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Combinatorial Optimization of Alternating Current Electric Power Systems

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Contents

Combinatorial Optimization of Alternating Current Electric Power Systems

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ABSTRACT

In the era of dynamic smart grid with fluctuating demands and uncertain renewable energy supplies, it is crucial to continuously optimize the operational cost and performance of electric power grid, while maintaining its state within the stable operating limits. Nonetheless, a major part of electric power grid consists of alternating current (AC) electric power systems, which exhibit complex behavior with non-linear operating constraints. The optimization of AC electric power systems with dynamic demands and supplies is a very challenging problem for electrical power engineers.

The hardness of optimization problems of AC electric power systems stems from two issues: (1) *non-convexity* involving complex-valued entities of electric power systems, and (2) combinatorial constraints involving *discrete* control variables. Without proper theoretical tools, heuristic methods or general numerical solvers had been utilized traditionally to tackle these problems, which do not provide theoretical guarantees of the achieved solutions with respect to the true optimal solutions. There have been recent advances in

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applying convex relaxations to tackle non-convex problems of AC electric power systems. On the other hand, discrete combinatorial optimization is rooted in theoretical computer science, which typically considers linear constraints, instead of those non-linear constraints in AC electric power systems.

To bridge power systems engineering and theoretical computer science, this monograph presents a comprehensive study of combinatorial optimization of AC electric power systems with (inelastic) discrete demands. The main idea of this monograph is to draw on new extensions of discrete combinatorial optimization with linear constraints, like knapsack and unsplittable flow problems. We present approximation algorithms and inapproximability results for various settings from (1) basic single-capacitated AC electric power systems, to (2) constant-sized AC electric grid networks with power flows, and (3) scheduling of AC electric power. This monograph aims to establish a foundation for the inter-disciplinary problems of power systems engineering and theoretical computer science.

1

Introduction

1.1 Need for Optimization in Smart Power Grid

The electric power grid has been an indispensable part of our society, empowering the economic and social activities in every aspect of our daily lives. Our society is consuming a tremendous amount of energy at an increasing rate. There has been a drastic surge in global energy consumption. As a result, the power grid needs to undergo transformations to meet the new challenges for a more sustainable society:

- *Deregulation of Power Industry*: Replacing the monopolized industry of power grid in generation, transmission and distribution by decentralized operators with heterogeneous requirements.
- *Decarbonization and Incorporation of Renewable Energy*: Transitioning from fossil fuel energy to environment-friendly but uncertain renewable energy supplies.
- *Demand Responsiveness*: Shifting the traditional power grid that is engineered for peak demands to be more demand responsive, such that grid operators and end users can react to variable grid resources by dynamic pricing and electricity markets.

- *Inefficiency Elimination*: Reducing the energy loss in power generation and transmission by employing technologies, such as Combined Heat-and-Power (CHP) generation and Flexible AC Transmission Systems (FACTS).

- *Disruption Protection*: Enabling more robust control against outage and power failures by incorporating autonomous microgrids and emergence demand response management.

These transformations will create a *smarter* power grid with improved energy-efficiency, responsiveness and stability. In particular, there is a need for *continuous optimization* in smart grid that can react rapidly to dynamic situations in presence of fluctuating demands and uncertain renewable energy. In the past, the operations of power grid relied on careful a-priori planning, under the assumptions of static demands and predictable circumstances. In the era of dynamic smart grid, self-optimization with adaptive control is more crucial to its operations.

There are several factors for consideration in the optimization of power grid operations:

- *Scale*: Power grid is connected to an increasing number of users and loads, with growing presence of electric vehicles and smart appliances. These demands have to be optimally coordinated and regulated in a large-scale manner.

- *Time*: The fluctuations of renewable energy supplies and demands under dynamic pricing occur more significantly in a shorter timescale. Power grid needs to adapt to intermittency rapidly.

- *Performance*: A variety of performance objectives ought to be considered by different parties among energy suppliers, transmitters, distributors, regulators and residential/commercial end users.

- *Stability*: The stability operating constraints of the power grid need be adhered and validated from time to time to ensure reliable operations.

Therefore, it is critical to continuously optimize the power grid under various performance objectives in a scalable and responsive manner, while maintaining its state within the stable operating limits.

However, a power grid is a large complex system. In particular, a major part of the power grid is composed of alternating current (AC) electric power systems, which exhibit complex behavior with non-linear operating constraints. The effective management and control operations of AC electric power systems involve very challenging problems that baffle electrical power engineers. The hardness of optimization problems in AC electric power systems stems mainly from two issues:

1. *Non-Convex Constraints* involving complex-valued variables and parameters of AC electric power systems.

2. *Combinatorial Constraints* involving *discrete* control variables for the operation of power systems.

Traditionally, heuristic methods or general numerical solvers had been utilized for the combinatorial optimization problems of AC electric power systems, without proper theoretical analyses on the performance, efficiency and optimality of the results. Some of these methods return inefficient algorithms that are not scalable in larger systems, or fail to provide guarantees on the deviation of output solutions from the true optimal solutions.

Combinatorial optimization has been extensively studied in theoretical computer science, with diverse applications in operations research and engineering science beyond computing systems. Hence, it is imperative to draw on the related tools from theoretical computer science to study the problems arising from smart grid. In particular, there are recent advances in approximation algorithms with provable approximation ratios that can be applied in combinatorial power systems problems.

This monograph aims to establish an interdisciplinary bridge between power systems engineering and theoretical computer science by relating the practical and challenging problems in electric power systems with the modern theoretical tools from computer science. The proper understanding of these hard problems in electric power systems can advance the frontiers of both communities. Particularly, this monograph is tailored for these two groups of audience:

- For *Power System Engineers*, it introduces the concepts and results of approximation algorithms, and applies them to solve electric power systems problems.
- For *Computer Scientists*, it provides an exposition of a class of challenging combinatorial problems in electric power systems.

Before presenting the approximation algorithms for AC power systems in the subsequent chapters, this section first explains the basics of AC electric power systems, and then some standard terminology of approximation algorithms in the literature.

1.2 Basics of AC Electric Power Systems

This section presents the basics of electric power systems. More details of electric power systems can be found in a standard power systems textbook (e.g., Grainger and Stevenson, 1994). For example, we consider scenario in Figure 1.1.

First, we give an example scenario of power consumption scheduling problem as illustrated in Figure 1.1. There are multiple households and electric vehicles connecting to the power grid with dynamic renewable energy supplies. In each household, there are electric appliances that can only be controlled by switching on or off. For charging electric vehicles, there are currently three main categories of charging infrastructure standards: *Level 1* charging with cord-set singe-phase connections to a regular household outlet, *Level 2* wall-mount three-phase connections, and *Level 3* DC fast charging. It is worth noting that none of these current popular charging standards allows continuously controllable charging power at an arbitrary rate. To ensure reliable charging, there requires a delicate control system for the supplied charging power. Hence, the charging power normally varies within a limited discrete set of nearly constant values (Gan *et al.*, 2012). The scheduling of power consumption with discrete controls is a natural combinatorial optimization.

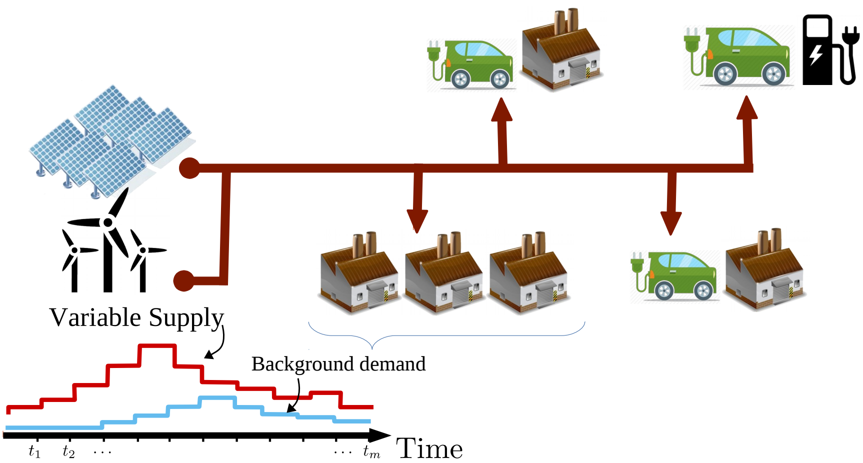

Figure 1.1: An illustration of power consumption scheduling problem.

1.2.1 Notations

An electric power system is characterized by an electric network with nodes (also called buses) and edges (also called lines). A power flow in an electric network is described by physical quantities such as current, voltage and power. We represent an electric network by a connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ denotes a set of nodes and $\mathcal{E}$ denotes a set of edges. We index the nodes in $\mathcal{V}$ by $\{0, 1..., m\}$, where $m \triangleq |\mathcal{V}|$. Node 0 usually carries a special meaning (called slack bus). If $\mathcal{G}$ represents an electric distribution network, then node 0 usually denotes the generation source or the feeder to the main grid. Let $\mathcal{V}^+ \triangleq \mathcal{V}\backslash\{0\}$. We fix an *arbitrary* orientation on the edges, and think of $\mathcal{G}$ as a directed graph. For convenience, we choose an orientation such that $\mathcal{G}$ forms a directed *acyclic* graph where the "power flow" from node 0 to the rest of nodes in $\mathcal{V}^+$[1]. Thus, in the rest of monograph, we assume that the orientation of a directed edge (i, j) designates that the current or power flows from i to j.

[1]Such orientation can always be obtained by first finding a spanning tree $\mathcal{T}$ on $\mathcal{V}$ and rooting it at node 0, then orienting all edges of $\mathcal{T}$ away from 0, with end points on directed paths in $\mathcal{T}$, and then orienting all other edges arbitrarily.

For node $i \in \mathcal{V}$, we denote its *voltage* by V_i. For each edge $e = (i,j) \in \mathcal{E}$, we denote its *current* from i to j by $I_{i,j}$, its *transmitted power* by $S_{i,j}$, and its *impedance* by $z_{i,j}$. In direct current (DC) electric systems, all quantities belong to the set of real numbers (denoted by $\mathbb{R}$); whereas in alternating current (AC) electric systems, these quantities belong to the set of complex numbers (denoted by $\mathbb{C}$). Usually, the voltage V_0 at node 0 is normalized as $V_0 = 1$.

For a complex number $\nu \in \mathbb{C}$, we denote the *magnitude* of ν by $|\nu|$, the *phase angle* (or argument) that ν makes with the real axis by $\angle\nu$, and the complex *conjugate* of ν by ν^*. We sometimes write $\nu^{\mathrm{R}} \triangleq \mathrm{Re}(\nu)$ for the real part and $\nu^{\mathrm{I}} \triangleq \mathrm{Im}(\nu)$ for the imaginary part of ν. For $\nu, \nu' \in \mathbb{C}$, we write $\nu \le \nu'$ to mean $\nu^{\mathrm{R}} \le \nu'^{\mathrm{R}}$ and $\nu^{\mathrm{I}} \le \nu'^{\mathrm{I}}$.

There are several basic laws governing the relationships of the quantities $V_i, I_{i,j}, z_{i,j}, S_{i,j}$ in an electric network:

- *Ohm's Law*: For each $(i,j) \in \mathcal{E}$,

$$V_i - V_j = z_{i,j} I_{i,j}. \tag{1.1}$$

- *Kirchhoff's Current Law*: For node $i \in \mathcal{V}$,

$$\sum_{(i,j)\in\mathcal{E}} I_{i,j} = 0. \tag{1.2}$$

- *Electric Power Formula*: For each $(i,j) \in \mathcal{E}$,

$$S_{i,j} = V_i I_{i,j}^*. \tag{1.3}$$

Additionally, by convention, the following *skew symmetry* relation holds:

$$I_{i,j} = -I_{j,i}. \tag{1.4}$$

Each node $i \in \mathcal{V}$ is associated with a power injection/extraction s_i, which represents the net power injecting to or extracting from the electric network at node i. The real part $\mathrm{Re}(s_i)$ represents the so-called *active* power, while the imaginary part $\mathrm{Im}(s_i)$ represents the *reactive* power. The *apparent* power is defined as the magnitude $|s_i| = \sqrt{(\mathrm{Re}(s_i))^2 + (\mathrm{Im}(s_i))^2}$ of s_i. For power injection (i.e., power generation), $\mathrm{Re}(s_i) \le 0$; whereas for power extraction (i.e., power demands or loads), $\mathrm{Re}(s_i) \geq 0$. We note the sign of power injection/extraction is sometimes reversed in the power systems literature. For an inductor, $\mathrm{Im}(s_i) \geq 0$; whereas for a capacitor, $\mathrm{Im}(s_i) \leq 0$. Note that transmission lines are usually resistive or inductive, namely, $\mathrm{Re}(z_{i,j}) \geq 0$ and $\mathrm{Im}(z_{i,j}) \geq 0$. The *power factor* of a power demand s_i is defined as $\mathtt{PF}(s_i) \triangleq \frac{\mathrm{Re}(s_i)}{|s_i|}$. As required by common power electronic standards (e.g., National Electrical Code, 2005), most appliances and equipment have a bounded power factor $\mathtt{PF}(s_i) \geq 0.8$, (roughly, $\angle s_i \leq \frac{\pi}{4}$).

1.2.2 Power Flow Model

A power flow model summarizes the state of power flows, considering Kirchhoff's current law with respect to the power injection/extraction. There are several ways of describing a power flow model.

Bus Injection Model

The *Bus Injection Model* (BIM) considers the power injection (or extraction), s_j, at each node (i.e., bus) $j \in \mathcal{V}^+$:

$$s_j = \sum_{(i,j)\in\mathcal{E}} V_j I_{i,j}^* - \sum_{(j,l)\in\mathcal{E}} V_j I_{j,l}^*, \qquad \forall j \in \mathcal{V}, \tag{1.5}$$

$$V_i - V_j = z_{i,j} I_{i,j}, \qquad \forall (i,j) \in \mathcal{E}. \tag{1.6}$$

Branch Flow Model

Alternatively, the *Branch Flow Model* (BFM) (Baran and Wu, 1989a; Baran and Wu, 1989b) considers the transmitted power ($S_{i,j}$) through each edge $(i,j) \in \mathcal{E}$:

$$s_j = \sum_{(i,j)\in\mathcal{E}} (S_{i,j} - z_{i,j}|I_{i,j}|^2) - \sum_{(j,l)\in\mathcal{E}} S_{j,l}, \qquad \forall j \in \mathcal{V}, \tag{1.7}$$

$$V_i - V_j = z_{i,j} I_{i,j}, \qquad \forall (i,j) \in \mathcal{E}, \tag{1.8}$$

$$S_{i,j} = V_i I_{i,j}^*, \qquad \forall (i,j) \in \mathcal{E}. \tag{1.9}$$

For completeness, set $s_0 = -\sum_{(0,i)\in\mathcal{E}} S_{0,i}$. Note that the power flows are interpreted as from node 0 toward the rest of nodes[2] in $\mathcal{V}^+$.

The Branch Flow Model provides a convenient way to simplify the notations. One can drop the phase angles, and replace $V_i = |V_i|e^{\angle V_i}$ and $I_{i,j} = |I_{i,j}|e^{\angle I_{i,j}}$ by simply $|V_i|$ and $|I_{i,j}|$, respectively. This gives us a relaxed model as follows.

Branch Flow Model with Angle Relaxation

Let $v_i = |V_i|^2$ and $\ell_{i,j} = |I_{i,j}|^2$. The *Branch Flow Model with angle relaxation* omits the phase angles:

$$s_j = \sum_{(i,j)\in\mathcal{E}} (S_{i,j} - z_{i,j}\ell_{i,j}) - \sum_{(j,l)\in\mathcal{E}} S_{j,l}, \qquad \forall j \in \mathcal{V}, \tag{1.10}$$

$$v_i - v_j = 2\mathrm{Re}(z_{i,j}^* S_{i,j}) - |z_{i,j}|^2 \ell_{i,j}, \qquad \forall (i,j) \in \mathcal{E}, \tag{1.11}$$

$$|S_{i,j}|^2 = v_i \ell_{i,j}, \qquad \forall (i,j) \in \mathcal{E}. \tag{1.12}$$

BFM with angle relaxation can be derived from BIM as follows. We rewrite (1.3) by taking the complex conjugate of both sides:

$$I_{i,j} = \frac{S_{i,j}^*}{V_i^*} \Rightarrow \ \ell_{i,j} = |I_{i,j}|^2 = \frac{|S_{i,j}|^2}{|V_i|^2} = \frac{|S_{i,j}|^2}{v_i}, \tag{1.13}$$

which is equivalent to (1.12). Substituting (1.9) in (1.8), we obtain

$$V_j = V_i - I_{i,j} z_{i,j} = V_i - \frac{S_{i,j}^*}{V_i^*} z_{i,j}. \tag{1.14}$$

Taking the magnitude square of both sides in (1.14), and using (1.13)[3]:

$$\begin{aligned} v_j = |V_j|^2 &= |V_i|^2 + |\tfrac{S_{i,j}^*}{V_i^*} z_{i,j}|^2 - 2\mathrm{Re}(V_i^* \tfrac{S_{i,j}^*}{V_i^*} z_{i,j}) \\ &= v_i^2 + \ell_{i,j}|z_{i,j}|^2 - 2\mathrm{Re}(z_{i,j}^* S_{i,j}), \end{aligned} \tag{1.15}$$

which is equivalent to (1.12).

[2] BFM can be also expressed using the opposite orientation toward node 0: $s_j = \sum_{(l,j)\in\mathcal{E}} \left(S_{l,j} - z_{l,j}|I_{l,j}|^2\right) - \sum_{(j,i)\in\mathcal{E}} S_{j,i}$. As shown in Low (2014a), there is a bijection between the models of the two orientations, since $S_{j,i} = -S_{i,j} + z_{i,j}|I_{i,j}|^2$ and $I_{i,j} = -I_{j,i}$.

[3] Using the relation $|a+b|^2 = (a+b)^*(a+b) = |a|^2 + |b|^2 + a^*b + b^*a = |a|^2 + |b|^2 + 2\mathrm{Re}(a^*b) = |a|^2 + |b|^2 + 2\mathrm{Re}(b^*a)$, for complex numbers $a, b \in \mathbb{C}$.

As shown in Farivar and Low (2013a) and Farivar and Low (2013b), it is always possible to recover $(V_i, I_{i,j})_{(i,j)\in\mathcal{E}}$ from $(v_i, \ell_{i,j})_{(i,j)\in\mathcal{E}}$, when $\mathcal{G}$ is a *tree* network.

In the rest of monograph, unless otherwise stated, we assume that $\mathcal{G}$ is a tree network, and hence, we will use BFM with angle relaxation (or simply called Branch Flow Model) for brevity.

Simplified DistFlow Model

In BFM with angle relaxation, if we assume $z_{i,j}\ell_{i,j} \to 0$, for example, because of negligible $z_{i,j}$ at each edge, then we obtain a simplified model called *DistFlow* model:

$$s_j = \sum_{(i,j)\in\mathcal{E}} S_{i,j} - \sum_{(j,l)\in\mathcal{E}} S_{j,l}, \qquad \forall j \in \mathcal{V}, \tag{1.16}$$

$$v_i - v_j = 2\mathrm{Re}(z_{i,j}^* S_{i,j}), \qquad \forall (i,j) \in \mathcal{E}, \tag{1.17}$$

$$|S_{i,j}|^2 = v_i \ell_{i,j}, \qquad \forall (i,j) \in \mathcal{E}. \tag{1.18}$$

The DistFlow model provides an "upper bound" for the power flow in BFM, because it ignores the power consumed on transmission lines.

1.2.3 Optimal Power Flow Problem

The *optimal power flow* (OPF) problem is a fundamental problem in power systems engineering, which was introduced in 1962 (Carpentier, 1962; Carpentier, 1979), and since then has received considerable attention (see Frank *et al.* (2012a) and Frank *et al.* (2012b) for a survey).

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a radial (tree) electric distribution network. Node 0 is called the *root*. Since $\mathcal{G}$ is a tree, $|\mathcal{V}^+| = |\mathcal{E}| = m$. We consider a particular tree topology in which a single feeder is attached to the root 0, via a single edge $(0, 1)$. See an illustration in Figure 1.2. Hence (1.10) in BFM (with angle relaxation) becomes

$$S_{i,j} = s_j + z_{i,j}\ell_{i,j} + \sum_{(j,l)\in\mathcal{E}} S_{j,l}, \qquad \forall (i,j) \in \mathcal{E}, \tag{1.19}$$

$$S_{0,1} = -s_0. \tag{1.20}$$

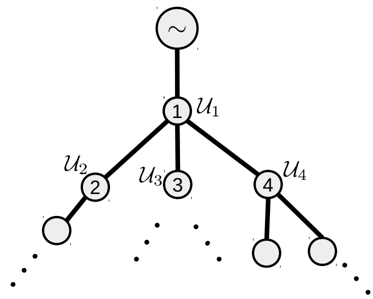

Figure 1.2: An illustration of the considered tree topology.

Control Variables

Instead of assigning a single power injection/extraction to each node, we consider a general setting where a set of users are attached to each node. We assume that the power demand of each user can be controlled individually. Let $\mathcal{N} = [n] \triangleq \{1, \ldots, n\}$ be the set of all users, where $|\mathcal{N}| = n$. Denote the set of users attached node j by $\mathcal{U}_j \subseteq \mathcal{N}$. Let $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i)$ be the subtree rooted at node i. Let the set of users within subtree $\mathcal{G}_j$ be $\mathcal{N}_j \triangleq \cup_{j \in \mathcal{V}_j} \mathcal{U}_j$.

By a slight abuse of notation, the demand for user k is represented by $s_k \in \mathbb{C}$. In this monograph, we consider only consumer users, such that $\mathrm{Re}(s_k) \geq 0$ (but $\mathrm{Im}(s_k)$ may be negative) $\forall k \in \mathcal{N}$. Hence, it follows that the total power injection $\mathrm{Re}(s_0) \leq 0$.

Among the users, some have *discrete* power demands, denoted by $\mathcal{I} \subseteq \mathcal{N}$. A discrete demand s_k, for $k \in \mathcal{I}$, takes values from a discrete set $\mathcal{S}_k \subseteq \mathbb{C}$. We assume that $\mathbf{0} \in \mathcal{S}_k$, for all $k \in \mathcal{I}$, so that a discrete demand can be completely shut off. A special case is the binary case $\mathcal{S}_k \triangleq \{\mathbf{0}, \overline{s}_k\}$, where a demand s_k can be either completely satisfied at level $\overline{s}_k \in \mathbb{C}$ or dropped, e.g., a piece of equipment that is either switched on with a fixed power consumption rate or completely off.

The rest of the users, denoted by $\mathcal{F} \triangleq \mathcal{N} \backslash \mathcal{I}$, have *continuous* demands, defined by *convex* sets $\mathcal{S}_k$, for $k \in \mathcal{F}$; a typical example is a set defined by box constraints: $\mathcal{S}_k \triangleq \{s_k \in \mathbb{C} : \underline{s}_k \leq s_k \leq \overline{s}_k\}$, for given lower and upper bounds $\underline{s}_k$ and $\overline{s}_k$.

Operating Constraints of Power Systems

Recall that $S_{i,j}$ is the power flowing from node i toward j. Note that $S_{i,j}$ is not symmetric, namely, $|S_{i,j}|$ is not equivalent to $|S_{j,i}|$, the power flowing in the opposite direction. There are the following common operating constraints of power systems:

- *Power Generation Constraint*: $|s_0| \leq \overline{s}_0$.
- *Power Capacity Constraints*: $|S_{i,j}| \leq \overline{S}_{i,j}$, $|S_{j,i}| \leq \overline{S}_{i,j}$, $\forall(i,j) \in \mathcal{E}$.
- *Current Thermal Constraints*: $\ell_{i,j} \leq \overline{\ell}_{i,j}$, $\forall(i,j) \in \mathcal{E}$.
- *Voltage Constraints*: $\underline{v}_j \leq v_j \leq \overline{v}_j$, $\forall j \in \mathcal{V}^+$.

In the above constraints, $\underline{v}_j, \overline{v}_j \in \mathbb{R}^+$ are the minimum and maximum allowable voltage magnitude squares at node j, and $\overline{S}_{i,j}, \overline{\ell}_{i,j} \in \mathbb{R}^+$ are the maximum allowable apparent power and current on edge (i,j), respectively. By (1.20), the power generation constraint is implicitly captured by power capacity constraints as $|s_0| = |S_{0,1}| \leq \overline{S}_{0,1}$.

Note that reverse power constraint $|S_{j,i}| \leq \overline{S}_{i,j}$ can be reformulated as $|S_{i,j} - z_{i,j}\ell_{i,j}| \leq \overline{S}_{i,j}$.

Objective Functions

In the following, a subscript is omitted from a variable to denote its vector form, for example, $S \triangleq (S_{i,j})_{(i,j)\in\mathcal{E}}, \ell \triangleq (\ell_{i,j})_{(i,j)\in\mathcal{E}}$, $s \triangleq (s_k)_{k\in\mathcal{N}}, v \triangleq (v_j)_{j\in\mathcal{V}^+}$.

The goal of OPF is to find an assignment for the demand vector s that optimizes a certain *non-negative* objective function. We consider two versions of objective functions: (1) a concave objective that represents the benefit (or utility) of power flow, and (2) a convex objective that represents the cost (or disutility) of power flow.

For utility based objective, we denote the objective function by:

$$f(s_0, s) = f_0(-\mathrm{Re}(s_0)) + \sum_{k\in\mathcal{N}} f_k(\mathrm{Re}(s_k)), \tag{1.21}$$

where $f_0 : \mathbb{R}^+ \to \mathbb{R}^+$ is non-negative and *non-increasing* (note that $\mathrm{Re}(s_0) \leq 0$), and f_k is non-negative and *non-decreasing* (a user's utility increases as more power is allocated to the user, while the generator's utility decreases as more power is generated).

For cost based objective, we denote the objective function by:

$$h(s_0, s) = h_0(-\operatorname{Re}(s_0)) + \sum_{k \in \mathcal{N}} h_k(\operatorname{Re}(s_k)), \tag{1.22}$$

where $h_0 : \mathbb{R}^+ \to \mathbb{R}^+$ is non-negative and *non-decreasing*, and f_k is non-negative and *non-increasing* (thus modeling the fact that each user prefers maximum demand).

Note that for finding an optimal solution, both versions are equivalent, as one can set $f_0(-\operatorname{Re}(s_0)) = C - h_0(-\operatorname{Re}(s_0))$ and $f_k(\operatorname{Re}(s_k)) = C - h_k(\operatorname{Re}(s_k))$, for $k \geq 1$, where C is a sufficiently large constant. Nonetheless, there is a significant difference in terms of finding an approximation solution. See Section 1.3.1 for details.

Problem Formulation

We formulate OPF using BFM (with angle relaxation). The goal of OPF is to maximize the utility objective function $f(s_0, s)$ (or minimize the cost objective function $h(s_0, s)$) subject to the operating constraints of power systems.

The inputs are the voltage, current and transmitted power limits $[v_0, (\underline{v}_j, \overline{v}_j)_{j \in \mathcal{V}^+}, (\overline{S}_{i,j}, \overline{\ell}_{i,j}, z_{i,j})_{(i,j) \in \mathcal{E}}, (\mathcal{S}_k)_{k \in \mathcal{N}}]$, whereas the outputs are the control decision variables and power flow states, (s_0, s, S, v).

The maximization version of OPF is defined by the mixed integer programming problem (OPF) with Cons. (1.23)–(1.31). To define the minimization version of OPF (denoted by $\text{OPF}_{\min}$), one replaces $\max_{s_0,s,S,v,\ell} f(s_0, s)$ by $\min_{s_0,s,S,v,\ell} h(s_0, s)$.

Note that there are two sources of *non-convexity* in this formulation: the quadratic equality constraints (1.23) and the discrete constraints for $k \in \mathcal{I}$ in (1.30).

1.3 Basics of Combinatorial Optimization

This monograph employs combinatorial optimization techniques to provide efficient approximation algorithms for AC electric power systems

$$
\begin{aligned}
\text{(OPF)} \quad & \max_{s_0, s, S, v, \ell} \; f(s_0, s) \\
\text{subject to} \quad & \ell_{i,j} = \frac{|S_{i,j}|^2}{v_i}, && \forall (i,j) \in \mathcal{E}, \quad (1.23) \\
& S_{i,j} = \sum_{k \in \mathcal{U}_j} s_k + \sum_{l:(j,l) \in \mathcal{E}} S_{j,l} + z_{i,j}\ell_{i,j}, && \forall (i,j) \in \mathcal{E}, \quad (1.24) \\
& S_{0,1} = -s_0, && (1.25) \\
& v_j = v_i + |z_{i,j}|^2 \ell_{i,j} - 2\mathrm{Re}(z_{i,j}^* S_{i,j}), && \forall (i,j) \in \mathcal{E}, \quad (1.26) \\
& \underline{v}_j \le v_j \le \overline{v}_j, && \forall j \in \mathcal{V}^+, \quad (1.27) \\
& |S_{i,j}| \le \overline{S}_{i,j}, \; |S_{j,i}| \le \overline{S}_{i,j}, && \forall (i,j) \in \mathcal{E}, \quad (1.28) \\
& \ell_{i,j} \le \overline{\ell}_{i,j}, && \forall (i,j) \in \mathcal{E}, \quad (1.29) \\
& s_k \in \mathcal{S}_k, && \forall k \in \mathcal{N}, \quad (1.30) \\
& v_j \in \mathbb{R}^+, \forall j \in \mathcal{V}^+, \ell_{i,j} \in \mathbb{R}^+, S_{i,j} \in \mathbb{C}, && \forall (i,j) \in \mathcal{E}. \quad (1.31)
\end{aligned}
$$

with discrete demands. The area of approximation algorithms is well-studied in theoretical computer science (see, e.g., Vazirani, 2010). In the following, we recall some standard terminology from this area.

1.3.1 Approximation Solutions

Consider a maximization problem $\mathcal{A}$ with *non-negative* objective function $f(\cdot)$, let F be a feasible solution to $\mathcal{A}$ and $F^\star$ be an optimal solution to $\mathcal{A}$. $f(F)$ refers to the objective value of F. Let $\textsc{Opt}(\mathcal{A}) = f(F^\star)$ be the objective value of $F^\star$. It is common to measure the quality of a proposed feasible solution F by the approximation ratio α between the objective of this solution and that of an optimal solution $F^\star$.

Definition 1.1. For $\alpha \in (0,1)$, an α-approximation to maximization problem $\mathcal{A}$ is a feasible solution F such that

$$f(F) \ge \alpha \cdot \textsc{Opt}(\mathcal{A}).$$

A (polynomial-time) algorithm that, for any given instance of the problem, produces a feasible solution achieving this ratio is called an α-approximation algorithm.

Similarly, for a minimization problem $\mathcal{B}$ with non-negative cost function $h(\cdot)$, let H be a feasible solution to and $H^\star$ be an optimal solution to $\mathcal{B}$. $h(H)$ refers to the cost of H. Let $\textsc{Opt}(\mathcal{B}) = h(H^\star)$ be the cost of $H^\star$.

Definition 1.2. For $\alpha' > 1$, an α'-approximation to minimization problem $\mathcal{B}$ is a feasible solution H such that

$$c(H) \leq \alpha' \cdot \textsc{Opt}(\mathcal{B}).$$

A (polynomial-time) algorithm that, for any given instance of the problem, produces a feasible solution achieving this ratio is called an α'-approximation algorithm.

Note that given a minimization problem $\mathcal{B}$, one can define a maximization problem $\mathcal{A}$, by setting $f(\cdot) = C - h(\cdot)$, for some constant C such that $f(\cdot)$ is non-negative. Although both problems are equivalent in the sense of finding an optimal solution, algorithms for finding α-approximation solutions may be very different in the two cases. In combinatorial optimization, there are numerous such examples of minimization and maximization versions of the same problems having completely different approximation algorithms and approximation ratios. One example is the minimum and maximum *traveling salesman* problems (see, e.g., Vazirani, 2010).

1.3.2 Resource-augmented Approximation Solutions

A more relaxed definition of an approximation solution is (α, β)-approximation, which also allows violation of certain constraints, parametrized by β. Consider a maximization problem $\mathcal{A}$ with a multivariate constraint function $g(\cdot)$. Suppose that a feasible solution F to $\mathcal{A}$ is required to satisfy $\underline{g} \leq g(F) \leq \overline{g}$.

Definition 1.3. For $\alpha \in (0, 1)$ and $\beta \geq 1$, an (α, β)-approximation solution to maximization problem $\mathcal{A}$ is a solution F such that

$$f(F) \geq \alpha \cdot \textsc{Opt}(\mathcal{A}),$$
$$\tfrac{1}{\beta} \cdot \underline{g} \leq g(F) \leq \beta \cdot \overline{g}.$$

A (polynomial-time) algorithm that, for any given instance of the problem, produces an (α, β)-approximation solution is called an (α, β)-approximation algorithm.

Definition 1.4. For $\alpha' > 1$ and $\beta \geq 1$, an (α', β)-approximation solution to minimization problem $\mathcal{B}$ is a solution H such that

$$h(H) \leq \alpha' \cdot \textsc{Opt}(\mathcal{B}),$$
$$\frac{1}{\beta} \cdot \underline{g} \leq g(H) \leq \beta \cdot \overline{g}.$$

A (polynomial-time) algorithm that, for any given instance of the problem, produces an (α, β)-approximation solution is called an (α, β)-approximation algorithm.

Note that α-approximation is $(\alpha, 1)$-approximation.

1.3.3 Polynomial-time Approximation Scheme (PTAS)

In particular, a *polynomial-time approximation scheme* (PTAS) is a $(1-\epsilon)$-approximation algorithm to a maximization problem, or a $(1+\epsilon)$-approximation algorithm to a minimization problem, for any $\epsilon > 0$. The running time of a PTAS is polynomial in the input size for every fixed $\epsilon > 0$, but the exponent of the polynomial might depend on $1/\epsilon$. Namely, a PTAS allows a parametrized approximation ratio in the running time.

A resource-augmented PTAS is a $(1 - \epsilon, 1 + \epsilon)$-approximation algorithm for a maximization problem, and a $(1 + \epsilon, 1 + \epsilon)$-approximation algorithm for a minimization problem, for any $\epsilon > 0$. Again the running time of such a PTAS is polynomial in the input size for every fixed $\epsilon > 0$.

1.3.4 Fully Polynomial-time Approximation Scheme (FPTAS)

An even stronger notion is a *fully polynomial-time approximation scheme* (FPTAS), which is the same as a PTAS but requires the running time to be polynomial in both input size and $1/\epsilon$.

Similarly, we define a resource augmented FPTAS, as a $(1-\epsilon, 1+\epsilon)$-approximation algorithm for a maximization problem, and a $(1+\epsilon, 1+\epsilon)$-approximation algorithm for a minimization problem, for any $\epsilon > 0$, with the running time being polynomial in the input size and $1/\epsilon$. We will refer to these as $(1-\epsilon, 1+\epsilon)$-FPTAS and $(1+\epsilon, 1+\epsilon)$-FPTAS, respectively.

1.3.5 Quasi Polynomial-time Approximation Scheme (QPTAS)

A weaker notion of a PTAS is a *quasi-polynomial-time approximation scheme* (QPTAS), which has time complexity $n^{\text{polylog}(n)}$ for each fixed $\epsilon > 0$, where n is the input size.

The notions of α-approximation, (α, β)-approximation, PTAS, FPTAS and QPTAS can be applied to OPF.

1.3.6 Polytopes and Linear Programming

A *convex polytope* $\mathcal{P}$ in $\mathbb{R}^n$ is the set of points satisfying a finite number of linear inequalities: $\mathcal{P} = \{x \in \mathbb{R}^n \mid Ax \le b\}$, for a given matrix $A \in \mathbb{R}^{m\times n}$ and vector $b \in \mathbb{R}^m$. Given a set of points $P = \{p_1, \ldots, p_r\} \subseteq \mathbb{R}^n$, the *convex hull* of P, denoted by cvxhull(P) is the set of all *convex combinations* of points in P:

$$\text{cvxhull}(P) \triangleq \Big\{ \sum_{i=1}^{r} \lambda_i p_i \mid \sum_{i=1}^{r} \lambda_i = 1,\ \lambda_i \ge 0\ \forall i \Big\}. \tag{1.32}$$

By the well-known Minkowski-Weyl theorem (see, e.g., Schrijver, 1986), any polytope $\mathcal{P} \subseteq \mathbb{R}^n$ can be represented as the *convex hull* of the set of its *extreme points*, also called *vertices* or *basic feasible solutions* (BFSs) of $\mathcal{P}$.

Linear programming (LP) is the problem of maximizing or minimizing a *linear* objective function subject to *linear* constraints. Linear programs (LPs) can be solved efficiently (in polynomial-time assuming rational input of finite precision), see Bertsimas and Tsitsiklis (1997) for an introduction to LP.

The following lemma will be used in our approximation algorithms.

Lemma 1.1 (see, e.g., Bertsimas and Tsitsiklis, 1997; Schrijver, 1986)**.** Consider the following LP:

$$\text{(LP)} \quad \max_{x \in [0,1]^n} \; c^T x \tag{1.33}$$

$$\text{subject to} \quad Ax \leq b, \tag{1.34}$$

where A is an $m \times n$ matrix and b is an m-dimensional vector. Then

1. there is an optimal basic feasible solution;
2. any basic feasible solution $x^\star$ has at most m *fractional* components. Namely, $|\{i \in \{1, ..., n\} \mid x_i^\star \in (0,1)\}| \leq m$.

1.3.7 Second Order Cone Programming

A *Second-order cone program* (SOCP) is a convex optimization problem in which a *linear* objective function is maximized or minimized subject to ℓ_2-*norm* constraints of the following form:

$$\text{(SOCP)} \quad \max_{x \in \mathbb{R}^n} \; c^T x \tag{1.35}$$

$$\text{subject to} \quad \|A_i x + b_i\|_2 \leq d_i^T x + f_i, \qquad \forall i \in \{1, ..., m\}, \tag{1.36}$$

where $A_i \in \mathbb{R}^{n_i \times n}$, $b_i \in \mathbb{R}^{n_i}$, $c, d_i \in \mathbb{R}^n$ and $f_i \in \mathbb{R}$.

There are also efficient polynomial-time algorithms for solving (approximately) SOCPs; (see, e.g., Boyd and Vandenberghe, 2004). In fact, such algorithms can find a near-feasible solution x' that satisfies the constraints within an absolute error $\delta > 0$ (that is, $\|A_i x + b_i\|_2 \leq d_i^T x + f_i + \delta$), such that $c^T x' \geq \text{OPT}^\star - \delta$, in polynomial time in the input size (including the bit complexity) and $\log \frac{1}{\delta}$, where $\text{OPT}^\star$ is the optimal objective value of (SOCP).

In many cases, it is possible to convert such an approximately feasible solution x' to an exactly feasible solution without losing much in the approximation guarantee; see, for example, Section 2.3.1. For simplicity in this monograph, unless otherwise stated, we will assume that the convex programming solver returns an *exact* optimal solution.

1.4 Organization

This monograph covers approximation algorithms and inapproximability results for various settings of AC electric systems in the following chapters:

- (Chapter 2) Basic single-capacitated AC electric power systems to establish the foundation of a more sophisticated electric grid.
- (Chapter 3) Constant-sized AC electric grid networks with power flows and common operating constraints of power systems.
- (Chapter 4) Scheduling of AC electric power that involves temporal optimization with heterogeneous users' preferences.

Moreover, we provide hardness results in Chapter 5 for the above settings to show that our approximation algorithms are among the best achievable in theory. We also provide simulation studies of our algorithms in several practical case studies in Chapter 6. Finally, we conclude this monograph with an outline of several on-going extensions and future work in Chapter 7.

1.5 Notes

The optimal power flow (OPF) problem was introduced in 1962 (Carpentier, 1962; Carpentier, 1979), and since then has been studied extensively (see Frank *et al.* (2012a) and Frank *et al.* (2012b) for a survey). There are several formulations of OPF, with subtle differences. For example, Gan *et al.* (2015) and Huang *et al.* (2017) adopt the opposite flow orientation from leaves to root. Also, Huang *et al.* (2017) implicitly considers power capacity constraints in one direction only. Our formulation explicitly considers bi-directional power capacity constraints. Although Gan *et al.* (2015) considers the possibility of discrete power injections, it provides efficient algorithm for finding the optimal solutions only in the *continuous* case, under some assumptions. For the minimization version of OPF, Gan *et al.* (2015) and Huang *et al.* (2017) consider only non-increasing objective functions for the exactness of convex relaxation. However, convex objective function is required for solving OPF.

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