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Rajna Gibson

University of Geneva and Swiss Finance Institute
Switzerland
Rajna.Gibson@unige.ch

Francois-Serge Lhabitant

Kedge Capital Fund Management, Ltd.
and EDHEC Business School
France
francois@lhabitant.net

Denis Talay

INRIA
France
denis.talay@inria.fr

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Modeling the Term Structure of Interest Rates: A Review of the Literature

Rajna Gibson[1], Francois-Serge Lhabitant[2] and Denis Talay[3]

[1] *University of Geneva and Swiss Finance Institute, Switzerland, Rajna.Gibson@unige.ch*
[2] *Kedge Capital Fund Management, Ltd. and EDHEC Business School, France, francois@lhabitant.net*
[3] *INRIA, France, denis.talay@inria.fr*

Abstract

The last decades have seen the development of a profusion of theoretical models of the term structure of interest rates. The aim of this survey is to provide a comprehensive review of these continuous time modeling techniques of the term structure applicable to value and hedge default-free bonds and other interest rate derivatives. The originality of the survey lies in the fact that it provides a unifying framework in which most continuous-time term structure models can be nested and thus related to each other. Thus, we not only present the most important continuous-time term structure models in the literature but also provide a mathematically rigorous and unifying setting in which these models can be compared in terms of their similarities, distinguished in terms of their idiosyncratic features and in which their main contributions and limitations can easily be highlighted.

Contents

1

Introduction

1.0.1 Objectives of this Monograph

Understanding and modeling the term structure of interest rates represents one of the most challenging topics in the recent financial economics literature. Judging by the proliferation of term structure models that have been proposed over the last decades, the subject seems to have been theoretically and empirically widely explored and with good reason since it lies at the core of the most basic and elaborate valuations problems encountered in finance. Indeed, all financial assets can be valued by the technique of discounting their expected future cash flows given an appropriate discount rate function that embeds an underlying theory about risk premia and the term structure. Moreover, since the introduction of option trading on bonds and other interest rate contingent claims, much attention has been given to the development of models to price and hedge interest rate derivatives as well as to manage the risk of interest rate contingent portfolios.

However, while the Black and Scholes (1973) model has rapidly established itself as "the" reference model for pricing and hedging stock contingent claims, none of the many continuous-time models that have been proposed by academics and used by practitioners to price and hedge interest rate contingent claims deserves the same qualification. It is indisputable that we benefit from the rich diversity of models of the term structure of interest rates but this variety comes at the expense of the lose of both consistency and harmonisation at the aggregate level of the positions that are being managed according to those various models.

The aim of this monograph is to provide a comprehensive review of the continuous-time modeling techniques of the term structure applicable to value and hedge default-free bonds and other interest rate derivatives. The originality of this monograph is the unified framework in which most continuous-time term structure models are nested and thus related to each other. Thus, we present the most important term structure models developed over the last three decades in a mathematically rigorous and unified setting which highlights similarities, idiosyncratic features, major contributions and limitations. In addition we show that most term structure models can be grouped into two main families, first the short-term rate-based univariate and multivariate term structure models and second the Heath, Jarrow and Morton (1992) forward rate based term structure models. We also provide conceptual bridges that allow us to redefine - under a certain set of assumptions - some of the models belonging to one family in terms of reciprocal models belonging to the other family. Finally, based on our own research, in Section 9 we characterize and quantify the profit and loss function due to model mis-specification when hedging interest rate contingent claims within both families of models.

Nice mathematical introductions to interest rates are the books by Privault (2008) and Filipović (2009). For advanced mathematical and financial surveys on term structure modeling, we highly recommend the in-depth book by Musiela and Rutkowski (2005) as well as Björk (1997)'s excellent short survey. The difficult subject of infinite dimensional interest rate models is treated in Filipović (2001) and Carmona and Tehranchi (2006). Brigo and Mercurio (2001) cover empirical and practical issues.

This monograph is organized as follows: Chapter 1 presents the main objectives and provides the definitions and notation used throughout the monograph. Chapter 2 proposes an interest rate model taxonomy and Chapter 3 introduces the mathematical framework used throughout the monograph. In chapter 4, we briefly present the main economic theories of the term structure of interest rates. Chapter 5 uses the mathematical framework to present the family of short-term rate-based term structure models. Chapter 6 uses the mathematical framework to present the family of forward rate based models. In Chapter 7, we briefly survey the empirical evidence on interest rate model estimation, discuss calibration issues and list selected empirical references for practitioners interested in the validity and the performance of these models. Chapter 8 introduces a novel approach to characterize and quantify the profit and loss function due to model mis-specification. Chapter 9 discusses some of the challenges in simulations of continuous - time term structure models. Chapter 10 concludes the survey. Finally, some useful mathematical results can be found in the Appendix of the survey.

A first difficulty associated with the term structure literature is related to its rich but often heterogeneous terminology. In order to minimize this problem, we will start by defining the terms and the notation used in the following sections.

1.0.2 Zero Coupon Bonds and Interest Rates

A **discount bond** (also called **zero-coupon bond**) with maturity T is a financial asset that pays to its holder one currency unit at time T with certainty. The price at time t of a discount bond with maturity date $T > t$ is denoted by $B(t,T)$. Hereafter, we will exclusively focus on bonds that do not have any default risk. Therefore, it follows immediately that $B(T,T) = 1$ for all T.

The **yield to maturity** at time t of a discount bond with maturity T is the constant and continuously compounded rate of return at which the discount bond price accrues from time t to time T to yield one currency unit at time T. The yield to maturity is sometimes called the spot rate and is denoted by $R(t,T)$. We have the following definition

$$B(t,T)e^{R(t,T)(T-t)} = 1.$$

Solving for the yield to maturity gives

$$R(t,T) = -\frac{\ln B(t,T)}{T-t}. \tag{1.1}$$

To be consistent with financial intuition, one should observe $R(t,T) > 0$ for any time t and $T \geq t$.

The **term structure of interest rates** at time t expresses the relationship between spot rates and their maturity dates as a graph of the function $T \longrightarrow R(t,T)$ for $T > t$. Hereafter, we will assume that a continuous set of bonds is traded, so that the term structure will be continuous with respect to the maturity date.

An interesting point on the term structure of interest rates is the **instantaneous risk-free interest rate** r_t, also called **short-term rate**. It is defined by

$$r_t = \lim_{T \to t} R(t,T),$$

which is the yield to maturity of an instantaneously maturing discount bond. Equivalently, it represents the interest rate on a risk-free investment over an infinitesimal time-period dt. We will see below that r_t is the state variable in many univariate models of the term structure. To be consistent with financial intuition, one should observe $r_t > 0$ for any time t.

Another interesting point on the term structure of interest rates is the **long-term rate** ℓ_t, also called **consol rate**. It is defined by

$$\ell_t = \lim_{T \to \infty} R(t,T),$$

but in practice the long-term rate can be approximated by the yield on a **consol bond** (an infinite time-to-maturity bond that pays a continuous coupon) which is quoted on some markets. To be consistent with financial intuition, one should observe $\ell_t > 0$ for any time t.

We denote by $f(t,T_1,T_2)$ the continuously compounded **forward rate** for a time interval $[T_1,T_2]$, i.e. the rate at time t for a risk-free loan starting at time T_1 and maturing at time T_2. One has

$$f(t,T_1,T_2) = \frac{\ln B(t,T_2) - \ln B(t,T_1)}{T_2 - T_1}.$$

Of particular interest is the **instantaneous forward rate** $f(t,T) = f(t,T,T)$, which is the rate that one contracts at time t for a loan starting at time T for an infinitesimal period of time dt. Assuming that bond prices are differentiable, we have

$$f(t,T) = -\frac{1}{B(t,T)}\frac{\partial B}{\partial T}(t,T).$$

Equivalently, one can define the bond price in terms of forward rates as

$$B(t,T) = \exp\left(-\int_t^T f(t,s)ds\right).$$

Note that the spot interest rate is also simply given by the forward rate for a maturity equal to the current date, that is,

$$r_t = f(t,t). \tag{1.2}$$

Not surprisingly, there exist some fundamental relationships between the dynamics of the short-term rate, the dynamics of the discount bond price and the dynamics of the forward rates. We will review them later on, when considering the Heath, Jarrow and Morton (1992) family of term structure models.

1.0.3 Simple Rates

In some cases discussed in Section 7 we will focus on simple interest rates rather than continuously compounded interest rates. We define the simple rate for a time interval $[t,T]$,the **Libor spot rate,** as

$$L(t,T) = -\frac{B(t,T)-1}{(T-t)B(t,T)}.$$

We can also define a **Libor forward rate** at time t for a time interval $[T_1,T_2]$ with $T_2 > T_1 > t$ as

$$L(t,T_1,T_2) = -\frac{B(t,T_2)-B(t,T_1)}{(T_2-T_1)B(t,T_1)}.$$

1.0.4 The Money Market Account

A rollover position at the short-term rate r_t will be called a **money market account**. By convention, we assume that the money market account was initialized at time 0 with a one currency unit (e.g. one dollar) investment, so that its value at time t is given by

$$B_t = \exp\left(\int_0^t r_s ds\right), \tag{1.3}$$

or equivalently, by

$$\begin{cases} dB_t = r_t dt, \\ B_0 = 1. \end{cases}$$

1.0.5 Remarks

Let us recall that there exist a set of discount bond price specific no-arbitrage restrictions:

- any discount bond price process has a non-stochastic terminal value at its maturity date.

$$B(T,T) = 1.$$

- a zero-coupon bond price is less than or equal to the price of another zero-coupon with a shorter maturity.
- interest rates (expressed in nominal terms) are not negative.
- the yield curve or term structure of interest rates is a smooth function of time to maturity.

In the following, we will assume that these restrictions are fulfilled, unless explicitly mentioned.

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