

---

# **Continuous-Time Linear Models**

---

# Continuous-Time Linear Models

---

**John H. Cochrane**

*University of Chicago Booth School of Business and  
NBER. 5807 S. Woodlawn  
Chicago, IL 60637  
USA*

*[john.cochrane@chicagobooth.edu](mailto:john.cochrane@chicagobooth.edu)*

**now**

the essence of **know**ledge

Boston – Delft

## Foundations and Trends<sup>®</sup> in Finance

*Published, sold and distributed by:*

now Publishers Inc.  
PO Box 1024  
Hanover, MA 02339  
USA  
Tel. +1-781-985-4510  
[www.nowpublishers.com](http://www.nowpublishers.com)  
[sales@nowpublishers.com](mailto:sales@nowpublishers.com)

*Outside North America:*

now Publishers Inc.  
PO Box 179  
2600 AD Delft  
The Netherlands  
Tel. +31-6-51115274

The preferred citation for this publication is J. H. Cochrane, Continuous-Time Linear Models, Foundations and Trends<sup>®</sup> in Finance, vol 6, no 3, pp 165–219, 2011

ISBN: 978-1-60198-586-6  
© 2012 J. H. Cochrane

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted in any form or by any means, mechanical, photocopying, recording or otherwise, without prior written permission of the publishers.

Photocopying. In the USA: This journal is registered at the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923. Authorization to photocopy items for internal or personal use, or the internal or personal use of specific clients, is granted by now Publishers Inc for users registered with the Copyright Clearance Center (CCC). The 'services' for users can be found on the internet at: [www.copyright.com](http://www.copyright.com)

For those organizations that have been granted a photocopy license, a separate system of payment has been arranged. Authorization does not extend to other kinds of copying, such as that for general distribution, for advertising or promotional purposes, for creating new collective works, or for resale. In the rest of the world: Permission to photocopy must be obtained from the copyright owner. Please apply to now Publishers Inc., PO Box 1024, Hanover, MA 02339, USA; Tel. +1-781-871-0245; [www.nowpublishers.com](http://www.nowpublishers.com); [sales@nowpublishers.com](mailto:sales@nowpublishers.com)

now Publishers Inc. has an exclusive license to publish this material worldwide. Permission to use this content must be obtained from the copyright license holder. Please apply to now Publishers, PO Box 179, 2600 AD Delft, The Netherlands, [www.nowpublishers.com](http://www.nowpublishers.com); e-mail: [sales@nowpublishers.com](mailto:sales@nowpublishers.com)

**Foundations and Trends<sup>®</sup> in  
Finance**

Volume 6 Issue 3, 2011

**Editorial Board**

**Editor-in-Chief:**

**George M. Constantinides**

*Leo Melamed Professor of Finance*

*The University of Chicago*

*Graduate School of Business*

*5807 South Woodlawn Avenue*

*Chicago IL 60637*

*USA*

*gmc@gsb.uchicago.edu*

**Editors**

**Franklin Allen**

*Nippon Life Professor of Finance and Economics,*

*The Wharton School, The University of Pennsylvania*

**Andrew W. Lo**

*Harris & Harris Group Professor, Sloan School of Management,*

*Massachusetts Institute of Technology*

**René M. Stulz**

*Everett D. Reese Chair of Banking and Monetary Economics,*

*Fisher College of Business, The Ohio State University*

## Editorial Scope

**Foundations and Trends<sup>®</sup> in Finance** will publish survey and tutorial articles in the following topics:

- Corporate Finance
  - Corporate Governance
  - Corporate Financing
  - Dividend Policy and Capital Structure
  - Corporate Control
  - Investment Policy
  - Agency Theory and Information
- Financial Markets
  - Market Microstructure
  - Portfolio Theory
  - Financial Intermediation
  - Investment Banking
  - Market Efficiency
  - Security Issuance
  - Anomalies and Behavioral Finance
- Asset Pricing
  - Asset-Pricing Theory
  - Asset-Pricing Models
  - Tax Effects
  - Liquidity
  - Equity Risk Premium
  - Pricing Models and Volatility
  - Fixed Income Securities
- Derivatives
  - Computational Finance
  - Futures Markets and Hedging
  - Financial Engineering
  - Interest Rate Derivatives
  - Credit Derivatives
  - Financial Econometrics
  - Estimating Volatilities and Correlations

### Information for Librarians

Foundations and Trends<sup>®</sup> in Finance, 2011, Volume 6, 4 issues. ISSN paper version 1567-2395. ISSN online version 1567-2409. Also available as a combined paper and online subscription.

Foundations and Trends<sup>®</sup> in  
Finance  
Vol. 6, No. 3 (2011) 165–219  
© 2012 J. H. Cochrane  
DOI: 10.1561/05000000037



## Continuous-Time Linear Models

John H. Cochrane

*University of Chicago Booth School of Business and NBER.  
5807 S. Woodlawn, Chicago, IL 60637, USA,  
[john.cochrane@chicagobooth.edu](mailto:john.cochrane@chicagobooth.edu)*

### Abstract

I translate familiar concepts of discrete-time time series to continuous-time equivalent. I cover lag operators, ARMA models, the relation between levels and differences, integration and cointegration, and the Hansen–Sargent prediction formulas.

## Contents

---

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Linear Models and Lag Operators</b>	<b>3</b>
2.1	Discrete Time Operators	3
2.2	A Note on Linear Processes	4
2.3	Continuous-time Operators	5
2.4	Laplace Transforms	8
<b>3</b>	<b>Moving Average Representation and Moments</b>	<b>11</b>
<b>4</b>	<b>ARMA Models</b>	<b>13</b>
4.1	Discrete Time	13
4.2	Continuous Time	14
4.3	How not to Define ARMA Models	15
<b>5</b>	<b>Differences</b>	<b>19</b>
5.1	Levels to Differences in Discrete Time	19
5.2	Levels to Differences in Continuous Time	20
<b>6</b>	<b>Impulse-response Function</b>	<b>23</b>
6.1	Discrete Time	23
6.2	Continuous Time	24

<b>7 Hansen–Sargent Formulas</b>	<b>29</b>
7.1 Discrete Time	29
7.2 Continuous Time	31
7.3 Derivation	33
<b>8 Integration and Cointegration</b>	<b>37</b>
8.1 Difference-stationary Series	37
8.2 Differences to Levels in Discrete Time; Beveridge and Nelson	38
8.3 Differences to Levels in Continuous Time	41
8.4 Cointegration	44
<b>9 Summary</b>	<b>49</b>
<b>Acknowledgments</b>	<b>57</b>
<b>References</b>	<b>59</b>



# 1

---

## Introduction

---

Discrete-time linear ARMA processes and lag operator notation are convenient for lots of calculations. Continuous-time representations often simplify economic models, and can handle interesting nonlinearities as well. But standard treatments of continuous-time processes typically don't mention how to adapt the discrete-time linear model concepts and lag operator methods to continuous time. Here I attempt that translation.

The point of this monograph is to exposit the techniques, understand the intuition, and to make the translation from familiar discrete-time ideas. I do not pretend to offer anything new. I also don't discuss the technicalities. Hansen and Sargent (1991) is a good reference. Heaton (1993) describes many of these methods and provides a useful application. I assume basic knowledge of discrete-time time-series representation methods and continuous-time representations. Cochrane (2005a,b) cover the necessary background, but any standard reference covers the same material.

The concluding section collects the important formulas in one place.

## References

---

- Beveridge, S. and C. R. Nelson (1981), ‘A new approach to decomposition of economic time series into permanent and transitory components with particular attention to measurement of the ‘business cycle’’. *Journal of Monetary Economics* **7**, 151–174. [http://dx.doi.org/10.1016/0304-3932\(81\)90040-4](http://dx.doi.org/10.1016/0304-3932(81)90040-4).
- Cochrane, J. H. (2005a), *Asset Pricing: Revised Edition*. Princeton: Princeton University Press. <http://press.princeton.edu/titles/7836.html>.
- Cochrane, J. H. (2005b), ‘Time series for macroeconomics and finance’. Manuscript, University of Chicago. [http://faculty.chicagobooth.edu/john.cochrane/research/papers/time\\_series\\_book.pdf](http://faculty.chicagobooth.edu/john.cochrane/research/papers/time_series_book.pdf).
- Cochrane, J. H. (2012), ‘A continuous-time asset pricing model with habits and durability’. Manuscript, University of Chicago. [http://faculty.chicagobooth.edu/john.cochrane/research/papers/linquad\\_asset\\_price\\_example.pdf](http://faculty.chicagobooth.edu/john.cochrane/research/papers/linquad_asset_price_example.pdf).
- Engle, R. F. and C. W. J. Granger (1987), ‘Co-integration and error correction: Representation, estimation, and testing’. *Econometrica* **55**, 251–276.

60 *References*

- Hansen, L. P. and T. J. Sargent (1980), 'Formulating and estimating dynamic linear rational-expectations models'. *Journal of Economic Dynamics and Control* **2**, 7–46.
- Hansen, L. P. and T. J. Sargent (1981), 'A note on Wiener–Kolmogorov prediction formulas for rational expectations models'. *Economics Letters* **8**, 255–260.
- Hansen, L. P. and T. J. Sargent (1991), 'Prediction formulas for continuous-time linear rational expectations models'. Chapter 8 of *Rational Expectations Econometrics*. <https://files.nyu.edu/ts43/public/books/TOMchpt.8.pdf>.
- Heaton, J. (1993), 'The interaction between time-nonseparable preferences and time aggregation'. *Econometrica* **61**, 353–385. <http://www.jstor.org/stable/2951555>.