Duality of the Max-Plus and Min-Plus Network Calculus

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Contents

Abstract

The network calculus is a framework for the analysis of communication networks, which exploits that many computer network models become tractable for analysis if they are expressed in a min-plus or max-plus algebra. In a min-plus algebra, the network calculus characterizes amounts of traffic and available service as functions of time. In a max-plus algebra, the network calculus works with functions that express the arrival and departure times or the required service time for a given amount of traffic. While the min-plus network calculus is more convenient for capacity provisioning in a network, the max-plus network calculus is more compatible with traffic control algorithms that involve the computation of timestamps. Many similarities and relationships between the two versions of the network calculus are known, yet they are largely viewed as distinct analytical approaches with different capabilities and limitations. We show that there exists a one-to-one correspondence between the min-plus and max-plus network calculus, as long as traffic and service are described by functions with real-valued domains and ranges. Consequently, results from one version of the network calculus can be readily applied for computations in the other version. The ability to switch between min-plus and max-plus analysis without any loss of accuracy provides additional flexibility for characterizing and analyzing traffic control algorithms. This flexibility is exploited for gaining new insights into link scheduling algorithms that offer rate and delay guarantees to traffic flows.

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Introduction

Network calculus is a methodology for performance evaluation of communication networks that expresses the analysis of networks in a min-plus or max-plus algebra. In these algebras, the conventional addition and multiplication operations are replaced by the minimum or maximum operation, respectively, and addition. On the one hand, algebras with a minimum or maximum operation have weaker properties than algebras endowed with an addition and a multiplication. For instance, the minimum and the maximum do not have inverse operations. On the other hand, taking minimums and maximums creates strong ordering properties that can be analytically exploited. Network algorithms that involve sequencing of traffic, *e.g.*, scheduling with a sorted queue, or ordering of events, *e.g.*, window flow control, can often be described by linear systems in a min-plus or max-plus algebra, but are non-linear in an algebra with addition and multiplication.

The deterministic analysis of networks by Cruz in [13, 14] and its application to Generalized Processor Sharing scheduling by Parekh and Gallager in [28, 29] mark the beginning of network calculus research. The research was motivated by the emergence of communication networks that provide service guarantees even in adversarial worst-case scenarios. Within a few years, researchers recognized that non-traditional algebras, so-called *dioids*, for modelling discrete-event dynamic systems [3] can provide the foundation for a systems theory for communication networks [1, 6, 8]. The dioid algebras are applied to non-decreasing functions that represent cumulative arrival, departure and service processes in a network. The essence of the systems theory is that the departure traffic at a network element can be characterized by a convolution of functions describing the cumulative arrivals and the available service. The convolution operation is performed either in a min-plus or max-plus dioid algebra, leading to the min-plus and max-plus versions of the network calculus. Detailed models have been developed for many types of network elements, such as buffered links with FIFO or more complex scheduling algorithms, delay elements, traffic regulators, and many more. Comprehensive discussions can be found in textbooks on the topic [7, 9].

Network calculus analysis can select either a min-plus or max-plus algebra setting, yet, overwhelmingly, the literature presents derivations in a min-plus framework. In such a setting, arrivals and departures are represented as functions of time, where a function value $F(t)$ represents the amount of arriving or departing traffic until time t. This representation is convenient when performing computations with multiplexed traffic flows, since an aggregate of traffic flows that are characterized by functions $F_1(t), F_2(t), \ldots, F_N(t)$ is simply the sum $\sum_j F_j(t)$. Expressions for multiplexed traffic flows are needed when determining capacity requirements for a network, *e.g.*, the maximum number of flows that can be supported in a network subject to given service requirements. The representation of traffic by functions of time is less ideal when describing network control algorithms that assign timestamps to traffic. An example is a traffic regulator that determines the earliest time when a packet can be admitted to a network, or a scheduling algorithm that assigns deadlines for the departure time of packets. Obtaining timestamps from a function of the form $F(t)$ requires to solve an inverse problem. In a max-plus framework, arrivals and departures are characterized by functions $F(\nu)$ that give the arrival time or departure time of the ν-th bit or packet. For example, at a traffic regulator, the timestamp that determines when the ν-th bit or packet can be admitted is simply the value of the departure time function at ν. On the other hand, expressions for multiplexed traffic in the max-plus algebra are cumbersome (as we will see in §3).

Ideally, network analysis should be able to reconcile the advantages of the min-plus and max-plus network calculus algebras. That is, it should be able to employ functions $F(\nu)$ when working with network mechanisms involving timestamps and functions $F(t)$ when multiplexing traffic. Such mix-and-match computations require that the functions $F(t)$ and $F(\nu)$ can be related to each other. Mappings between expressions in the min-plus and max-plus network calculus, and vice versa, exist in the literature (see §13), however, since the mappings are (generally) not one-to-one, performing them comes at a loss of accuracy. At present, the prevailing view is that "many concepts [of the min-plus algebra] can be mirrored in the max-plus algebra," [20, p. 63], but also that not every result in the min-plus algebra can be extended to a max-plus setting [9, Remark 6.2.7] and that there is a lacking correspondence between concepts in the min-plus and max-plus algebra [7, §1.10]. On the other hand, according to dioid theory, the underlying min-plus and max-plus algebras of integer or real numbers are isomorphic [3, 21]. Thus, the question arises why the isomorphism does not extend to the min-plus and max-plus network calculus, which are based on these algebras? Our objective is to explore this question. We find that there exists a one-to-one relationship between the min-plus and max-plus network calculus, as long as both approaches are using functions that have a real-valued, that is, continuous-time or continuous-space, domains. Some of the previously observed differences between max-plus and min-plus analysis can be traced to the use of functions with a discrete-valued domain. After establishing the duality between the two versions of the network calculus, we proceed to characterize scheduling algorithms with rate and delay guarantees by service curves of the network calculus.

The remainder is structured as follows. In §2, we show that the max-plus convolution operation emerges when we describe the departures at a work-conserving link in terms of the arrivals and the link capacity. We observe that the expression for the departures is sensitive to the choice of measuring traffic in discrete units (bits, bytes, or packets) or by a real-valued metric. In §3–8, we present a self-contained description of the max-plus network calculus. In §9, we summarize the definitions and main results of the min-plus network calculus, which are later used for comparisons between the two network calculus versions. In §10, we show that the min-plus algebra and max-plus algebra for non-decreasing functions endowed with a minimum (or maximum) and a convolution operation are isomorphic to each other. We use the isomorphism in §11 to establish a duality of service curves, traffic envelopes, and performance bounds. In §12, we express scheduling algorithms for rate guarantees in terms of the continuous-space max-plus network calculus, and establish a connection between well-known scheduling algorithms and expressions in the max-plus algebra. In §13, we discuss the related literature with a focus on prior work on the max-plus network calculus, existing mappings between the min-plus and max-plus network calculus, and its relationship to lattice theory. We present conclusions in §14.

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