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On the Sensitivity of the Critical Transmission Range: Lessons from the Lonely Dimension

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Abstract

We consider geometric random graphs where n points are distributed independently on the unit interval $[0,1]$ according to some probability distribution function F with density function f. Two nodes communicate with each other if their distance is less than some transmission range. For this class of random graphs, we survey results concerning the existence of zero-one laws for graph connectivity, the type of the zero-one law obtained under specific assumptions on the density function f, the form of its critical scaling and its dependence on f, and the width of the corresponding phase transitions. This is motivated by the desire to understand how node distribution affects the critical transmission range as specified by the disk model. Engineering implications are discussed for power allocation.

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1

Introduction

1.1 Modeling Wireless Communication Constraints — The Disk Model

By now the disk model has become a commonly used framework for modeling communication constraints in wireless networks: The setting is that of n users (interchangeably referred to as nodes) which are distributed over some region Γ of the plane $\mathbb{R}^2$.[1] The nodes, labelled $1, 2, \ldots, n$, are placed at the random locations $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$, respectively, in Γ. This reflects a common situation where node locations are not available, especially when mobility is involved. A simplified pathloss model is assumed, and there is no user interference and no fading. Users all transmit at the same power level P, and do not exercise power control. For distinct users i and j located at $\boldsymbol{X}_i$ and $\boldsymbol{X}_j$, their received power $P_{i,j}$ is given by

$$P_{i,j} := P \cdot \|\boldsymbol{X}_i - \boldsymbol{X}_j\|^{-\nu}$$

[1] The model can be defined more generally on $\mathbb{R}^d$ with $d \geq 1$; see the monograph [48]. The literature on wireless networking focuses on the case $d = 2$, but many results are proved for arbitrary $d \geq 2$.

for some pathloss exponent $\nu > 0$.[2] Under these assumptions, the disk model posits that nodes i and j are able to communicate with each other if $P_{i,j} \geq \Gamma$ for some threshold $\gamma > 0$ (whose selection is guided by bit error rate considerations, among others). This condition is equivalent to requiring

$$\|\boldsymbol{X}_i - \boldsymbol{X}_j\| \leq \rho \quad \text{with} \quad \rho := \left(\frac{P}{\gamma}\right)^{1/\nu}, \tag{1.1}$$

and points to the *transmission range* ρ as a convenient proxy for the common transmit power P used by this homogeneous population of users.

1.2 Critical Power Levels for Network Connectivity

Given a transmission range $\rho > 0$, we can view the relation (1.1) as defining a notion of adjacency amongst nodes whereby an edge exists between nodes i and j if (1.1) holds. Let $\mathbb{G}(n;\rho)$ denote the resulting undirected *geometric random graph* on the set of nodes $1,\ldots,n$.

In this model, the presence of an edge between two nodes captures their ability to communicate directly and reliably with each other. However, viewed as systems, networks are "greater than the sum of their parts," and "network connectivity" emerges from one-hop connectivity as network resources collectively enable end-to-end data transfer between all participating nodes. It is customary to identify this desired network connectivity with the usual notion of *graph connectivity* in $\mathbb{G}(n;\rho)$ according to which every pair of nodes is linked by at least one path over the edges of the graph.

A natural question consists in determining the *minimum* power level needed to ensure (network) connectivity amongst the nodes located at $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n$. This quantity is given by the *critical power level* P_n defined by

$$P_n := \min\left(P > 0: \ \mathbb{G}(n;\rho) \text{ is connected}\right) \tag{1.2}$$

[2] Here $\|\boldsymbol{x}\|$ denotes the Euclidean norm of the vector $\boldsymbol{x}$ in $\mathbb{R}^2$. Other choices for the norm have been considered in problems of computational geometry and in the general context of geometric random graphs; see, for example, the papers [2, 12], the monograph [48] and references therein.

with parameters P and ρ related as in the second half of (1.1). Expressed in terms of the transmission range, this amounts to considering the *critical transmission range* R_n defined by

$$R_n := \min\left(\rho > 0 : \ \mathbb{G}(n;\rho) \text{ is connected}\right). \tag{1.3}$$

This quantity is also known as the *connectivity distance* [2]. The critical power level P_n and the critical transmission range R_n are simply related by

$$P_n = \gamma R_n^\nu. \tag{1.4}$$

Knowledge of P_n, or equivalently R_n, has obvious engineering implications since any information concerning them should be of help in dimensioning systems resources which are often scarce. This was the very issue considered by Gupta and Kumar in [23], a paper which revived interest in the disk model as a framework for studying wireless ad-hoc networks.

The quantity R_n being a function of the random locations $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$, it is of *limited* operational use since node locations are neither available, nor should their knowledge be expected, especially in the presence of node mobility. Moreover, its probability distribution function

$$P(n;\rho) = \mathbb{P}\left[R_n \le \rho\right], \quad \rho \ge 0$$

is usually not known in closed form. To the best of our knowledge, the only possible exception is to be found in the one-dimensional case under independent and identically distributed (i.i.d.) uniform node placement; see the discussion in Section 3.1. Even there, the available expression yields no insights on the distributional behavior of R_n.

1.3 The Case of Many, Many Users

Fortunately a case can be made that efficient power allocation matters only when dealing with a *very large* number of users. After all this is a regime where the problem assumes added relevance (as well as some urgency) since energy resources are always painfully finite. In that asymptotic regime it is hoped that limiting results would be available, leading to a reasonably good approximation to R_n by a *non*-random and easily *computable* quantity $\rho_n^\star$, say

$$R_n \simeq \rho_n^\star \quad \text{with very high probability.} \tag{1.5}$$

A possible formalization of this idea is provided by the convergence (in probability)[3]

$$\frac{R_n}{\rho_n^\star} \xrightarrow{P}_n 1. \tag{1.6}$$

Such a result immediately suggests a similar approximation to the critical power level P_n by means of the *non*-random quantity $\pi_n^\star$ given by

$$\pi_n^\star = \gamma \left(\rho_n^\star\right)^\nu .$$

Since the critical transmission range and the critical power level are quantities recoverable from each other, from now on we shall focus exclusively on the former.

As we shall see shortly, developments such as (1.5)–(1.6) are indeed possible under appropriate assumptions. The relevant results have been obtained from several complementary viewpoints which can be reconciled upon noting that $\mathbb{G}(n;\rho)$ is connected if and only if $R_n \leq \rho$, so that

$$\mathbb{P}\left[\mathbb{G}(n;\rho) \text{ is connected}\right] = P(n;\rho), \quad \rho > 0. \tag{1.7}$$

The validity of (1.5)–(1.6) is then seen to be equivalent[4] to the *zero-one* law

$$\lim_{n\to\infty} P(n;\rho_n) = 0 \quad \text{if } \rho_n \text{ “(much) smaller than” } \rho_n^\star$$

and

$$\lim_{n\to\infty} P(n;\rho_n) = 1 \quad \text{if } \rho_n \text{ “(much) larger than” } \rho_n^\star.$$

The approximation $\rho_n^\star$ to the critical transmission range R_n acts as a *boundary* in the space of scalings, and is often referred to as a *critical* scaling.

[3] See later in the chapter for the notation and conventions used.
[4] See Proposition 2.1.

1.4 Zero-one Laws

In the many node regime there are settings where good approximations to R_n can indeed be derived in terms of a *non*-random quantity $\rho_n^\star$ which is explicitly computable. For instance, consider the standard case when the random locations $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ are mutually independent and uniformly distributed over a closed bounded region Γ of the plane. This is the setting most commonly used when discussing the disk model. If the transmission range is scaled with the number of users according to

$$\pi\rho_n^2 = \frac{\log n + \alpha_n}{n}, \quad n = 1, 2, \ldots \tag{1.8}$$

for some sequence $\alpha : \mathbb{N}_0 \to \mathbb{R}$, then the zero-one law

$$\lim_{n\to\infty} P(n;\rho_n) = \begin{cases} 0 & \text{if } \lim_{n\to\infty} \alpha_n = -\infty \\ 1 & \text{if } \lim_{n\to\infty} \alpha_n = \infty \end{cases} \tag{1.9}$$

is known to hold. This result was obtained independently by Gupta and Kumar [23], and by Penrose [48] (and references therein).

When the scaling $\rho : \mathbb{N}_0 \to \mathbb{R}_+$ is selected so that

$$\pi\rho_n^2 \sim c\,\frac{\log n}{n} \tag{1.10}$$

for some $c > 0$, the zero-one law (1.8)–(1.9) (applied with $\alpha_n \sim (c-1)\log n$) readily implies

$$\lim_{n\to\infty} P(n;\rho_n) = \begin{cases} 0 & \text{if } 0 < c < 1 \\ 1 & \text{if } 1 < c. \end{cases} \tag{1.11}$$

Both zero-one laws (1.8)–(1.9) and (1.10)–(1.11) suggest a central role for the scaling $\rho^\star : \mathbb{N}_0 \to \mathbb{R}_+$ determined by

$$\pi\rho_n^{\star 2} = \frac{\log n}{n}, \quad n = 1, 2, \ldots \tag{1.12}$$

This scaling is indeed the critical scaling in this case, and with this choice, the zero-one law (1.10)–(1.11) is equivalent to (1.6).

1.5 Sensitivity to Statistical Assumptions

Given these results, a natural question arises as to their *dependence* on, and therefore *sensitivity* to, the statistical assumptions enforced on the node locations. For instance, if one accepts that nodes are indeed placed in an i.i.d. manner across Γ,[5] there is however no good reason to believe that they should be placed uniformly over this region. A typical example where this assumption will be challenged occurs when nodes are mobile, say according to the random waypoint mobility model [17, 52, 53]. Under these circumstances, do zero-one laws still hold and if so, in what form and under what assumptions?

In [47] Penrose partially addressed this issue; see also [46]: There the locations $\boldsymbol{X}_1,\ldots,\boldsymbol{X}_n$ were assumed i.i.d. rvs distributed over the domain Γ in $\mathbb{R}^2$ according to some probability distribution F with density function f. Under mild assumptions of continuity on f and smoothness on Γ, Penrose showed [47, Thm. 1.1, p. 247][6] that (1.11) still holds if the scaling $\rho : \mathbb{N}_0 \to \mathbb{R}_+$ satisfies

$$\pi\rho_n^2 \sim c\,\frac{1}{M(F)} \cdot \frac{\log n}{n} \tag{1.13}$$

for some $c > 0$, with the constant $M(F)$ determined by the minima of f on Γ and on its boundary. The critical scaling $\rho^\star : \mathbb{N}_0 \to \mathbb{R}_+$ is now determined through

$$\pi\rho_n^{\star 2} = \frac{1}{M(F)} \cdot \frac{\log n}{n}, \quad n = 1,2,\ldots \tag{1.14}$$

and provides a non-random approximation to the critical transmission range.

There remains open the question as to what is the analog of the zero-one law (1.8)–(1.9) under *non*-uniform node placement distributions, or what happens when $M(F) = 0$ (since (1.14) is now meaningless). Interest in these questions stems from the fact that such zero-one laws express (extreme) sensitivity to deviations from the critical scaling, and suggest the presence of (sharp) *phase transitions* with potential implications for power allocation. Yet, to the best of our knowledge, no results

[5] The independence assumption between users fails to hold in a number of practical scenarios, for example, in the presence of group mobility. Recently, La and Seo [39] have dispensed with the i.i.d. assumption in simple one-dimensional situations with group mobility. For these networks, a more intricate picture emerges and points to the subtle impact that correlations between node locations can have on the nature of the results. The heterogeneous case, still under the independence assumption, is considered by La in [38].

[6] Penrose already considered the d-dimensional case with $d \geq 2$.

have been reported on analogs of (1.8)–(1.9) in the non-uniform setting in dimension two and higher. If the analysis in [47] provides already any indication, establishing such analogs will be technically involved, possibly requiring additional assumptions on the density function f.[7] Furthermore, the case $M(F) = 0$ has not received any attention in the higher-dimensional setting.

1.6 Enter One-dimensional Networks

In this volume we turn to the *one*-dimensional setting where n points are distributed independently on the (generic) unit interval $[0,1]$ according to some probability distribution function F with probability density function f. Thus, we are interested in understanding how the underlying distribution F affects connectivity in the induced geometric random graphs. In particular, under various assumptions on the density function f, we discuss (i) the *existence* of zero-one laws for graph connectivity, (ii) the *type* of the zero-one law obtained under the specific assumptions made, and (iii) the *form* of the corresponding critical scaling (when available). Ultimately, such results should help generate approximations to the critical transmission range by means of an appropriate critical scaling.

A basic reason for considering one-dimensional networks lies in the fact that geometry in one dimension is much simpler than in higher dimensions, holding up the promise that many of the alluded technical difficulties will not be present. One-dimensional models are arguably the least geometric in nature. They indeed occupy a somewhat singular place in the literature on geometric random graphs [48, p. 283] as reflected by the continuing attention given to one-dimensional random graphs in several research communities with various (non-geometric) perspectives.

Already in the uniform case, several complementary approaches are available: The monograph by Godehardt [19] deals with applications to cluster analysis, and the exhaustive study in [20] provides a direct combinatorial analysis of many results of interest. Appel and Russo

[7] As discussed in Chapter 8, the appropriate version of (1.8)–(1.9) in one dimension does require additional structural assumptions on the density function f.

[2, p. 352] leverage the connection with maximal spacings, Han and Makowski [30] derive zero-one laws by applying the method of first and second moments to the number of breakpoint users, while Muthukrishnan and Pandurangan [44] make use of bin-covering techniques.

As a result of these and related efforts, many questions concerning graph connectivity have by now been given answers in various forms of completeness, in both the uniform and non-uniform cases. Results have sometimes been obtained independently by several authors, are scattered in multiple literatures and are not always couched in graph-theoretic terms. With this in mind we provide here a unified presentation of these results, both old and new, in their sharpest form known to us. Before providing highlights of the discussion in Section 1.7, we close with additional reasons for considering the one-dimensional case:

A complete set of results A fairly complete picture of zero-one laws is now available in the one-dimensional setting, even under non-uniform node placement. For the most part, this can be traced back to the fact that connectivity in such random graphs can be expressed in terms of the *maximal spacings* associated with the i.i.d. node locations. Much is known about the asymptotic properties of these quantities, eliminating many of the technical difficulties associated with higher-dimensional geometry, see, for example, [46, 47, 48] vs. [26, 30, 31].

Transfer to higher dimensions Thus far, whenever a one-dimensional result is known to have a higher-dimensional counterpart, they are structurally similar, for example, (1.8)–(1.9) vs. Theorem 3.4, or (1.13) vs. Theorem 7.2. We expect that this similarity will continue to hold when a one-dimensional result has no known (as of yet) analog in higher dimensions. For instance, consider the very strong zero-one law of Theorem 8.1; hopefully this easier-to-prove one-dimensional result might suggest the appropriate version in higher dimensions, possibly by formal transfer.

One-dimensional modeling One-dimensional random networks may be deemed less physically relevant than their two-dimensional counterparts. However, they are of interest in their own right as simple models of wireless ad-hoc networks constrained over "linear" highways. They have been discussed in that context by a number of authors mostly under uniform node placement, see, for example, [11, 16, 17, 18, 20, 22, 26, 28, 30, 43, 44, 52, 53, 54] (and references therein).

1.7 Lessons Learned from the Lonely Dimension

As we discuss graph connectivity in one-dimensional networks under the i.i.d. node placement assumption, we will be putting the emphasis on the non-uniform case. A single *unifying* framework is developed to present available results, some classical and some recently obtained by the authors. Two complementary viewpoints are used, each based on a different characterization of graph connectivity: The first approach, already mentioned earlier, relies on asymptotic properties of the maximal spacings induced by i.i.d. variates on the unit interval. This naturally gives rise to the notions of *weak*, *strong*, and *very strong* zero-one laws, and attending critical thresholds; this classification is at the heart of some of our conclusions. The second approach, developed mostly in the references [26, 30, 33], exploits the asymptotics for the counts of *breakpoint* nodes in the graph. A large portion of this work was developed in Han's Ph.D. thesis [25]. Many of the results by the authors were reported in the conference papers [27, 28, 29], and in the journal papers [30, 31, 33]. This monograph expands on the earlier survey paper [32].

The non-vanishing case As we shall see shortly, a key role is played by the minimumm $f_\star$ of f. In the non-vanishing case (that is, $f_\star > 0$), a version of (1.11) is shown to hold with (1.13): The critical scaling $\rho^\star_F : \mathbb{N}_0 \to \mathbb{R}_+$ depends on the *inverse* of $f_\star$ through

$$\rho^\star_{F,n} = \frac{1}{f_\star} \cdot \frac{\log n}{n}, \quad n = 1, 2, \ldots \tag{1.15}$$

Note the similarity with (1.13). When nodes are uniformly placed on the unit interval, then $f_\star = 1$ and the critical scaling $\rho^\star_U : \mathbb{N}_0 \to \mathbb{R}_+$ reduces to

$$\rho^\star_{U,n} = \frac{\log n}{n}, \quad n = 1, 2, \ldots \tag{1.16}$$

The appropriate version of (1.8)–(1.9) is given in Theorem 8.1.

Estimating $f_\star$ might be an issue The value of $f_\star$ is typically not known to the users (and to the network operator, if any present), and there seems to be little operational reason for them to have this knowledge (especially when nodes are mobile). Since $f_\star$ is the minimum of a density function, estimating it will be fraught with difficulties akin to those encountered in the estimation of probabilities of rare events. This is especially so when $f_\star$ is very small. In particular, the unavailability of data sets large enough could lead to poor estimates.

Only weak laws are operationally relevant In light of this difficulty, when $f_\star > 0$ it is therefore not *practically* feasible to rely on strong/very strong critical scalings for determining effective power allocations. From a practical viewpoint, we are left only with weak zero-one laws as we note that the scaling $\rho^\star_U$ is a weak critical scaling, a *robust*, albeit weak, conclusion which holds across *all* distributions F satisfying (2.7). Ultimately this leads one to use power allocations that are far more conservative as we take transmission range which are orders of magnitude larger than $\frac{\log n}{n}$.

This conservative approach is unavoidable when the density function f vanishes at isolated points (that is, $f_\star = 0$). As shown on an example [29],[8] a weak zero-one law is nevertheless available in a form much weaker than either (1.8)–(1.9) or (1.11) with (1.13). In particular, the appropriate notion of critical scaling does not have the functional form $\Theta(\frac{\log n}{n})$ any more. In sum, critical scalings are very sensitive to whether $f_\star > 0$ or $f_\star = 0$, and only weak zero-one laws can be leveraged in any practical sense.

Robustness and phase transitions Within the confines of the one-dimensional disk model, critical scalings provide a baseline for determining power allocations that support connectivity. In many situations when $f_\star > 0$, sharp phase transitions are shown to exist, and this is certainly theoretically pleasing from a mathematical standpoint. Unfortunately, sharp phase transitions express *strong* sensitivity through very strong zero-one laws. As a result, small deviations from

[8] See Chapter 10.

the critical scaling can easily lead to power allocations under which the network fails to be connected a.s. in the many node limit. Such deviations can be created unwittingly if the estimates used for the parameters defining the critical scaling are poor, as is likely to be the case in practice for reasons discussed earlier.

Large scale wireless ad-hoc networks are expected to be deployed under very diverse environmental conditions, resulting in large variations in critical system parameters. Sound engineering practice requires that performance should not heavily depend on parameters which are either unrealistic to estimate or hard to obtain. In all cases considered here, either with $f_\star > 0$ or $f_\star = 0$, the mandate for connectivity leads to overprovisioning by orders of magnitude above the (minimal) power allocations in the range $\Theta(\frac{\log n}{n})$. These issues hold irrespectively of the dimension of the disk model being used. The one-dimensional model, through the present survey, helps make the case. The higher-dimensional case is technically more involved and is not completely understood as of this writing. We hope that the discussion given here will stimulate work along these lines. The journey goes on!

1.8 A Roadmap

To help the reader navigate this monograph, we provide a roadmap to its various chapters. Usually, proofs of results are relegated at the end of the chapter where they appear; sometimes they have been collected in separate chapters. These technical arguments can be omitted in a first reading.

Chapter 2: We introduce the one-dimensional model together with various assumptions on the underlying node location distribution F through its probability density function f. Connectivity is related to the maximal spacings associated with i.i.d. variates on the unit interval. The notions of weak and strong zero-one laws (for connectivity) are presented, and given simple characterizations in terms of the asymptotics of these maximal spacings.

Chapter 3: When F is the uniform distribution on $[0,1]$, the key results are derived from classical results for maximal spacings due to Lévy [40]. The notion of a very strong zero-one law emerges as a by-product of a double-exponential result similar to the one available for Erdős–Renyi graphs. This leads very naturally to an estimate for the width of the associated phase transition.

Chapter 4: We revisit the results of Chapter 3 by relating graph connectivity to the number of breakpoint nodes in the graph. The very strong zero-one law is derived this time by making use of the method of first and second moments applied to this count. We also recover the double-exponential result mentioned earlier by showing a Poisson convergence result for the number of breakpoints under an appropriate scaling. This turns out to be an easy application of the Stein-Chen method for constructing Poisson approximations [5].

Chapter 5: Most proofs of the results discussed in Chapter 4 are given here.

Chapter 6: We summarize the key ideas which underly many of the results presented here under non-uniform node placement. In particular, we show that the maximal spacings under a non-uniform probability distribution F can be expressed in terms of the order statistics for independent and uniformly distributed variates.

Chapter 7: Using the ideas discussed in Chapter 6 we establish a strong zero-one law when $f_\star > 0$. This is the one-dimensional analog of (1.11) with (1.13). To do this, we rely on the characterizations of graph connectivity from Chapter 2 in terms of maximal spacings. This leads to generalizing a standard result of Lévy [40] to a broad class of non-uniform distributions [31].

Chapter 8: The results concerning very strong zero-one laws are discussed when $f_\star > 0$. We give conditions on f to ensure the validity of the one-dimensional analog of (1.10)–(1.11), and the original proof is outlined in Chapter 9. We also provide a second, and much shorter, proof using a result of Barbe [3] concerning the asymptotics of maximal spacings under a non-uniform probability distribution F. This stronger result also leads to an estimate of the width of the phase transition.

Chapter 9: We outline a proof of the very strong zero-one laws when $f_\star > 0$ along the lines originally given in [33]. The approach is based on counting breakpoint nodes as was done in Chapter 4 for the uniform case. This is accomplished by developing a more involved variant of the method of first and second moments.

Chapter 10: We discuss an instance when the probability density function f vanishes at an isolated point on the interval $[0,1]$. Only a weak zero-one law is shown to exist, and its critical scaling is identified.

1.9 Notation and Conventions

Throughout, $\mathbb{R}$ and $\mathbb{R}_+$ will stand for the set of real numbers, and for the set of non-negative numbers, respectively. We use $\mathbb{N}$ to denote the set of non-negative integers $\{0,1,2,\ldots\}$ and the symbol $\mathbb{N}_0$ is reserved for the set of positive integers $\{1,2,\ldots\}$.

All statements involving limits, including asymptotic equivalences, are always understood with n going to infinity.

Almost everywhere is abbreviated as a.e., and all such statements are made with respect to Lebesgue measure λ on the unit interval $[0,1]$.

The random variables (rvs) under consideration are all defined on the same probability triple $(\Omega,\mathcal{F},\mathbb{P})$. All probabilistic statements are made with respect to this probability measure $\mathbb{P}$, and we denote the corresponding expectation operator by $\mathbb{E}$. Thus, almost sure(ly) (under $\mathbb{P}$) is abbreviated as a.s.. The notation $\stackrel{P}{\to}_n$ (resp. $\Longrightarrow_n$) is used to signify convergence in probability (resp. convergence in distribution) with n going to infinity. Also, we use the notation $=_{st}$ to indicate distributional equality.

The indicator function of an event E is simply denoted by $\mathbf{1}[E]$, and we let $|S|$ denote the cardinality of any discrete set S.

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