Marketing Dynamics: A Primer on Estimation and Control

Prasad A. Naik
University of California,
Davis, USA
panaik@ucdavis.edu

the essence of knowledge
Boston — Delft

Foundations and Trends® in Marketing

Published, sold and distributed by:
now Publishers Inc.
PO Box 1024
Hanover, MA 02339
United States
Tel. +1-781-985-4510
www.nowpublishers.com
sales@nowpublishers.com

Outside North America:
now Publishers Inc.
PO Box 179
2600 AD Delft
The Netherlands
Tel. +31-6-51115274

The preferred citation for this publication is

P. A. Naik. *Marketing Dynamics: A Primer on Estimation and Control.* Foundations and Trends® in Marketing, vol. 9, no. 3, pp. 175–266, 2014.

This Foundations and Trends® issue was typeset in LaTeX using a class file designed by Neal Parikh. Printed on acid-free paper.

ISBN: 978-1-68083-067-5

Foundations and Trends® in Marketing
Volume 9, Issue 3, 2014

Editorial Board

Editorial Scope

Topics

Foundations and Trends® in Marketing publishes survey and tutorial articles in the following topics:

- B2B marketing
- Bayesian models
- Behavioral decision making
- Branding and brand equity
- Channel management
- Choice modeling
- Comparative market structure
- Competitive marketing strategy
- Conjoint analysis
- Customer equity
- Customer relationship management
- Game theoretic models
- Group choice and negotiation
- Discrete choice models
- Individual decision making
- Marketing decisions models
- Market forecasting
- Marketing information systems
- Market response models
- Market segmentation
- Market share analysis
- Multi-channel marketing
- New product diffusion
- Pricing models
- Product development
- Product innovation
- Sales forecasting
- Sales force management
- Sales promotion
- Services marketing
- Stochastic model

Information for Librarians

Foundations and Trends® in Marketing, 2014, Volume 9, 4 issues. ISSN paper version 1555-0753. ISSN online version 1555-0761. Also available as a combined paper and online subscription.

Foundations and Trends® in Marketing
Vol. 9, No. 3 (2014) 175–266

DOI: 10.1561/1700000031

Marketing Dynamics: A Primer on Estimation and Control

Prasad A. Naik
University of California,
Davis, USA
panaik@ucdavis.edu

Contents

Abstract

This primer provides a gentle introduction to the estimation and control of dynamic marketing models. It introduces dynamic models in discrete- and continuous-time, scalar and multivariate settings, with observed outcomes and unobserved states, as well as random and/or time-varying parameters. It exemplifies how various dynamic models can be cast into the unifying state space framework, the benefit of which is to use one common algorithm to estimate all dynamic models.

The primer then focuses on the estimation part, which answers questions such as: how much is the sales elasticity of advertising? How much sales lift can managers expect for a certain level of price promotion? What is the best sales forecast for the next quarter? The estimation relies on two principles: Kalman filtering and the likelihood principle. The Kalman filter recursively infers the means and covariances of an unobserved state vector as the observed outcomes arrive over time. This evolution of moments is then embedded in the likelihood function to obtain parameter estimates and their statistical significance.

Next, the primer elucidates the control part, which answers questions such as: how much should managers spend on advertising over time and across regions? What is the best promotional timing and depth? How should managers optimally respond to competing brands' actions and resulting outcomes? The control part relies on the maximum principle and the optimality principle. Pontryagin's maximum principle allows managers to determine the optimal course of action (for example, the optimal levels and timing of advertising spends or price promotions) to attain a specified goal, such as profit maximization. Bellman's optimality principle, on the other hand, offers insights into optimal course correction when implementing the best plan as the state of a system varies dynamically and/or stochastically. Finally, the

primer presents three examples on the application of optimal control, differential games, and stochastic control theory to marketing problems, and illustrates how to discover novel insights into managerial decision-making.

P. A. Naik. *Marketing Dynamics: A Primer on Estimation and Control*. Foundations and Trends® in Marketing, vol. 9, no. 3, pp. 175–266, 2014.

DOI: 10.1561/1700000031.

1

Introduction

In 1696, Johann Bernoulli posed a challenge to his contemporary "sharpest mathematical minds of the globe" with the following problem: If in a vertical plane two points A and B are given, then what is the trajectory of an object C starting from A to arrive at B in the shortest possible time falling under its own weight? He added that "this problem ... is not ... purely speculative and without practical use ... Rather it even appears ... that it is very useful also for other branches of science than mechanics." This problem is called *Brachystochrone* problem because, in Greek, $\beta\rho\alpha\chi\iota\sigma\tau\zeta$ = shortest and $\chi\rho o\nu o\zeta$ = time. Leibniz described this problem as "splendid" and furnished the solution (a cycloid) in a letter to Bernoulli, while Newton presented his solution to the Royal Society anonymously.[1] For the quoted text and a definitive account of the intellectual history, see Sussmann and Willems [1997].

[1] Johann Bernoulli ascribes the anonymous solution to Newton because he noted that "you can tell the lion by its claws" (ex ungue leonem). The solution is the cycloid curve for the point C whose coordinates $(x(t),\, y(t))$ evolve, starting from the point A at (0,0) at $t = 0$, according to $x(\theta) = \alpha(\theta - \sin(\theta))$ and $y(\theta) = \alpha(\cos(\theta) - 1)$, where $\theta(t) = t\sqrt{g/\alpha}, g = 9.8\,\mathrm{m/sec}^2$, and the parameters $(\alpha,\, T)$ are determined by the terminal condition $(x(\theta(T)), y(\theta(T))) = B$.

This event gave birth to the mathematical branch, later known as the calculus of variation, due to Leonhard Euler, who was Bernoulli's student and who discovered what is now called the Euler's equation to solve such dynamic optimization problems. Intense mathematical research over the subsequent three hundred years culminated into the modern control theory. Consistent with Bernoulli's prognosis, this theory found use in launching man on Moon, landing Curiosity on Mars, deploying unmanned drones or designing driverless cars, high precision manufacturing using robots, providing navigation guidance turn by turn to users on roads, seas, and air, besides numerous applications in "other branches of science" including Operations Research (for example, Berstekas [2005]), Economics (for example, Stokey and Lucas [1989], Aghion and Howitt [1998], Ljungqvist and Sargent [2004], Weber [2011], Kamien and Schwartz [2012]), Management Science (for example, Sethi and Thompson [2000], Dockner et al. [2000]), and Marketing (for example, Erickson [2003], Jørgensen and Zaccour [2004]).

In Marketing, the point A represents the current state of the company's brand sales or consumer's utility. The point B marks the desired state the decision-maker wants to arrive at. The object C is the decision-maker (for example, CEO, brand managers, consumers), and its use of own weight denotes the set of actions (for example, price, advertising, brand choices) available for transitioning the system from state A to state B. The shortest time specifies the decision-maker's objectives (for example, maximize the stream of future profit or utility). In Section 2, I clarify the terms state, system, transition, actions (or controls), and objectives, but note here that this simple abstraction is "splendid" because it not only unifies diverse problems across many applications, but also offers a systematic approach for solving them.

The purpose of this primer is to impart this systematic approach for solving dynamic marketing problems. To pursue this pedagogical focus, this article does not aim to review dynamic models in the extant marketing literature, for which readers are referred to Bowman and Gatignon [2010], Neslin and van Heerde [2009], Shankar [2008], Hanssens et al. [2003], and Leeflang et al. [2000].

Solving dynamic problems involves two distinct topics: parameter estimation and optimal control. The former refers to *describing* *the system* of relations among current states, past states, and actions via econometric time-series models (for example, Pauwels [2004], Steenkamp et al. [2005]). The latter refers to *managing the system* by determining the optimal course of actions to execute over the future planning horizon (for example, Kumar et al. [2008], Esteban-Bravo et al. [2014]). The foundation for parameter estimation stands on the Kalman Filter and the Likelihood Principle, whereas that for optimal control stands on the Pontryagin's Maximum Principle and Bellman's Principle of Optimality.

This monograph elucidates these four principles and related concepts, focusing on how to estimate dynamic models in Sections 3 and 4, and how to solve the control problems in Sections 5 and 6. But first, I clarify the terms, present the diversity of dynamic models, unify them via the canonical state space model, and highlight the value of unification.

References

P. Aghion and P. Howitt. *Endogenous Growth Theory*. MIT Press, Cambridge, MA, 1998.

B. D. O. Anderson and J. Moore. *Optimal Filtering*. Dover Publications, Mineola, NY, 2012.

C. Andrieu and A. Doucet. Particle filtering for partially observed gaussian state space models. *Journal of the Royal Statistical Society, Series B*, 64 (4):827–836, 2002.

I. Arasaratnam and S. Haykin. Squareroot quadrature kalman filtering. *IEEE Transactions on Signal Processing*, 56(6):2589–2593, 2008.

A. Aravindakshan and P. A. Naik. How does awareness evolve when advertising stops? the role of memory. *Marketing Letters*, 22(3):315–326, 2011.

A. Aravindakshan and P. A. Naik. Understanding the memory effects in pulsing advertising. *Operations Research*, 63(1):35–47, 2015.

A. Aravindakshan, K. Peters, and P. A. Naik. Spatiotemporal allocation of advertising budgets. *Journal of Marketing Research*, 49(1):1–14, 2012.

A. Aravindakshan, O. Rubel, and O. Rutz. Managing blood donations with marketing. *Marketing Science*, 34(2):269–280, 2015.

K. J. Astrom. *Introduction to Stochastic Control Theory*. Dover Publications, Mineola, NY, 2006.

M. B. Ataman, C. F. Mela, and H. J. van Heerde. Building brands. *Marketing Science*, 27(6):1036–1054, 2008.

M. B. Ataman, H. J. van Heerde, and C. F. Mela. The long-term effect of marketing strategy on brand sales. *Journal of Marketing Research*, 47(5): 866–882, 2010.

T. Başar and G. J. Olsder. *Dynamic Noncooperative Game Theory*. Society for Industrial and Applied Mathematics, Philadelphia, 2nd edition, 1999.

T. E. Barry and D. J. Howard. A review and critique of the hierarchy of effects in advertising. *International Journal of Advertising*, 9(2):121–135, 1990.

F. M. Bass. A new product growth model for consumer durables. *Management Science*, 15(5):215–227, 1969.

F. M. Bass, T. V. Krishnan, and D. C. Jain. Why the bass model fits without decision variables. *Marketing Science*, 13(3):203–223, 1994.

F. M. Bass, A. Krishnamoorthy, A. Prasad, and S. P. Sethi. Advertising competition with market expansion for finite horizon firms. *Journal of Industrial and Management Optimization*, 1(1):1–19, 2013.

G. Belch and M. Belch. *Advertising & Promotion: An Integrated Marketing Communications Perspective*. McGraw-Hill Companies, 6th edition, 2004.

D. P. Berstekas. *Dynamic Programming and Optimal Control,* vol. 1 and 2. Athena Scientific, Belmont, MA, 2005.

E. Biyalogorsky and P. A. Naik. Clicks and mortar: The effect of online activities and offline sales. *Marketing Letters*, 14(1):21–32, 2003.

D. Bowman and H. Gatignon. Market response and marketing mix models: Trends and research opportunities. *Now Publishers, Boston, USA*, 4(3): 129–207, 2010.

B. J. Bronnenberg. Advertising frequency decisions in a discrete markov process under a budget constraint. *Journal of Marketing Research*, 35(3): 399–406, 1998.

N. Bruce, N. Z. Foutz, and C. Kolsarici. Dynamic effectiveness of advertising and word of mouth in sequential distribution of new products. *Journal of Marketing Research*, 49(4):469–486, 2012a.

N. I. Bruce. Pooling and dynamic forgetting effects in multi-theme advertising: Tracking the advertising sales relationship with particle filters. *Marketing Science*, 27(4):659–673, 2008.

N. I. Bruce, K. Peters, and P. A. Naik. Discovering how advertising grows sales and builds brands. *Journal of Marketing Research*, 49(6):793–806, 2012b.

A. E. Bryson and Y. C. Ho. *Applied Optimal Control: Optimization, Estimation and Control.* Taylor and Francis, Boston, MA, 1975.

K. P. Burnham and D. R. Anderson. *Model Selection and Multimodel Inference: A Practical Information-theoretic Approach.* Springer Science & Business Media, New York, NY, 2002.

P. K. Chintagunta and D. C. Jain. Empirical analysis of a dynamic duopoly model of competition. *Journal of Economics & Management Strategy*, 4 (1):109–131, 1995.

P. K. Chintagunta and V. R. Rao. Pricing strategies in a dynamic duopoly: A differential game model. *Management Science*, 42(11):1501–1514, 1996.

P. K. Chintagunta and N. Vilcassim. An empirical investigation of advertising strategies in a dynamic duopoly. *Management Science*, 38(9):1230–1244, 1992.

P. K. Chintagunta, V. R. Rao, and N. J. Vilcassim. Equilibrium pricing and advertising strategies for nondurable experience products in a dynamic duopoly. *Managerial and Decision Economics*, 14(3):221–234, 1993.

M. G. Dekimpe, P. H. Franses, D. M. Hanssens, and P. A. Naik. Time-series models in marketing. In B. Wierenga, editor, *Handbook of Marketing Decision Models*, pages 373–398, Berlin, 2008. Springer Verlag.

A. K. Dixit and R. S. Pindyck. *Investement Under Uncertainty.* Princeton University Press, Princeton, NJ, 1994.

E. J. Dockner, S. Jørgensen, N. V. Long, and G. Sorger. *Differential Games in Economics and Management Science.* Cambridge University Press, Cambridge, UK, 2000.

R. Dorfman and P. O. Steiner. Optimal advertising and optimal quality. *The American Economic Review*, 44(5):826–836, 1954.

R. Du and W. Kamakura. Improving the statistical performance of tracking studies based on repeated cross-sections with primary dynamic factor analysis. *International Journal of Research in Marketing*, 32(1):94–112, 2015.

J. P. Dubé, G. J. Hitsch, and P. Manchanda. An empirical model of advertising dynamics. *Quantitative Marketing and Economics*, 3(2):107–144, 2005.

J. Durbin and S. J. Koopman. *Time Series Analysis by State Space Methods.* Oxford University Press, Oxford, UK, 2012.

R. Elliott, L. Aggoun, and J. Moore. *Hidden Markov Models: Estimation and Control.* Springer, New York, N.Y., 1995.

G. M. Erickson. *Dynamic Models of Advertising Competition.* Kluwer, Norwell, MA, 2nd edition, 2003.

M. Esteban-Bravo, J. M. Vidal-Sanz, and G. Yildirim. Valuing customer portfolios with endogenous mass and direct marketing interventions using a stochastic dynamic programming decomposition. *Marketing Science*, 33 (5):621–640, 2014.

F. M. Feinberg. Pulsing policies for aggregate advertising models. *Marketing Science*, 11(3):221–234, 1992.

F. M. Feinberg. On continuous-time optimal advertising under s-shaped response. *Management Science*, 47(12):1476–1487, 2001.

R. Feynman, R. Leighton, and M. Sands. *The Feynman Lectures on Physics,* Vol. 2. Chapter 25-2, http://www.feynmanlectures.caltech.edu/ (HTML edition), 2006.

M. Freimer and D. Horsky. Periodic advertising pulsing in a competitive market. *Marketing Science*, 31(4):637–648, 2012.

G. E. Fruchter. The many-player advertising game. *Management Science*, 45: 1609–1611, 1999.

G. E. Fruchter and S. Kalish. Closed-loop advertising strategies in a duopoly. *Management Science*, 43(1):54–63, 1997.

G. E. Fruchter and C. Van den Bulte. Why the generalized bass model leads to odd optimal advertising policies. *International Journal of Research in Marketing*, 28(3):218–230, 2011.

P. Gill, W. Murray, and M. H. Wright. *Practical Optimization.* Academic Press, London, UK, 1982.

M. Grigoriu. *Stochastic Calculus: Applications in Science and Engineering.* Birkhauser, Boston, MA, 2002.

D. M. Hanssens, L. J. Parsons, and R. L. Schultz. Market response models: Econometric and time series analysis. In *International Series in Quantitative Marketing*, Massachusetts, USA: Kluwer, 2003. Academic Publishers.

J. Harrison and M. West. *Bayesian Forecasting and Dynamic Models.* Springer, New York, NY, 2013.

R. F. Hartl. A simple proof of the monotonicity of the state trajectories in autonomous control problems. *Journal of Economic Theory*, 41:211–215, 1987.

A. C. Harvey. *Forecasting, Structural Time Series Models and the Kalman Filter.* Cambridge University Press, Cambridge, UK, 2001.

S. Hasegawa, N. Terui, and G. M. Allenby. Dynamic brand satiation. *Journal of Marketing Research*, 49(6):842–853, 2012.

Y. Hu, R. Y. Du, and S. Damangir. Decomposing the impact of advertising: Augmenting sales with online search data. *Journal of Marketing Research*, 51(3):300–319, 2014.

P. J. Huber. The behavior of maximum likelihood estimates under nonstandard conditions. *Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability*, 1(1):221–233, 1967.

S. D. Jap and P. Naik. Bid analyzer: A method for estimation and selection of dynamic bidding models. *Marketing Science*, 27(6):949–960, 2008.

A. H. Jazwinski. *Stochastic Processes and Filtering Theory.* Dover Publications, Mineola, N.Y., USA, 2007.

J. P. Jones. *What's in a Name: Advertising and the Concept of Brand.* D. C. Heath and Company, Lexington, MA, 1986.

J. P. Jones. Ad spending: Maintaining market share. *Harvard Business Review*, 68(1):38–42, 1990.

S. Jørgensen and G. Zaccour. Differential games in marketing. In *International Series in Quantitative Marketing.* Kluwer Academic Publishers, USA, 2004.

S. J. Julier and J. K. Uhlmann. Unscented filtering and nonlinear estimation. *Proceedings of the IEEE*, 92(3):401–422, 2004.

K. Kalaignanam, T. Kushwaha, and M. Eilert. The impact of product recalls on future product reliability and future accidents: Evidence from the automobile industry. *Journal of Marketing*, 77(2):41–57, 2013.

S. Kalish. Monopolist pricing with dynamic demand and production cost. *Marketing Science*, 2(2):135–159, 1983.

R. E. Kalman and R. S. Bucy. New results in linear filtering and prediction theory. *Journal of Basic Engineering*, 83(1):95–108, 1961.

M. I. Kamien and N. L. Schwartz. *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management.* Dover Publications, USA, 2nd edition, 2012.

C. Kolsarici and D. Vakratsas. Category-versus brand-level advertising messages in a highly regulated environment. *Journal of Marketing Research*, 47(6):1078–1089, 2010.

G. Koop and D. Korobilis. Bayesian multivariate time-series methods for empirical macroeconomics. *Foundations and Trends in Econometrics*, 3(4): 267–358, 2009. Now Publishers, Boston, USA.

V. Kumar, R. Venkatesan, T. Bohling, and D. Beckmann. The power of CLV: Managing customer lifetime value at IBM. *Marketing Science*, 27(4): 585–599, 2008.

H. J. Kushner. On the differential equations satisfied by conditional probability densities of markov processes with applications. *Journal of SIAM Control, Series A*, 2(1):106–119, 1964.

M. Lachaab, A. Ansari, K. Jedidi, and A. Trabelsi. Modeling preference evolution in discrete choice models: A bayesian state-space approach. *Quantitative Marketing and Economics*, 4(1):57–81, 2006.

L. Lapidus and G. F. Pinder. *Numerical Solution of Partial Differential Equations in Science and Engineering.* John Wiley and Sons, New York, NY, 1999.

R. J. Lavidge and G. A. Steiner. A model for predictive measurements of advertising effectiveness. *Journal of Marketing*, 25(6):59–62, 1961.

P. S. H. Leeflang, D. R. Wittink, M. Wedel, and P. A. Naert. Building models for marketing decisions. In *International Series in Quantitative Marketing vol. 9.* Springer, USA, 2000.

Y. Liu and V. Shankar. *The Dynamic Effects of Product Harm Crises on Brand Preference and Advertising Effectiveness: An Empirical Analysis of the Automobile Industry.* forthcoming, 2015.

L. Ljungqvist and T. J. Sargent. *Recursive Macroeconomic Theory.* MIT Press, Cambridge. MA, 2nd edition, 2004.

V. Mahajan and E. Muller. Advertising pulsing policies for generating awareness for new products. *Marketing Science*, 5(2):89–111, 1986.

V. Mahajan, E. Muller, and S. Sharma. An empirical comparison of awareness forecasting models of new product introduction. *Marketing Science*, 3(3): 179–197, 1984.

A. G. Malliaris and W. A. Brock. *Stochastic Methods in Economics and Finance.* North Holland, 1982.

A. McQuarie and C.-L. Tsai. *Regression and Time Series Model Selection.* World Scientific, Singapore, 1998.

H. I. Mesak. An aggregate advertising pulsing model with wearout effects. *Marketing Science*, 11(3):310–326, 1992.

R. Montoya, O. Netzer, and K. Jedidi. Dynamic allocation of pharmaceutical detailing and sampling for long-term profitability. *Marketing Science*, 29 (5):909–924, 2010.

P. A. Naik and K. Raman. Understanding the impact of media synergy in multimedia communications. *Journal of Marketing Research*, 40(4):375–388, 2003.

P. A. Naik, M. K. Mantrala, and A. Sawyer. Planning pulsing media schedules in the presence of dynamic advertising quality. *Marketing Science*, 17(3): 214–235, 1998.

P. A. Naik, K. Raman, and R. Winer. Planning marketing-mix strategies in the presence of interactions. *Marketing Science*, 24(1):25–34, 2005.

P. A. Naik, P. Shi, and C.-L. Tsai. Extending the akaike information criterion to mixture regression models. *Journal of the American Statistical Association*, 102(477):244–254, 2007.

P. A. Naik, A. Prasad, and S. P. Sethi. Building brand awareness in dynamic oligopoly markets. *Management Science*, 54(1):129–138, 2008.

M. Nerlove and K. Arrow. Optimal advertising policy under dynamic conditions. *Economica*, 29(114):129–142, 1962.

S. A. Neslin and H. J. van Heerde. *Promotion Dynamics.* Now Publishers, Boston, USA, 2009.

O. Netzer, J. M. Lattin, and V. Srinivasan. A hidden markov model of customer relationship dynamics. *Marketing Science*, 27(2):185–204, 2008.

J. Nocedal and S. J. Wright. *Numerical Optimization.* New York, NY: Springer-Verlag, 2006.

E. C. Osinga, P. S. H. Leeflang, and J. E. Wieringa. Early marketing matters: A time-varying parameter approach to persistence modeling. *Journal of Marketing Research*, 47(1):173–185, 2010.

S. Park and M. Hahn. Pulsing in a discrete model of advertising competition. *Journal of Marketing Research*, 28(4):397–405, 1991.

K. Pauwels. How dynamic consumer response, competitor response, company support, and company inertia shape long-term marketing effectiveness. *Marketing Science*, 23(4):596–610, 2004.

A. Prasad and S. P. Sethi. Integrated marketing communications in markets with uncertainty and competition. *Automatica*, 45(3):601–610, 2009.

K. Raman. Boundary value problems in stochastic optimal control of advertising. *Automatica*, 42(8):1357–1362, 2006.

K. Raman and R. Chatterjee. Optimal monopolist pricing under demand uncertainty in dynamic markets. *Management Science*, 41(1):144–162, 1995.

K. Raman and P. A. Naik. Long-term profit impact of integrated marketing communications program. *Review of Marketing Science*, 2(1), 2004. Article 8, DOI: 10.2202/1546-5616.1014.

O. J. Rubel, P. A. Naik, and S. Srinivasan. Optimal advertising when envisioning a product-harm crisis. *Marketing Science*, 30(6):1048–1065, 2011.

O. J. Rutz and R. Bucklin. From generic to branded: A model of spillover dynamics in paid search advertising. *Journal of Marketing Research*, 48(1): 87–102, 2011.

O. J. Rutz and G. P. Sonnier. Modeling the evolution of internal market structure. *Marketing Science*, 30(2):274–289, 2011.

M. Sasieni. Optimal advertising expenditure. *Management Science*, 18:64–72, December 1971.

A. Seierstad and K. Sydsaeter. Sufficient conditions in optimal control theory. *International Economic Review*, 18(2):367–391, 1977.

S. P. Sethi. Optimal advertising for the nerlove-arrow model under a budget constraint. *Operations Research Quarterly*, 28(3):683–693, 1977.

S. P. Sethi. Deterministic and stochastic optimization of a dynamic advertising model. *Optimal Control Applications and Methods*, 4(2):179–184, 1983.

S. P. Sethi and G. L. Thompson. A tutorial on optimal control theory. *INFOR*, 19(4):279–291, 1981.

S. P. Sethi and G. L. Thompson. *Optimal Control Theory: Applications to Management Science and Economics.* Kluwer Academic Publishers, Boston, 2nd edition, 2000.

V. Shankar. Strategic allocation of marketing resources: Methods and insights. In Roger Kerin and Rob O'Regan, editors, *Marketing Mix Resource Allocation and Planning: New Perspectives and Practices*, pages 154–183. American Marketing Association, 2008.

R. H. Shumway and D. S. Stoffer. *Time Series Analysis and Its Applications: With R Examples.* Springer, New York, NY, 2011.

D. Simon. *Optimal State Estimation.* John-Wiley and Sons, Hoboken, NJ, USA, 2006.

A. Smith, P. A. Naik, and C.-L. Tsai. Markov-switching model selection using kullback–leibler divergence. *Journal of Econometrics*, 134(2):553–577, 2006.

G. D. Smith. *Numerical Solution of Partial Differential Equations: Finite Difference Methods.* Oxford University Press, Oxford, UK, 2003.

G. Sorger. Competitive dynamic advertising: A modification of the case game. *Journal of Economics Dynamics and Control*, 13:55–80, 1989.

S. Sridhar, M. K. Mantrala, P. A. Naik, and E. Thorson. Dynamic marketing budgeting for platform firms: Theory, evidence, and application. *Journal of Marketing Research*, 48(6):929–943, 2011.

S. Sriram and M. U. Kalwani. Optimal advertising and promotion budgets in dynamic markets with brand equity as a mediating variable. *Management Science*, 53(1):46–60, 2007.

S. Sriram, S. Balachander, and M. U. Kalwani. Monitoring the dynamics of brand equity using store-level data. *Journal of Marketing*, 71(2):61–78, 2007.

J.-B. E. M. Steenkamp, V. R. Nijs, D. M. Hanssens, and M. G. Dekimpe. Competitive reactions to advertising and promotion attacks. *Marketing Science*, 24(1):35–54, 2005.

N. L. Stokey and R. E. Lucas. *Recursive Methods in Economic Dynamics.* Harvard University Press, Cambridge, MA, 1989.

H. J. Sussmann and J. C. Willems. 300 years of optimal control: From the brachystochrone to the maximum principle. *IEEE Control Systems Magazine*, pages 33–44, June 1997.

T. S. Teixeira, M. Wedel, and R. Pieters. Moment-to-moment optimal branding in TV commercials: Preventing avoidance by pulsing. *Marketing Science*, 29(5):783–804, 2010.

G. J. Tellis. *Advertising and Sales.* Reading, MA: Addison-Wesley, 1998.

D. Vakratsas and T. Ambler. How advertising works: What do we really know? *Journal of Marketing*, 63(1):26–43, 1999.

D. Vakratsas and C. Kolsarici. A dual market diffusion model for a new prescription pharmaceutical. *International Journal of Research in Marketing*, 25(4):282–293, 2008.

H. van Heerde, K. Helsen, and M. G. Dekimpe. The impact of product-harm crisis on marketing effectiveness. *Marketing Science*, 26(2):230–245, 2007.

R. L. Vaughn. How advertising works: A planning model. *Journal of Advertising Research*, 20(5):27–33, 1980.

R. L. Vaughn. How advertising works: A planning model revisited. *Journal of Advertising Research*, 26(1):57–66, 1986.

M. L. Vidale and H. B. Wolfe. An operations research study of sales response to advertising. *Operations Research*, 5:370–381, 1957.

T. A. Weber. *Optimal Control Theory with Applications in Economics*. MIT Press, Cambridge, MA, 2011.

H. White. A heteroskedasticity-consistent covariance matrix estimator and a direct test for heteroskedasticity. *Econometrica*, 48(4):817–838, 1980.

M. Zakai. On the optimal filtering of diffusion processes. *Probability Theory and Related Fields*, 11(3):230–243, 1969.

W. Zucchini and I. L. MacDonald. *Hidden Markov Models for Time Series: An Introduction Using R*. Boca Raton, FL: Chapman & Hall, 2009.