varepsilon_i)\right) = 1 - \tau.
\end{aligned}
$$

[1] Note the interplay between realizations, *conditional* quantiles, and additivity.

Following as before we have, in the case of a cost frontier,

$$\varepsilon_i = q_{v\,|\,\varepsilon}(1-\tau\,|\,\varepsilon_i) \,+\, q_{u\,|\,\varepsilon}(\tau\,|\,\varepsilon_i).$$

Here, under correct specification we obtain the statistic

$$|\widehat{\varepsilon}_i(\widehat{\tau}_{DF}) - (\hat{q}_{v\,|\,\varepsilon}(1-\tau\,|\,\varepsilon_i) \,+\, \hat{q}_{u\,|\,\varepsilon}(\tau\,|\,\varepsilon_i))| \underset{\mathrm{H}_0}{\longrightarrow_p} 0, \quad \forall\, i, \quad \forall\, \tau.$$

Using Estimates of the Constant Term of the Regression at Different Quantile Probabilities

As we have said, the importance of having a consistent predictor series for the residuals and the location of the deterministic frontier, makes the execution of the Q-estimator at different quantiles problematic in SFA. But we can use multiple-quantiles estimation to construct another specification test.

Assume that we estimate the deterministic frontier as described earlier, and so we have

$$\hat{\alpha}(\widehat{\tau}_{DF}) \longrightarrow_p \alpha.$$

Then by executing the Q-estimator at other τ values $\tau_1,\ldots,\tau_j,\ldots\tau_m$, we can obtain the series

$$\hat{\alpha}(\tau_j) - \hat{\alpha}(\widehat{\tau}_{DF}) = q_{\varepsilon}(\tau_j) + o_p(1), \quad j = 1,\ldots,m.$$

At the same time, we can compute estimates for these quantiles by using the assumed distribution for the composite error term,

$$G_{\varepsilon}^{-1}(\tau_j;\hat{\theta}) = q_{\varepsilon}(\tau_j) + o_p(1),$$

where $\hat{\theta}$ represents the vector of estimated distribution parameters. Under the null hypothesis of correct specification we will have

$$|G_{\varepsilon}^{-1}(\tau_j;\hat{\theta}) - \alpha(\widehat{\tau}_j) + \alpha(\widehat{\tau}_{DF})| \underset{\mathrm{H}_0}{\longrightarrow_p} 0, \quad \forall\, j.$$

This is another route to construct a specification test for the distributional assumptions, here using the quantiles of the error term and multiple-quantiles estimation by the Q-estimator, perhaps along the lines of a Kolmogorov-Smirnov approach, and the use of the suprema of these absolute values. The asymptotic theory for these tests would likely be complicated and, since we have to use estimated quantities, bootstrapping will be needed to construct variances and/or critical values.

Handling Panel Data

We briefly review here the small literature related to quantile regression in a panel data setting, in order to highlight the challenges that arise but also the avenues that one could take. Quantile regression with panel data is conceptually and technically complicated, and still very much a rather unsettled endeavor.

The wheels of research started to turn many years after the introduction of quantile regression, with Koenker (2004) who presented a fixed-effects model where the individual effects were time-invariant, not depending on the conditioning quantile probability, and therefore representing only a location shift, just an individual intercept, and not a distributional shift. The author deployed a weighted and penalized Q-estimator where we pool and weight sample information over several τ's in order to improve the estimation of the individual effects. But the model is applicable conditional on a single τ also (at the researcher's peril). Lamarche (2010) established further that there exists an optimal value for the regularization parameter attached to the penalty term and provided the related formula. Galvao and Montes-Rojas (2010) extended the penalized fixed effects model to a dynamic setting while Galvao (2011) revisited the dynamic model but this time without a penalty term. Harding and Lamarche (2009) considered a model with endogeneity and IV estimation where the individual effect is estimated on its own with the use of the two-stage method of Chernozhukov and Hansen (2008).

Kato *et al.* (2012) examined carefully the asymptotic theory for the fixed effects Q-estimator for large-N/large-T panels (for the static and for the dynamic case). A main finding is that asymptotic Normality requires the T-dimension to grow much faster than the N-dimension. Galvao and Kato (2016) smoothed the objective function and obtain asymptotic Normality when N and T grow at the same rate. A bias remains in the asymptotic mean and they propose a bias-reduction scheme.

Graham *et al.* (2018) presented a quantile correlated random coefficients panel data model (in a "fixed-effects" approach). Gu and Volgushev (2019) examined a model with grouped fixed effects. Zhang *et al.* (2019) had similar concerns and proposed a new quantile-regression-based clustering method for panel data. These approaches could link to an SFA model with group-frontiers and the meta-frontier approach.

Abrevaya and Dahl (2008) developed an early model with random effects, where the individual effect is partly a function of the regressors, which is essentially the formulation of Mundlak (1978). Rosen (2012) was concerned with relaxing the i.i.d assumption related to the error term across the temporal dimension. He shows that, left completely unrestricted, dependence leads to loss of identification. He derives a set of restrictions weaker than full independence that restore identification.

Canay (2011) started with a random-coefficients panel data model that is a mean-conditional model with a mean-independent and conditionally heteroskedastic error term. After estimating the individual effect, he proposed a quantile regression where we subtract the estimated individual effect from the dependent variable. We note that in a SFM the estimate of the individual effect will include also the non-zero mean of the composite error term, hence it is not clear how one could proceed from there.

Besstremyannaya and Golovan (2019) detected two errors in Canay (2011). One relates to the needed rates of convergence to infinity of the two panel dimensions. They show that the condition that guarantees asymptotically valid inference requires a much higher growth of the T dimension than what Canay asserted, leading again to the conclusion that quantile regression requires long panels to be trusted. The second error relates to the estimation and inference of the constant term of the model. Chen and Huo (2021) elaborated further on the problems that plague Canay's model and estimation method, and offer an alternative "simple" approach, by combining the first step of Canay's estimator with a "smoothed quantile regression" as proposed in Galvao and Kato (2016). This second step is no longer a linear programming problem but a non-linear minimization one, albeit estimating fewer parameters than in Galvao and Kato's method.

A main lesson from all these studies is that once we move away from the most naive dependence structure (i.i.d data across both dimensions of the panel), established techniques like data transformation do not work: in the complete i.i.d. case, we could certainly apply first-differencing to obtain estimates of the slope regression coefficients using the Q-estimator, estimates that would be the same across quantiles. But then again, the complete i.i.d. case essentially reduces panel data to a large cross-sectional sample.[2] In an i.i.d.-fixed effects setting we exploit neither the panel structure nor the potential of the Q-estimator for different results per different quantile. Once some form of quantile dependence is allowed, the quantile regression coefficients will differ per τ, and then we have to work with untransformed data, and/or multiple-stage procedures.

A second message is that the asymptotics of the Q-estimator with panel data are opaque and need special attention and examination in more depth than usual.

A third result is that currently, the estimators need long panels to claim asymptotic validity, in fact samples where the temporal dimension dominates the cross-sectional. This limits their practical reliability. In this respect the sensible approach of Galvao and Kato (2016) that restored balance between the two dimensions and attempts to correct for the resulting bias looks promising.

Finally, the case of a quantile-dependent individual effect has been left essentially unexplored, or put aside by treating models and estimators "conditional on the individual effect" (which then can be conveniently left uncharacterized).[3] Only the model of Abrevaya and Dahl (2008) of those mentioned above treat the issue directly, by making the individual effect partly dependent on the regressors. This leads to a quantile-dependent individual effect. It also provides an obvious

[2] This was the case with Knox *et al.* (2007) that we presented in Section 4, which applied standard quantile regression on a pooled panel data sample.

[3] Machado and Silva (2019) proposed a novel "quantiles via moments" approach to estimate panel data models with individual effects that are, eventually, quantile-dependent.

connection to the "determinants of inefficiency" SFA model, at least if we are willing to consider a set of covariates (possibly including the regressors) that operate as "determinants of the individual effect".

In the SFA literature, Laporte and Dass (2016) and Hsu *et al.* (2017) made a first attempt at applying panel-data quantile regression, initially using simulated data and in the second article an empirical data set. They equated the individual effect with inefficiency, and then applied both the Schmidt and Sickles (1984) approach to compute inefficiency, and also the Mundlak (1978) approach where the individual effect is a function of regressors. Identifying inefficiency with the individual effect has been criticized in the literature, see for example Greene (2005).

Colombi *et al.* (2014) have proposed a four component model that nests most of the SF panel data models that have been published. Their model separates the individual effect from time-invariant inefficiency, and also includes time varying inefficiency and stochastic noise (hence four components).[4] By nesting the restricted formulations, this four component model also provides a useful compact typology for researchers to decide, conceptually and structurally in the first place, what is more suitable for each applied study. We should also anticipate that quantile estimation methods may have different particularities, different asymptotics and different results as we choose different assumptions on the unobservables of a SFM with panel data.

Nonparametric Methods

The discussion so far has focused on parametric specification of the production frontier. However, there have been major developments in the area of nonparametric quantile estimation and those methods could also be deployed here to estimate the frontier. See Li and Racine (2007) for a textbook treatment of nonparametric quantile regression. One early paper in this area is Wang *et al.* (2014). The authors used convexification and monotonization to construct a piecewise linear representation of the true, unknown quantile process. Consequently, if the axioms of production underlying these restrictions do not hold then

[4] Under specific distributional assumptions the model has a Closed Skew Normal likelihood.

this estimator will not be consistent. An alternative approach, which is nonparametric in nature, but does not rely on axioms of production to pull out the shape of the frontier is kernel smoothing. This is akin in the conditional mean setting to the work of Fan *et al.* (1996) (see Parmeter and Kumbhakar, 2014, for a review).

In the presence of determinants of inefficiency, one problem with direct nonparametric quantile estimation is that the object that is returned is not a quantile frontier, but a conditional quantile. Consider the nonparametric quantile estimator of Li and Racine (2008). With this approach one would estimate the conditional CDF of output (or cost) conditional on both $\boldsymbol{x}$ and $\boldsymbol{z}$. However, because the quantile is recovered by inverting the estimated conditional CDF, there is no way to distinguish between the impact of $\boldsymbol{x}$ on the frontier and $\boldsymbol{z}$ on inefficiency.

An alternative approach would be to try to follow the setup of Tran and Tsionas (2009) and Parmeter *et al.* (2017) which exploits the partial linear structure of the stochastic frontier model in the presence of determinants of inefficiency. This would entail estimation of the quantile model

$$y_i = \boldsymbol{x}_i'\boldsymbol{\beta}(\tau) + \sigma(\boldsymbol{z}_i, \tau)\varepsilon_i, \tag{12.1}$$

where $\mathrm{Var}(\varepsilon_i) = 1$ and so $\sigma(\boldsymbol{z}_i, \tau)$ is the standard deviation of the composite error, something that does not allow separate identification of the distribution parameters of the components of ε_i.

For this "partly linear" setup a quantile estimator based on B-splines (Wang *et al.*, 2009) can be constructed. Let $N = N_n$ be the number of interior knots and let q be the spline order. Divide $[0,1]$ (we can always rescale any of our variables that do not live on $[0,1]$ accordingly) into $(N+1)$ subintervals $I_j = [r_j, r_{j+1})$, $j = 0, \ldots, N-1$, $I_N = [r_N, 1]$, where $\{r_j\}_{j=1}^N$ is a sequence of interior knots, given as

$$r_{-(q-1)} = \cdots = r_0 = 0 < r_1 < \cdots < r_N < 1 = r_{N+1} = \cdots = r_{N+q}.$$

Define the q-th order B-spline basis as $B_{s,q} = \{B_j(x_s) : 1-q \le j \le N\}'$ (de Boor, 2001, Page 89). Let $G_{s,q} = G_{s,q}^{(q-2)}$ be the space spanned by $B_{s,q}$, and let G_q be the tensor product of $G_{1,q}, \ldots, G_{d,q}$, which is the space of functions spanned by

$$\begin{aligned}\mathcal{B}_q(\boldsymbol{z}) &= B_{1,q} \otimes \cdots \otimes B_{d,q} \\ &= \left[\left\{\prod_{s=1}^{d} B_{j_s,q}(z_s) : 1-q \leq j_s \leq N, 1 \leq s \leq d\right\}'\right]_{\boldsymbol{K}_n \times 1} \\ &= \left[\{\mathcal{B}_{j_1,\ldots,j_d,q}(\boldsymbol{z}) : 1-q \leq j_s \leq N, 1 \leq s \leq d\}'\right]_{\boldsymbol{K}_n \times 1},\end{aligned}$$

where $\boldsymbol{z} = (z_1, \ldots, z_d)'$ and $\boldsymbol{K}_n = (N+q)^d$. Let $\mathbf{B}_q = [\{\mathcal{B}_q(\boldsymbol{z}_1), \ldots, \mathcal{B}_q(\boldsymbol{z}_n)\}']_{n \times \boldsymbol{K}_n}$. Then $\sigma(\boldsymbol{z}, \tau)$ can be approximated by $\mathcal{B}_q(\boldsymbol{z})'\theta$, where θ is a $\boldsymbol{K}_n \times 1$ vector. With this notation in tow we can estimate our partly linear quantile stochastic frontier model as

$$\min_{\beta,\theta} \sum_{i=1}^{n} \rho_\tau \left(\frac{y_i - \boldsymbol{x}_i'\boldsymbol{\beta}(\tau)}{\mathcal{B}_q(\boldsymbol{z}_i)'\theta(\tau)} \right). \tag{12.2}$$

To our knowledge this type of an approach has not appeared in the quantile estimation literature but does follow the parametric approach in Jung *et al.* (2015). The theoretical properties of this estimator are left for future work.

13

Summary and Concluding Remarks

In this work we started a bit grimly, sounding the alarm and showing the incompatibilities that exist between quantile regression and SFA, by using theory but also by reviewing published applied studies (Sections 2–6). But then, we provided new estimation and inference tools that surmount these obstacles and allow the valid use of quantile regression and the quantile approach more generally in SFA (Sections 7–11). Specifically, we developed the quantile-based Corrected Q-estimator, that is simple to implement and performed well in the empirical illustration of Section 11, and, for the first time in the SFA literature, we constructed valid quantile-dependent measures of efficiency, both at the sample level but also for individual observations. We showed how the availability of determinants of inefficiency in a data set can be exploited by the implementation of a non-linear quantile model. We also proved an important theoretical result as regards the composed error term of stochastic frontier analysis: large positive values of the composed error in a production model have a large probability of containing *low* values of inefficiency, when noise and inefficiency are independent. This provides a link between conditional and unconditional quantiles that could be further explored. . . alongside the many other open research topics, that we have mainly collected in Section 12. The empirical application of Section 11 we believe vindicates our belief that the quantile approach is an interesting and useful path to take in order to enhance stochastic frontier and efficiency analysis. Yes, the approach needs to be modified; yes, there are still many things we do not yet know; but there are illuminating ways to do quantile-based applied stochastic frontier and efficiency analysis *now*, while we expand this frontier also.

References

Abrevaya, J. and C. M. Dahl (2008). “The effects of birth inputs on birthweight: Evidence from quantile estimation on panel data”. *Journal of Business & Economic Statistics*. 26(4): 379–397.

Aigner, D. J., C. A. K. Lovell, and P. Schmidt (1977). “Formulation and estimation of stochastic frontier production functions”. *Journal of Econometrics*. 6(1): 21–37.

Amemiya, T. (1982). “Two stage least absolute deviations estimators”. *Econometrica*. 50(3): 689–711.

Amsler, C., A. Papadopoulos, and P. Schmidt (2021). “Evaluating the CDF of the Skew normal distribution”. *Empirical Economics*. 60: 3171–3202.

Aragon, Y., A. Daouia, and C. Thomas-Agnan (2005). “Nonparametric Frontier estimation: A conditional quantile-based approach”. *Econometric Theory*. 21(2): 358–389.

Assaf, A. G., M. G. Tsionas, and F. Kock (2020). “Dynamic quantile stochastic frontier models”. *International Journal of Hospitality Management*. 89: 102588.

Azzalini, A. (1986). “Further results on a class of distributions which includes the normal ones”. *Statistica*. 46(2): 199–208.

Azzalini, A. and A. Capitanio (2014). *The Skew-Normal and Related Families*. Cambridge University Press.

Badunenko, O. and D. J. Henderson (2021). "Production analysis with asymmetric error". MPRA Paper No. 110888. URL: https://mpra.ub.uni-muenchen.de/110888.

Battese, G. E. and T. J. Coelli (1988). "Prediction of firm-level technical efficiencies with a generalized frontier production function and panel data". *Journal of Econometrics*. 38: 387–399.

Behr, A. (2010). "Quantile regression for robust bank efficiency score estimation". *European Journal of Operational Research*. 200: 568–581.

Bera, A. K., A. F. Galvao, G. V. Montes-Rojas, and S. Y. Park (2016). "Asymmetric Laplace regression: Maximum likelihood, maximum entropy and quantile regression". *Journal of Econometric Methods*. 5: 79–101.

Berger, A. N. and D. B. Humphrey (1991). "The dominance of inefficiencies over scale and product mix economies in banking". *Journal of Monetary Economics*. 28(1): 117–148.

Bernini, C., M. Freo, and A. Gardini (2004). "Quantile estimation of frontier production function". *Empirical Economics*. 29: 373–381.

Bernstein, D., C. F. Parmeter, and M. Tsionas (2021). On Bayesian quantile estimation of the stochastic Frontier model. Mimeo.

Besstremyannaya, G. and S. Golovan (2019). "Reconsideration of a simple approach to quantile regression for panel data". *The Econometrics Journal*. 22(3): 292–308.

Bonanno, G., D. De Giovanni, and F. Domma (2015). "The 'wrong skewness' problem: A re-specification of stochastic frontiers". *Journal of Productivity Analysis*. 47(1): 49–64.

Buchinsky, M. (1998). "Recent advances in quantile regression models: a practical guideline for empirical research". *Journal of Human Resources*: 88–126.

Canay, I. A. (2011). "A simple approach to quantile regression for panel data". *The Econometrics Journal*. 14: 368–386.

Capitanio, A. (2010). "On the approximation of the tail probability of the scalar skew-normal distribution". *Metron*. 68(3): 299–308.

Chen, L. and Y. Huo (2021). "A simple estimator for quantile panel data models using smoothed quantile regressions". *The Econometrics Journal*. 24(2): 247–263.

Chernozhukov, V., I. Fernández-Val, and A. Galichon (2010). "Quantile and probability curves without crossing". *Econometrica*. 78(3): 1093–1125.

Chernozhukov, V. and C. Hansen (2008). "Instrumental variable quantile regression: A robust inference approach". *Journal of Econometrics*. 142(1): 379–398.

Chidmi, B., D. Solís, and V. E. Cabrera (2011). "Analyzing the sources of technical efficiency among heterogeneous dairy farms: A quantile regression approach". *Journal of Development and Agricultural Economics*. 3(7): 318–324.

Coelli, T. J. (1995). "Estimators and hypothesis tests for a stochastic Frontier function: A Monte Carlo analysis". *Journal of Productivity Analysis*. 6(4): 247–268.

Colombi, R., S. Kumbhakar, G. Martini, and G. Vittadini (2014). "Closed-skew normality in stochastic Frontiers with individual effects and long/short-run efficiency". *Journal of Productivity Analysis*. 42(2): 123–136.

Daouia, A. and L. Simar (2007). "Nonparametric efficiency analysis: A multivariate conditional quantile approach". *Journal of Econometrics*. 140(2): 375–400.

David, F. N. (1953). "A note on the evaluation of the multivariate Normal integral". *Biometrika*. 40(3–4): 458–459.

de Boor, C. (2001). *A Practical Guide to Splines*. New York: Springer.

Fan, Y., Q. Li, and A. Weersink (1996). "Semiparametric estimation of stochastic production frontier models". *Journal of Business & Economic Statistics*. 14(4): 460–468.

Firpo, S., N. M. Fortin, and T. Lemieux (2009). "Unconditional quantile regressions". *Econometrica*. 77(3): 953–973.

Fitzenberger, B., R. Koenker, and J. A. F. Machado (2002). *Economic Applications of Quantile Regression*. Springer Verlag.

Fitzenberger, B., R. Koenker, J. Machado, and B. Melly (2022). "Economic applications of quantile regression 2.0". *Empirical Economics*. 62(1): 1–6.

Frumento, P. and M. Bottai (2016). "Parametric modeling of quantile regression coefficient functions". *Biometrics*. 72(1): 74–84.

Fusco, E., R. Benedetti, and F. Vidoli (2022). "Stochastic frontier estimation through parametric modelling of quantile regression coefficients". *Empirical Economics*. URL: https://doi.org/10.1007/s00181-022-02273-x.

Galvao, A. F. (2011). "Quantile regression for dynamic panel data with fixed effects". *Journal of Econometrics*. 164(1): 142–157.

Galvao, A. F. and K. Kato (2016). "Smoothed quantile regression for panel data". *Journal of Econometrics*. 193(1): 92–112.

Galvao, A. F. and G. V. Montes-Rojas (2010). "Penalized quantile regression for dynamic panel data". *Journal of Statistical Planning and Inference*. 140(11): 3476–3497.

Godfrey, L. G. and C. D. Orme (1991). "Testing for skewness of regression disturbances". *Economics Letters*. 37(1): 31–34.

Goldberger, A. S. (1968). "The interpretation and estimation of Cobb-Douglas functions". *Econometrica*. 36(3/4): 464–472.

Graham, B. S., J. Hahn, A. Poirier, and J. L. Powell (2018). "A quantile correlated random coefficients panel data model". *Journal of Econometrics*. 206(2): 305–335.

Greene, W. H. (1980). "Maximum likelihood estimation of econometric frontier functions". *Journal of Econometrics*. 13(1): 27–56.

Greene, W. H. (2005). "Fixed and random effects in stochastic frontier models". *Journal of Productivity Analysis*. 23(1): 7–32.

Gregg, D. and J. Rolfe (2016). "The value of environment across efficiency quantiles: A conditional regression quantiles analysis of rangelands beef production in north Eastern Australia". *Ecological Economics*. 128: 44–54.

Griliches, Z. (1957). "Specification bias in estimates of production functions". *Journal of Farm Economics*. 39(1): 8–20.

Gu, J. and S. Volgushev (2019). "Panel data quantile regression with grouped fixed effects". *Journal of Econometrics*. 213(1): 68–91.

Gupta, R. D. and D. Kundu (1999). "Generalized exponential distributions". *Australian and New Zealand Journal of Statistics*. 41(2): 173–188.

Harding, M. and C. Lamarche (2009). "A quantile regression approach for estimating panel data models using instrumental variables". *Economics Letters*. 104(3): 133–135.

Horrace, W. C. and C. F. Parmeter (2018). “A Laplace stochastic frontier model”. *Econometric Reviews*. 37: 260–280.

Horrace, W. C., C. F. Parmeter, and I. Wright (2021). On asymmetry and quantiles in the stochastic Frontier model. Mimeo.

Hsu, A., A. R. Dass, W. Bert, P. Coyte, and A. Laporte (2017). “Efficiency estimation with panel quantile regression: An application using longitudinal data from nursing homes in Ontario, Canada”. *Working Paper* No. 170003. Canadian Centre for health economics.

Jondrow, J., C. A. K. Lovell, I. S. Materov, and P. Schmidt (1982). “On the estimation of technical efficiency in the stochastic frontier production function model”. *Journal of Econometrics*. 19(2/3): 233–238.

Jradi, S., C. F. Parmeter, and J. Ruggiero (2019). “Quantile estimation of the stochastic Frontier model”. *Economics Letters*. 182: 15–18.

Jradi, S., C. F. Parmeter, and J. Ruggiero (2021). “Quantile Estimation of Stochastic Frontiers with the Normal-Exponential Specification”. *European Journal of Operational Research*. 295(2): 475–483.

Jradi, S. and J. Ruggiero (2019). “Stochastic data envelopment analysis: A quantile regression approach to estimate the production frontier”. *European Journal of Operational Research*. 278: 385–393.

Jung, Y., Y. Lee, and S. N. MacEachern (2015). “Efficient quantile regression for heteroscedastic models”. *Journal of Statistical Computation and Simulation*. 85(13): 2548–2568.

Kaditi, E. A. and E. I. Nitsi (2010). “Applying regression quantiles to farm efficiency estimation”. *Discussion Paper* No. 112. Centre of Planning and Economic Research (KEPE). URL: https://www.kepe.gr/index.php/en/research/recent-publications/discussion-papers.

Kato, K., A. F. Galvao, and G. V. Montes-Rojas (2012). “Asymptotics for panel quantile regression models with individual effects”. *Journal of Econometrics*. 170(1): 76–91.

Knox, K. J., E. C. Blankmeyer, and J. R. Stutzman (2007). “Technical efficiency in Texan nursing facilities: A stochastic production forntier approach”. *Journal of Economics and Finance*. 31(1): 75–86.

Koenker, R. (2004). “Quantile regression for longitudinal data”. *Journal of Multivariate Analysis*. 91(1): 74–89.

Koenker, R. (2005). *Quantile Regression*. Cambridge University Press.

Koenker, R. and G. Bassett (1978). "Regression quantiles". *Econometrica*. 46(1): 33–50.

Koenker, R. and G. Bassett (1982). "Robust tests for heteroscedasticity based on regression quantiles". *Econometrica*. 50: 43–61.

Komunjer, I. (2005). "Quasi-maximum likelihood estimation for conditional quantiles". *Journal of Econometrics*. 128(1): 137–164.

Kumbhakar, S. C., R. Amundsveen, H. M. Kvile, and G. Lien (2015). "Scale economies, technical change and efficiency in Norwegian electricity distribution, 1998–2010". *Journal of Productivity Analysis*. 43(3): 295–305.

Kumbhakar, S. C. and C. A. K. Lovell (2000). *Stochastic Frontier Analysis*. Cambridge University Press.

Kumbhakar, S. C., C. F. Parmeter, and V. Zelenyuk (2020). "Stochastic Frontier analysis: Foundations and advances II". In: *Handbook of Production Economics*. Ed. by S. Ray, R. Chambers, and S. C. Kumbhakar. Vol. 1. Springer.

Kumbhakar, S. C. and H.-J. Wang (2010). "Estimation of technical inefficiency in production Frontier models using cross-sectional data". *Indian Economic Review*. 45(2): 7–77.

Lamarche, C. (2010). "Robust penalized quantile regression estimation for panel data". *Journal of Econometrics*. 157(2): 396–408.

Land, K. C., C. K. Lovell, and S. Thore (1993). "Chance-constrained data envelopment analysis". *Managerial and Decision Economics*. 14(6): 541–554.

Laporte, A. and A. R. Dass (2016). "The use of panel quantile regression for efficiency measurement: Insights from Monte Carlo simulations". *Working Paper* No. 160005. Canadian Centre for health economics.

Lee, T.-H., A. Ullah, and H. Wang (2018). "The second-order bias of quantile estimators". *Economics Letters*. 173: 143–147.

Li, Q. and J. Racine (2007). *Nonparametric Econometrics: Theory and Practice*. Princeton University Press.

Li, Q. and J. S. Racine (2008). "Nonparametric estimation of conditional CDF and quantile functions with mixed categorical and continuous data". *Journal of Business & Economic Statistics*. 26(4): 423–434.

Lien, G., S. Størdal, and S. Baardsen (2007). "Technical efficiency in timber production and effects of other income sources". *Small-Scale Forestry*. 6: 65–78.

Liu, C., A. Laporte, and B. S. Ferguson (2008). "The quantile regression approach to efficiency measurement: Insights from Monte Carlo simulations". *Health Economics*. 17: 1073–1087.

Liu, J. and H. A. David (1989). "Quantiles of sums and expected values of ordered sums". *Australian Journal of Statistics*. 31(3): 469–474.

Machado, J. A. and J. S. Silva (2019). "Quantiles via moments". *Journal of Econometrics*. 213(1): 145–173.

Maneejuk, P. and W. Yamaka (2021). "Copula-based stochastic Frontier quantile model with unknown quantile". In: *Data Science for Financial Econometrics*. Ed. by N. Ngoc Thach, V. Kreinovich, and N. D. Trung. Cham: Springer International Publishing. 445–458.

Martins-Filho, C. and F. Yao (2008). "A smooth nonparametric conditional quantile frontier estimator". *Journal of Econometrics*. 143(2): 317–333.

Meeusen, W. and J. van den Broeck (1977). "Efficiency estimation from Cobb-Douglas production functions with composed error". *International Economic Review*. 18(2): 435–444.

Mundlak, Y. (1978). "On the pooling of time series and cross section data". *Econometrica*. 46(1): 69–85.

Nelsen, R. B. (1993). "Some concepts of bivariate symmetry". *Journal of Nonparametric Statistics*. 3(1): 95–101.

Oberhofer, W. (1982). "The consistency of nonlinear regression minimizing the L1-norm". *The Annals of Statistics*: 316–319.

Olson, J. A., P. Schmidt, and D. A. Waldman (1980). "A Monte Carlo study of estimators of stochastic frontier production functions". *Journal of Econometrics*. 13: 67–82.

Ondrich, J. and J. Ruggiero (2001). "Efficiency measurement in the stochastic frontier model". *European Journal of Operational Research*. 129(3): 434–442.

Papadopoulos, A. (2021a). "Measuring the effect of management on production: a two-tier stochastic frontier approach". *Empirical Economics*. 60: 3011–3041.

Papadopoulos, A. (2021b). "Stochastic frontier models using the Generalized Exponential distribution". *Journal of Productivity Analysis*. 55(1): 15–29.

Papadopoulos, A. (2022a). "The noise error component in Stochastic Frontier Analysis". Paper presented at EWEPA XVII, Porto, Portugal.

Papadopoulos, A. (2022b). "Trade liberalization and growth: a quantile moderator for Hoyos'(2021) replication study of Estevadeordal and Taylor (2013)". *Empirical Economics*. 63: 549–563.

Papadopoulos, A. and C. F. Parmeter (2021). "Type II failure and specification testing in the stochastic Frontier model". *European Journal of Operational Research*. 293(3): 990–1001.

Parmeter, C. F. (2021). "Is it MOLS or COLS?" *Tech. rep.* University of Oviedo, Department of Economics, Oviedo Efficiency Group (OEG).

Parmeter, C. F. and S. C. Kumbhakar (2014). "Efficiency analysis: A primer on recent advances". *Foundations and Trends in Econometrics*. 7(3–4): 191–385.

Parmeter, C. F., H.-J. Wang, and S. C. Kumbhakar (2017). "Nonparametric estimation of the determinants of inefficiency". *Journal of Productivity Analysis*. 47(3): 205–221.

Paul, S. and S. Shankar (2018). "On estimating efficiency effects in a stochastic frontier model". *European Journal of Operational Research*. 271(2): 769–774.

Poiraud-Casanova, S. and C. Thomas-Agnan (2000). "About monotone regression quantiles". *Statistics & Probability Letters*. 48(1): 101–104.

Rosen, A. M. (2012). "Set identification via quantile restrictions in short panels". *Journal of Econometrics*. 166(1): 127–137.

Schennach, S. M. (2005). "Bayesian exponentially tilted empirical likelihood". *Biometrika*: 31–46.

Schmidt, P. and R. C. Sickles (1984). "Production Frontiers and panel data". *Journal of Business & Economic Statistics*. 2(2): 367–374.

Sickles, R. C. and V. Zelenyuk (2019). *Measurement of Productivity and Efficiency*. Cambridge University Press.

Stead, A. D., P. Wheat, and W. H. Greene (2022a). "Quantile regression estimation of the stochastic frontier model: Choice of quantile and robustness". Paper presented at EWEPA XVII, Porto, Portugal.

Stead, A. D., P. Wheat, and W. H. Greene (2022b). "Robustness in stochastic Frontier analysis". In: *Advanced Mathematical Methods in Economic Efficiency Analysis, Lecture Notes in Economics and Mathematical Systems*. Ed. by P. Macedo, V. F. Moutinho, and M. Madaleno. Cham: Springer International Publishing.

Stigler, S. M. (1980). "Stigler's law of eponymy". *Transactions of the New York Academy of Sciences*. 39(1 Series II): 147–157.

Tran, K. C. and E. G. Tsionas (2009). "Estimation of nonparametric inefficiency effects stochastic frontier models with an application to British manufacturing". *Economic Modelling*. 26: 904–909.

Tsionas, E. G., A. G. Assaf, and A. Andrikopoulos (2020). "Quantile stochastic frontier models with endogeneity". *Economics Letters*. 188: 108964.

Tsionas, M. G. (2020a). "Quantile stochastic frontiers". *European Journal of Operational Research*. 282(3): 1177–1184.

Tsionas, M. G. (2020b). "Bounded rationality and thick frontiers in stochastic frontier analysis". *European Journal of Operational Research*. 284(2): 762–768.

Van den Broeck, J., G. Koop, J. Osiewalski, and M. F. Steel (1994). "Stochastic frontier models: A Bayesian perspective". *Journal of Econometrics*. 61(2): 273–303.

Wang, H.-J. and P. Schmidt (2002). "One-step and two-step estimation of the effects of exogenous variables on technical efficiency levels". *Journal of Productivity Analysis*. 18: 129–144.

Wang, H. J., Z. Zhu, and J. Zhou (2009). "Quantile regression in partially linear varying coefficient models". *The Annals of Statistics*. 37(6B): 3841–3866.

Wang, W. S. and P. Schmidt (2009). "On the distribution of estimated technical efficiency in stochastic frontier models". *Journal of Econometrics*. 148(1): 36–45.

Wang, Y., S. Wang, C. Dang, and W. Ge (2014). "Nonparametric quantile frontier estimation under shape restriction". *European Journal of Operational Research*. 232(3): 671–678.

Watson, R. and I. Gordon (1986). “On quantiles of sums”. *Australian Journal of Statistics*. 28(2): 192–199.

Wei, Z., X. Zhu, and T. Wang (2021). “The extended skew-normal-based stochastic frontier model with a solution to ‘wrong skewness’ problem”. *Statistics*. 55(6): 1387–1406.

Wichitaksorn, N., S. B. Choy, and R. Gerlach (2014). “A generalized class of skew distributions and associated robust quantile regression models”. *Canadian Journal of Statistics*. 42(4): 579–596.

Wüthrich, K. (2020). “A comparison of two quantile models with endogeneity”. *Journal of Business & Economic Statistics*. 38(2): 443–456.

Zeebari, Z. (2021). “Conditional quantile estimators of unit inefficiency in stochastic frontier analysis with application to electricity distribution market”. *Paper presented at the 1st Virtual African Productivity Conference.*

Zhang, N., X. Huang, and Y. Liu (2021). “The cost of low-carbon transition for China’s coal-fired power plants: A quantile frontier approach”. *Technological Forecasting and Social Change*. 169: 120809.

Zhang, Y., H. J. Wang, and Z. Zhu (2019). “Quantile-regression-based clustering for panel data”. *Journal of Econometrics*. 213(1): 54–67.

Zhao, S. (2021). “Quantile estimation of stochastic frontier models with the normal-half normal specification: A cumulative distribution function approach”. *Economics Letters*: 109998.