# Chordal Graphs and Semidefinite Optimization

**Lieven Vandenberghe**
University of California, Los Angeles
vandenbe@ucla.edu

**Martin S. Andersen**
Technical University of Denmark
mskan@dtu.dk

the essence of knowledge
Boston — Delft

# Foundations and Trends® in Optimization

*Published, sold and distributed by:*
now Publishers Inc.
PO Box 1024
Hanover, MA 02339
United States
Tel. +1-781-985-4510
www.nowpublishers.com
sales@nowpublishers.com

*Outside North America:*
now Publishers Inc.
PO Box 179
2600 AD Delft
The Netherlands
Tel. +31-6-51115274

The preferred citation for this publication is

L. Vandenberghe and M. S. Andersen. *Chordal Graphs and Semidefinite Optimization*. Foundations and Trends® in Optimization, vol. 1, no. 4, pp. 241–433, 2014.

*This Foundations and Trends® issue was typeset in LaTeX using a class file designed by Neal Parikh. Printed on acid-free paper.*

ISBN: 978-1-68083-039-2

Full text available at: http://dx.doi.org/10.1561/2400000006

# Foundations and Trends® in Optimization

Volume 1, Issue 4, 2014

## Editorial Board

# Editorial Scope

## Topics

Foundations and Trends® in Optimization publishes survey and tutorial articles on methods for and applications of mathematical optimization, including the following topics:

- Algorithm design, analysis, and implementation (especially on modern computing platforms)
- Models and modeling systems
- New optimization formulations for practical problems
- Applications of optimization in:
  - Machine learning
  - Statistics
  - Data analysis
  - Signal and image processing
  - Computational economics and finance
  - Engineering design
  - Scheduling and resource allocation
  - and other areas

## Information for Librarians

Foundations and Trends® in Optimization, 2014, Volume 1, 4 issues. ISSN paper version 2167-3888. ISSN online version 2167-3918. Also available as a combined paper and online subscription.

Foundations and Trends® in Optimization
Vol. 1, No. 4 (2014) 241–433

DOI: 10.1561/2400000006

# Chordal Graphs and Semidefinite Optimization

Lieven Vandenberghe
University of California, Los Angeles
vandenbe@ucla.edu

Martin S. Andersen
Technical University of Denmark
mskan@dtu.dk

# Contents

## Abstract

Chordal graphs play a central role in techniques for exploiting sparsity in large semidefinite optimization problems and in related convex optimization problems involving sparse positive semidefinite matrices. Chordal graph properties are also fundamental to several classical results in combinatorial optimization, linear algebra, statistics, signal processing, machine learning, and nonlinear optimization. This survey covers the theory and applications of chordal graphs, with an emphasis on algorithms developed in the literature on sparse Cholesky factorization. These algorithms are formulated as recursions on elimination trees, supernodal elimination trees, or clique trees associated with the graph. The best known example is the multifrontal Cholesky factorization algorithm, but similar algorithms can be formulated for a variety of related problems, including the computation of the partial inverse of a sparse positive definite matrix, positive semidefinite and Euclidean distance matrix completion problems, and the evaluation of gradients and Hessians of logarithmic barriers for cones of sparse positive semidefinite matrices and their dual cones. The purpose of the survey is to show how these techniques can be applied in algorithms for sparse semidefinite optimization, and to point out the connections with related topics outside semidefinite optimization, such as probabilistic networks, matrix completion problems, and partial separability in nonlinear optimization.



L. Vandenberghe and M. S. Andersen. *Chordal Graphs and Semidefinite Optimization*. Foundations and Trends® in Optimization, vol. 1, no. 4, pp. 241–433, 2014.

DOI: 10.1561/2400000006.

# 1

## Introduction

This survey gives an introduction to techniques from graph and sparse matrix theory that are important in semidefinite optimization. The results from graph theory we discuss are related to *chordal graphs* and *graph elimination*.

A *chordal graph* is an undirected graph with the property that every cycle of length greater than three has a chord (an edge between nonconsecutive vertices in the cycle). Chordal graphs have attracted interest in graph theory because several combinatorial optimization problems that are very difficult in general turn out to be easy for chordal graphs and solvable by simple greedy algorithms. Examples are the graph coloring problem and the problem of finding the largest clique in a graph. Chordal graphs have been studied extensively since the 1950s and their history shares some key events with the history of semidefinite optimization. In particular, it was Shannon's 1956 paper [203] that led Berge to the definition of perfect graphs, of which chordal graphs are an important subclass [28], and Lovász to one of the most famous early applications of semidefinite optimization [158].

Chordal graphs in applications often result from *graph elimination*, a process that converts a general undirected graph into a chordal graph by adding edges. Graph elimination visits the vertices of the graph in a certain order, called the *elimination order*. When vertex $v$ is visited, edges are added between the vertices that are adjacent to $v$, follow $v$ in the elimination order, and are not yet mutually adjacent. If no edges are added during graph elimination the elimination order is called a *perfect elimination order*. It has been known since the 1960s that chordal graphs are exactly the graphs for which a perfect elimination order exists. A variety of algorithms based on different forms of 'variable elimination' can be described and analyzed via graph elimination. Examples include the solution of sparse linear equations (Gauss elimination), dynamic programming (eliminating optimization variables by optimizing over them), and marginalization of probability distributions (eliminating variables by summation or integration). Variable elimination is a natural approach in many applications and this partly explains the diversity of the disciplines in which chordal graphs have been studied.

The first part of this survey (Chapters 2–7) covers the basic theory of chordal graphs and graph elimination, with an emphasis on tree data structures developed in sparse matrix theory (elimination trees and supernodal elimination trees) and efficient analysis algorithms based on elimination trees.

The second part (Chapters 8–11) describes applications of chordal graphs to sparse matrices. The sparsity pattern of a symmetric sparse matrix can be represented by an undirected graph and graph elimination describes the fill-in during Cholesky factorization of a sparse positive definite matrix. Hence, the sparsity pattern of a Cholesky factor is chordal, and positive definite matrices with chordal sparsity patterns can be factored with zero fill-in. This fact underlies several classical decomposition results, discovered in the 1980s, that characterize sparse positive semidefinite matrices, sparse matrices with a positive semidefinite completion, and sparse matrices with a Euclidean distance matrix completion, when the sparsity pattern is chordal. We give an overview of these chordal decompositions and completion problems. We also present practical algorithms for solving them and several related problems, including computing the partial inverse of a sparse positive definite matrix, the inverse factorization of the maximum determinant positive definite completion, and calculating derivatives of logarithmic barriers for cones of sparse symmetric matrices. We refer to these algorithms as *multifrontal* algorithms because they use similar recursions on elimination trees as the multifrontal Cholesky factorization algorithm. A library with implementations of most of the algorithms described in these chapters can be found at `http://cvxopt.github.io/chompack` [13].

In the last three chapters (Chapters 12–14) we discuss applications of chordal sparse matrix techniques in continuous optimization. The main focus is on decomposition results that exploit *partial separability* in nonlinear and conic optimization problems, and on techniques that exploit sparsity in interior-point methods for semidefinite optimization.

A summary of the notation used in the paper can be found on page 173.

We assume the reader is familiar with semidefinite optimization and its applications. Introductions and surveys of semidefinite optimization can be found in several books and articles, including [15, 223, 235, 24, 217, 226, 43]. On the other hand we do not assume any background in graph or sparse matrix theory. The main purpose of this survey is to give an extended introduction to the theory of chordal graphs and graph elimination, and the algorithms and elimination tree structures developed in the literature on sparse Cholesky factorization, and to describe the different ways in which these techniques can be applied in semidefinite and conic optimization algorithms. In addition, we aim to show the connections with related topics outside semidefinite optimization, such as probabilistic networks, matrix completion problems, and partial separability in nonlinear optimization.

# References


[1] J. Agler, J. W. Helton, S. McCullough, and L. Rodman. Positive semidefinite matrices with a given sparsity pattern. *Linear Algebra and Its Applications*, 107:101–149, 1988.

[2] S. M. Aji and J. McEliece. The generalized distributive law. *IEEE Transactions on Information Theory*, 46(2):335–343, 2000.

[3] A. Alfakih and H. Wolkowicz. Matrix completion problems. In H. Wolkowicz, R. Saigal, and L. Vandenberghe, editors, *Handbook of Semidefinite Programming. Theory, Algorithms, and Applications*, chapter 18, pages 533–545. Kluwer Academic Publishers, 2000.

[4] F. Alizadeh. Interior point methods in semidefinite programming with applications to combinatorial optimization. *SIAM Journal on Optimization*, 5(1):13–51, February 1995.

[5] F. Alizadeh, J.-P. A. Haeberly, and M. L. Overton. Primal-dual interior-point methods for semidefinite programming: convergence rates, stability and numerical results. *SIAM J. on Optimization*, 8(3):746–768, 1998.

[6] P. R. Amestoy, I. S. Duff, J.-Y. L'Excellent, Y. Robert, F.-H. Rouet, and B. Uçar. On computing inverse entries of a sparse matrix in an out-of-core environment. *SIAM Journal on Scientific Computing*, 34(4):A1975–A1999, 2012.

[7] M. Andersen, L. Vandenberghe, and J. Dahl. Linear matrix inequalities with chordal sparsity patterns and applications to robust quadratic optimization. In *Proceedings of the IEEE International Symposium on Computer-Aided Control System Design (CACSD)*, 2010.

[8] M. S. Andersen, J. Dahl, and L. Vandenberghe. Implementation of nonsymmetric interior-point methods for linear optimization over sparse matrix cones. *Mathematical Programming Computation*, 2:167–201, 2010.

[9] M. S. Andersen, J. Dahl, and L. Vandenberghe. Logarithmic barriers for sparse matrix cones. *Optimization Methods and Software*, 28(3):396–423, 2013.

[10] M. S. Andersen, J. Dahl, and L. Vandenberghe. CVXOPT: A Python package for convex optimization, version 1.1.7. Available at `cvxopt.org`, 2014.

[11] M. S. Andersen, A. Hansson, and L. Vandenberghe. Reduced-complexity semidefinite relaxations of optimal power flow problems. *IEEE Transactions on Power Systems*, 29:1855–1863, 2014.

[12] M. S. Andersen and L. Vandenberghe. Support vector machine training using matrix completion techniques. 2010.

[13] M. S. Andersen and L. Vandenberghe. *CHOMPACK: A Python Package for Chordal Matrix Computations, Version 2.1*, 2014. `cvxopt.github.io/chompack`.

[14] M. S. Andersen and L. Vandenberghe. *SMCP: Python Extension for Sparse Matrix Cone Programs, Version 0.4*, 2014. `cvxopt.github.io/smcp`.

[15] M. F. Anjos and J. B. Lasserre, editors. *Handbook on Semidefinite, Conic and Polynomial Optimization.* Springer US, 2012.

[16] S. Arnborg, D. Corneil, and A. Proskurowski. Complexity of finding embeddings in a k-tree. *SIAM J. on Algebraic Discrete Methods*, 8(2):277–284, 1987.

[17] X. Bai, H. Wei, K. Fujisawa, and Y. Wang. Semidefinite programming for optimal power flow problems. *International Journal of Electrical Power & Energy Systems*, 30(6):383–392, 2008.

[18] M. Bakonyi and C. R. Johnson. The Euclidian distance matrix completion problem. *SIAM Journal on Matrix Analysis and Applications*, 16(2):646–654, 1995.

[19] M. Bakonyi and H. J. Woerdeman. *Matrix Completions, Moments, and Sums of Hermitian Squares.* Princeton University Press, 2011.

[20] W. W. Barrett, C. R. Johnson, and M. Lundquist. Determinantal formulation for matrix completions associated with chordal graphs. *Linear Algebra and Appl.*, 121:265–289, 1989.

[21] H. H. Bauschke and P. L. Combettes. *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. Springer, 2011.

[22] C. Beeri, R. Fagin, D. Maier, and M. Yannakakis. On the desirability of acyclic database schemes. *Journal of the Association for Computing Machinery*, 30(3):479–513, 1983.

[23] S. Bellavia, J. Gondzio, and B. Morini. A matrix-free preconditioner for sparse symmetric positive definite systems and least-squares problems. *SIAM Journal on Scientific Computing*, 35(1):A192–A211, 2013.

[24] A. Ben-Tal and A. Nemirovski. *Lectures on Modern Convex Optimization. Analysis, Algorithms, and Engineering Applications*. SIAM, 2001.

[25] S. J. Benson and Y. Ye. Algorithm 875: DSDP5—software for semidefinite programming. *ACM Trans. Math. Softw.*, 34(3):16:1–16:20, May 2008.

[26] S. J. Benson, Y. Ye, and X. Zhang. Solving large-scale sparse semidefinite programs for combinatorial optimization. *SIAM Journal on Optimization*, 10:443–461, 2000.

[27] C. Berge. Some classes of perfect graphs. In F. Harary, editor, *Graph Theory and Theoretical Physics*, pages 155–165. Academic Press, 1967.

[28] C. Berge and J. L. Ramírez Alfonsín. Origins and genesis. In J. L. Ramírez Alfonsín and B. A. Reed, editors, *Perfect Graphs*, pages 1–12. John Wiley & Sons, 2001.

[29] A. Berry, J. R. S. Blair, P. Heggernes, and B. W. Peyton. Maximum cardinality search for computing minimal triangulations of graphs. *Algorithmica*, 39:287–298, 2004.

[30] A. Berry, J.-P. Bordat, P. Heggernes, G. Simonet, and Y. Villanger. A wide-range algorithm for minimal triangulation from an arbitrary ordering. *Journal of Algorithms*, 58:33–66, 2006.

[31] M. W. Berry, M. T. Heath, I. Kaneko, M. Lawo, R. J. Plemmons, and R. C. Ward. An algorithm to compute a sparse basis of the null space. *Numerische Mathematik*, 47(4):483–504, 1985.

[32] U. Bertelè and F. Brioschi. *Nonserial Dynamic Programming*. Academic Press, 1972.

[33] D. P. Bertsekas. *Dynamic Programming and Optimal Control*, volume 1. Athena Scientific, Belmont, Massachusetts, second edition, 2000.

[34] P. Biswas, T.-C. Lian, T.-C. Wang, and Y. Ye. Semidefinite programming based algorithms for sensor network localization. *ACM Transactions on Sensor Networks*, 2(2):188–220, 2006.

[35] P. Biswas, T.-C. Liang, K.-C. Toh, Y. Ye, and T.-C. Wang. Semidefinite programming approaches for sensor network localization with noisy distance measurements. *IEEE Transactions on Automation Science and Engineering*, 3(4):360–371, 2006.

[36] P. Biswas and Y. Ye. A distributed method for solving semidefinite programs arising from ad hoc wireless sensor network localization. In W. W. Hager, S.-J. Huang, P. M. Pardalos, and O. A. Prokopyev, editors, *Multiscale Optimization Methods and Applications*, volume 82 of *Nonconvex Optimization and Its Applications*, pages 69–94. Springer US, 2006.

[37] J. R. S. Blair, P. Heggerness, and J. A. Telle. A practical algorithm for making filled graphs minimal. *Theoretical Computer Science*, 250:125–141, 2001.

[38] J. R. S. Blair and B. Peyton. An introduction to chordal graphs and clique trees. In A. George, J. R. Gilbert, and J. W. H. Liu, editors, *Graph Theory and Sparse Matrix Computation*. Springer-Verlag, 1993.

[39] H. L. Bodlaender. A tourist guide through treewidth. *Acta Cybernetica*, 11(1-2):1–21, 1993.

[40] H. L. Bodlaender, J. R. Gilbert, H. Hafsteinsson, and T. Kloks. Approximating treewidth, pathwidth, frontsize, and shortest elimination tree. *Journal of Algorithms*, 18(2):238–255, 1995.

[41] B. Borchers. CSDP, a C library for semidefinite programming. *Optimization Methods and Software*, 11(1):613–623, 1999.

[42] S. Boyd, P. Diaconis, and L. Xiao. Fastest mixing Markov chain on a graph. *SIAM Review*, 46(4):667–689, 2004.

[43] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*, volume 15 of *Studies in Applied Mathematics*. SIAM, 1994.

[44] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein. Distributed optimization and statistical learning via the alternating direction method of multipliers. *Foundations and Trends in Machine Learning*, 3(1):1–122, 2011.

[45] A. Brandstädt, V. B. Le, and J. P. Spinrad. *Graph Classes. A Survey.* Society for Industrial and Applied Mathematics, 1999.

[46] P. Buneman. A characterization of rigid circuit graphs. *Discrete Mathematics*, 9:205–212, 1974.

[47] S. Burer. Semidefinite programming in the space of partial positive semidefinite matrices. *SIAM Journal on Optimization*, 14(1):139–172, 2003.

[48] S. Burer and R. D. C. Monteiro. A nonlinear programming algorithm for solving semidefinite programs via low-rank factorization. *Mathematical Programming (Series B)*, 95(2), 2003.

[49] Y. E. Campbell and T. A. Davis. Computing the sparse inverse subset: an inverse multifrontal approach. Technical Report TR-95-021, Computer and Information Sciences Department, University of Florida, 1995.

[50] Y. Censor and S. A. Zenios. *Parallel Optimization: Theory, Algorithms, and Applications*. Numerical Mathematics and Scientific Computation. Oxford University Press, New York, 1997.

[51] P. R. Chares. *Cones and Interior-Point Algorithms for Structured Convex Optimization Involving Powers and Exponentials*. PhD thesis, Université Catholique de Louvain, 2009.

[52] T. Coleman and A. Pothen. The null space problem I. Complexity. *SIAM J. on Algebraic Discrete Methods*, 7(4):527–537, 1986.

[53] T. Coleman and A. Pothen. The null space problem II. Algorithms. *SIAM J. on Algebraic Discrete Methods*, 8(4):544–563, 1987.

[54] P. L. Combettes and J.-C. Pesquet. Proximal splitting methods in signal processing. In H. H. Bauschke, R. S. Burachik, P. L. Combettes, V. Elser, D. R. Luke, and H. Wolkowicz, editors, *Fixed-Point Algorithms for Inverse Problems in Science and Engineering*, pages 185–212. Springer, 2011.

[55] P. L. Combettes and V. R. Wajs. Signal recovery by proximal forward-backward splitting. *Multiscale Modeling and Simulation*, 4(4):1168–1200, 2005.

[56] A. R. Conn, N. I. M. Gould, and Ph. L. Toint. *LANCELOT: A Fortran Package for Large-Scale Nonlinear Optimization (Release A)*, volume 17 of *Springer Series in Computational Mathematics*. Springer-Verlag, 1992.

[57] A. R. Conn, N. I. M. Gould, and Ph. L. Toint. *Trust-Region Methods*. SIAM, 2000.

[58] R. G. Cowell, A. P. Dawid, S. L Lauritzen, and D. J. Spiegelhalter. *Probabilistic Networks and Expert Systems: Exact Computational Methods for Bayesian Networks*. Information Science and Statistics. Springer, 2007.

[59] Y. Crama, P. Hansen, and B. Jaumard. The basic algorithm for pseudo-Boolean programming revisited. *Discrete Applied Mathematics*, 29:171–185, 1990.

[60] CVX Research, Inc. *CVX: Matlab Software for Disciplined Convex Programming, version 2.0*, 2012.

[61] J. Dahl, L. Vandenberghe, and V. Roychowdhury. Covariance selection for non-chordal graphs via chordal embedding. *Optimization Methods and Software*, 23(4):501–520, 2008.

[62] Y. H. Dai and N. Yamashita. Analysis of sparse quasi-Newton updates with positive definite matrix completion. *Journal of the Operations Research Society of China*, 2(1):39–56, 2014.

[63] E. Dall'Anese, H. Zhu, and G. B. Giannakis. Distributed optimal power flow for smart microgrids. *IEEE Transactions on Smart Grid*, 4(3):1464–475, 2013.

[64] A. Darwiche. *Modeling and Reasoning with Bayesian Networks*. Cambridge University Press, 2009.

[65] T. A. Davis. *Direct Methods for Sparse Linear Systems*. SIAM, 2006.

[66] T. A. Davis and Y. Hu. The University of Florida sparse matrix collection. *ACM Transactions on Mathematical Software*, 38:1–25, 2011.

[67] G. De Micheli. *Synthesis and Optimization of Digital Circuits*. McGraw-Hill, 1994.

[68] A. P. Dempster. Covariance selection. *Biometrics*, 28:157–175, 1972.

[69] Z. Deng, M. Gu, and M. L. Overton. Structured primal-dual interior-point methods for banded semidefinite programming. In I. Gohberg, editor, *Topics in Operator Theory*, volume 202 of *Operator Theory: Advances and Applications*, pages 111–141. Birkhäuser Basel, 2010.

[70] J. E. Dennis and R. B. Schnabel. *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*. Prentice-Hall, 1983.

[71] I. S. Dhillon and J. A. Tropp. Matrix nearness problems with Bregman divergences. *SIAM Journal on Matrix Analysis and Applications*, 29(4), 2007.

[72] S. Diamond, E. Chu, and S. Boyd. CVXPY: a Python-embedded modeling language for convex optimization, version 0.2. `cvxpy.org`, 2014.

[73] Y. Ding, N. Krislock, J. Qian, and H. Wolkowicz. Sensor network localization, Euclidean distance matrix completion, and graph realization. *Optimization and Engineering*, 11(1):45–66, 2010.

[74] G. A. Dirac. On rigid circuit graphs. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 25(1-2):71–76, 1961.

[75] I. S. Duff and J. K. Reid. The multifrontal solution of indefinite sparse symmetric linear equations. *ACM Transactions on Mathematical Software*, 9(3):302–325, 1983.

[76] H. Dym and I. Gohberg. Extensions of band matrices with band inverses. *Linear Algebra and Appl.*, 36:1–24, 1981.

[77] J. Eckstein and D. Bertsekas. On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators. *Mathematical Programming*, 55:293–318, 1992.

[78] A. M. Erisman and W. F. Tinney. On computing certain elements of the inverse of a sparse matrix. *Communications of the ACM*, 18(3):177–179, 1975.

[79] J. Faraut and A. Korányi. *Analysis on Symmetric Cones*. Oxford University Press, 1994.

[80] L. Faybusovich. Euclidean Jordan algebras and interior-point algorithms. *Positivity*, 1(4):331–357, 1997.

[81] S. Fine and K. Scheinberg. Efficient SVM training using low-rank kernel representations. *Journal of Machine Learning Research*, 2:243–264, 2002.

[82] P. C. Fishburn. *Interval Orders and Interval Graphs*. John Wiley & Sons, 1985.

[83] R. Fletcher. A new variational result for quasi-Newton formulae. *SIAM J. on Optimization*, pages 18–21, 1991.

[84] R. Fletcher. An optimal positive definite update for sparse Hessian matrices. *SIAM Journal on Optimization*, 5(1):192–218, February 1995.

[85] A. B. Frakt, H. Lev-Ari, and A. S. Willsky. A generalized Levinson algorithm for covariance extension with application to multiscale autoregressive modeling. *IEEE Transactions on Information Theory*, 49(2):411–424, 2003.

[86] K. Fujisawa, S. Kim, M. Kojima, Y. Okamoto, and M. Yamashita. User's manual for SparseCoLO: Conversion methods for sparse conic-form linear optimization problems. Technical Report B-453, Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, 2009.

[87] M. Fukuda, M. Kojima, K. Murota, and K. Nakata. Exploiting sparsity in semidefinite programming via matrix completion I: general framework. *SIAM Journal on Optimization*, 11:647–674, 2000.

[88] D. R. Fulkerson and O. Gross. Incidence matrices and interval graphs. *Pacific Journal of Mathematics*, 15(3):835–855, 1965.

[89] D. Gabay. Applications of the method of multipliers to variational inequalities. In M. Fortin and R. Glowinski, editors, *Augmented Lagrangian methods: Applications to the numerical solution of boundary-value problems*, Studies in Mathematics and Its Applications, pages 299–331. North-Holland, 1983.

[90] D. Gabay and B. Mercier. A dual algorithm for the solution of nonlinear variational problems via finite element approximation. *Computers and Mathematics with Applications*, 2:17–40, 1976.

[91] F. Gavril. Algorithms for minimum coloring, maximum clique, minimum covering by cliques, and maximum independent set of a chordal graph. *SIAM Journal on Computing*, 1(2):180–187, 1972.

[92] F. Gavril. The intersection graphs of subtrees in trees are exactly the chordal graphs. *Journal of Combinatorial Theory Series B*, 16:47–56, 1974.

[93] J. Gilbert and M. Heath. Computing a sparse basis for the null space. *SIAM J. on Algebraic Discrete Methods*, 8(3):446–459, 1987.

[94] J. R. Gilbert, E. G. Ng, and B. W. Peyton. An efficient algorithm to compute row and column counts for sparse Cholesky factorization. *SIAM Journal on Matrix Analysis and Applications*, 15(4):1075–1091, 1994.

[95] F. Glineur and T. Terlaky. Conic formulations for $\ell_p$-norm optimization. *Journal of Optimization Theory and Applications*, 122(2):285–307, 2004.

[96] R. Glowinski and A. Marrocco. Sur l'approximation, par éléments finis d'ordre un, et la résolution, par pénalisation-dualité d'une classe de problèmes de Dirichlet non linéaires. *Revue française d'automatique, informatique, recherche opérationnelle*, 9(2):41–76, 1975.

[97] M. Goemans and F. Rendl. Combinatorial optimization. In H. Wolkowicz, R. Saigal, and L. Vandenberghe, editors, *Handbook of Semidefinite Programming*, chapter 12, pages 343–360. Kluwer Academic Publishers, 2000.

[98] M. X. Goemans and D. P. Williamson. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *Journal of the Association for Computing Machinery*, 42(6):1115–1145, 1995.

[99] G. H. Golub and R. J. Plemmons. Large-scale geodetic least-squares adjustment by dissection and orthogonal decomposition. *Linear Algebra and Its Applications*, 34(3):3–27, 1980.

[100] G. H. Golub and C. F. Van Loan. *Matrix Computations.* Johns Hopkins University Press, 3rd edition, 1996.

[101] M. C. Golumbic. *Algorithmic Graph Theory and Perfect Graphs.* Elsevier, second edition, 2004.

[102] J. Gondzio. Matrix-free interior point method. *Computational Optimization and Applications*, 51(2):457–480, 2012.

[103] M. Grant and S. Boyd. Graph implementations for nonsmooth convex programs. In V. Blondel, S. Boyd, and H. Kimura, editors, *Recent Advances in Learning and Control (a tribute to M. Vidyasagar)*, pages 95–110. Springer, 2008.

[104] A. Griewank and Ph. L. Toint. On the unconstrained minimization of partially separable functions. In M. J. D. Powell, editor, *Nonlinear Optimization 1981*, pages 301–312. Academic Press, 1982.

[105] A. Griewank and Ph. L. Toint. Partitioned variable metric updates for large structured optimization problems. *Numerische Mathematik*, 39:119–137, 1982.

[106] A. Griewank and Ph. L. Toint. On the existence of convex decompositions of partially separable functions. *Mathematical Programming*, 28:25–49, 1984.

[107] A. Griewank and A. Walther. *Evaluating Derivatives: Principles and Techniques of Algorithmic Differentiation.* Society for Industrial and Applied Mathematics, second edition, 2008.

[108] R. Grone, C. R. Johnson, E. M Sá, and H. Wolkowicz. Positive definite completions of partial Hermitian matrices. *Linear Algebra and Appl.*, 58:109–124, 1984.

[109] R. Grone, S. Pierce, and W. Watkins. Extremal correlation matrices. *Linear Algebra and Its Applications*, 134:63–70, 1990.

[110] M. Grötschel, L. Lovász, and A. Schrijver. Polynomial algorithms for perfect graphs. *Annals of Discrete Mathematics*, 21:325–356, 1984.

[111] M. Grötschel, L. Lovász, and A. Schrijver. *Geometric Algorithms and Combinatorial Optimization*, volume 2 of *Algorithms and Combinatorics*. Springer-Verlag, 1988.

[112] A. Hajnal and J. Surányi. Über die Auflösung von Graphen in vollständige Teilgraphen. *Annales Universitatis Scientiarum Budapestinensis de Rolando Eötvös Nominatae — Sectio Mathematica*, 1:113–121, 1958.

[113] P. Heggernes. Minimal triangulation of graphs: a survey. *Discrete Mathematics*, 306:297–317, 2006.

[114] P. Heggernes and B. W. Peyton. Fast computation of minimal fill inside a given elimination ordering. *SIAM Journal on Matrix Analysis and Applications*, 30(4):1424–1444, 2008.

[115] C. Helmberg and F. Rendl. A spectral bundle method for semidefinite programming. Technical Report SC 97-37, Konrad-Zuse Zentrum fuer Informationstechnik Berlin, 1997.

[116] C. Helmberg, F. Rendl, R. J. Vanderbei, and H. Wolkowicz. An interior-point method for semidefinite programming. *SIAM J. on Optimization*, 6(2):342–361, 1996.

[117] N. J. Higham. Analysis of the Cholesky decomposition of a semi-definite matrix. In M. G. Cox and S. J. Hammarling, editors, *Reliable Numerical Computation*, pages 161–185. Oxford University Press, 1990.

[118] P. L. Hammer (Ivănescu) and S. Rudeanu. *Boolean Methods in Operations Research and Related Areas*. Springer-Verlag New York, 1968. With a preface by Richard Bellman.

[119] R. A. Jabr. Exploiting sparsity in SDP relaxations of the OPF problem. *IEEE Transactions on Power Systems*, 27(2):1138–1139, 2012.

[120] F. Jarre and F. Rendl. An augmented primal-dual method for linear conic programs. *SIAM Journal on Optimization*, 19(2):808–823, 2008.

[121] C. R. Johnson. Matrix completion problems: A survey. In C. R. Johnson, editor, *Matrix Theory and Applications*, volume 40 of *Proceedings of Symposia in Applied Mathematics*, pages 171–189. American Mathematical Society, 1990.

[122] N. Kakimura. A direct proof for the matrix decomposition of chordal-structured positive semidefinite matrices. *Linear Algebra and Its Applications*, 433:819–823, 2010.

[123] T. Kanamori and A. Ohara. A Bregman extension of quasi-Newton updates I: an information geometrical framework. *Optimization Methods and Software*, 28(1):96–123, 2013.

[124] T. Kanamori and A. Ohara. A Bregman extension of quasi-Newton updates II: analysis of robustness properties. *Journal of Computational and Applied Mathematics*, 253:104–122, 2013.

[125] S. Kim and M. Kojima. Exploiting sparsity in SDP relaxation of polynomial optimization problems. In M. F. Anjos and J. B. Lasserre, editors, *Handbook on Semidefinite, Conic and Polynomial Optimization*, volume 166 of *International Series in Operations Research & Management Science*, chapter 18, pages 499–531. Springer, 2012.

[126] S. Kim, M. Kojima, M. Mevissen, and M. Yamashita. Exploiting sparsity in linear and nonlinear matrix inequalities via positive semidefinite matrix completion. *Mathematical Programming*, 129:33–68, 2011.

[127] S. Kim, M. Kojima, and P. Toint. Recognizing underlying sparsity in optimization. *Mathematical Programming*, 119:273–303, 2009.

[128] S. Kim, M. Kojima, and H. Waki. Generalized Lagrangian duals and sums of squares relaxations of sparse polynomial optimization problems. *SIAM Journal on Optimization*, 15(3):697–719, 2005.

[129] S. Kim, M. Kojima, and H. Waki. Exploiting sparsity in SDP relaxations for sensor network localization. *SIAM Journal on Optimization*, 20(1):192–215, 2009.

[130] K. Kobayashi, S. Kim, and M. Kojima. Correlative sparsity in primal-dual interior-point methods for LP, SDP, and SOCP. *Applied Mathematics and Optimization*, 58(1):69–88, 2008.

[131] M. Kojima, S. Kim, and H. Waki. Sparsity in sums of squares of polynomials. *Mathematical Programming*, 103:45–62, 2005.

[132] M. Kojima, S. Shindoh, and S. Hara. Interior-point methods for the monotone linear complementarity problem in symmetric matrices. *SIAM J. on Optimization*, 7:86–125, February 1997.

[133] D. Koller and N. Friedman. *Probabilistic Graphical Models. Principles and Techniques.* MIT Press, 2009.

[134] M. Kočvara and M. Stingl. PENNON: a code for convex nonlinear and semidefinite programming. *Optimization Methods and Software*, 18(3):317–333, 2003.

[135] M. Kočvara and M. Stingl. On the solution of large-scale SDP problems by the modified barrier method using iterative solvers. *Mathematical Programming, Series B*, 109(2-3):413–444, 2007.

[136] N. Krislock and H. Wolkowicz. Euclidean distance matrices and applications. In M. F. Anjos and J. B. Lasserre, editors, *Handbook on Semidefinite, Conic and Polynomial Optimization*, chapter 30, pages 879–914. Springer, 2012.

[137] S. Kullback. *Information Theory and Statistics*. Dover Publications, 1997. Originally published by John Wiley & Sons, 1959.

[138] A. Y. S. Lam, B. Zhang, and D. Tse. Distributed algorithms for optimal power flow problem, 2011. `arxiv.org/abs/1109.5229`.

[139] G. Lan, Z. Lu, and R. D. C. Monteiro. Primal-dual first-order methods with $o(1/\epsilon)$ iteration-complexity for cone programming. *Mathematical Programming*, 126(1):1–29, 2011.

[140] J. B. Lasserre. Convergent SDP-relaxations in polynomial optimization with sparsity. *SIAM J. on Optimization*, 17(3):822–843, 2006.

[141] M. Laurent. A tour d'horizon on positive semidefinite and Euclidean distance matrix completion problems. In P. M. Pardalos and H. Wolkowicz, editors, *Topics in Semidefinite and Interior-Point Methods*, volume 18 of *Fields Institute Communications*, pages 51–76. The American Mathematical Society, 1998.

[142] M. Laurent. Matrix completion problems. In C. A. Floudas and P. M. Pardalos, editors, *Encyclopedia of Optimization*, volume III, pages 221–229. Kluwer, 2001.

[143] M. Laurent and S. Poljak. On a positive semidefinite relaxation of the cut polytope. *Linear Algebra and Its Applications*, 223:439–461, 1995.

[144] M. Laurent and S. Poljak. On the facial structure of the set of correlation matrices. *SIAM Journal on Matrix Analysis and Applications*, 17(3):530–547, 1996.

[145] M. Laurent and A. Varvitsiotis. A new graph parameter related to bounded rank positive semidefinite matrix completions. *Mathematical Programming*, 145(1-2):291–325, 2014.

[146] S. L. Lauritzen. *Graphical Models*. Oxford University Press, Oxford, 1996.

[147] S. L. Lauritzen and F. V. Jensen. Local computation with valuations from a commutative semigroup. *Annals of Mathematics and Artificial Intelligence*, 21:51–69, 1997.

[148] J. Lavaei and S. H. Low. Zero duality gap in optimal power flow problem. *IEEE Trans. Power Systems*, 27(1):92–107, February 2012.

[149] C. G. Lekkerkerker and J. C. Boland. Representation of a finite graph by a set of intervals on the real line. *Fundamenta Mathematicae*, 51:45–64, 1962.

[150] J. G. Lewis, B. W. Peyton, and A. Pothen. A fast algorithm for reordering sparse matrices for parallel factorization. *SIAM Journal on Scientific and Statistical Computing*, 10(6):1146–1173, 1989.

[151] P. L. Lions and B. Mercier. Splitting algorithms for the sum of two nonlinear operators. *SIAM Journal on Numerical Analysis*, 16(6):964–979, 1979.

[152] J. W. H. Liu. A compact row storage scheme for Cholesky factors using elimination trees. *ACM Transactions on Mathematical Software*, 12:127–148, 1986.

[153] J. W. H. Liu. The role of elimination trees in sparse factorization. *SIAM Journal on Matrix Analysis and Applications*, 11:134–172, 1990.

[154] J. W. H. Liu. The multifrontal method for sparse matrix solution: theory and practice. *SIAM Review*, 34:82–109, 1992.

[155] J. W. H. Liu, E. G. Ng, and B. W. Peyton. On finding supernodes for sparse matrix computations. *SIAM Journal on Matrix Analysis and Applications*, 14(1):242–252, 1993.

[156] J. Löfberg. *YALMIP : A Toolbox for Modeling and Optimization in MATLAB*, 2004.

[157] L. Lovász. Normal hypergraphs and the perfect graph conjecture. *Discrete Mathematics*, 2:253–267, 1972.

[158] L. Lovász. On the Shannon capacity of a graph. *IEEE Transactions on Information Theory*, 25:1–7, 1979.

[159] S. H. Low. Convex relaxation of optimal power flow—part I: Formulations and equivalence. *IEEE Transactions on Control of Network Systems*, 1(1):15–27, March 2014.

[160] S. H. Low. Convex relaxation of optimal power flow—part II: Exactness. *IEEE Transactions on Control of Network Systems*, 1(2):177–189, June 2014.

[161] Z. Lu, A. Nemirovski, and R. D.C. Monteiro. Large-scale semidefinite programming via a saddle point mirror-prox algorithm. *Mathematical Programming*, 109(2-3):211–237, 2007.

[162] D. J. C. MacKay. *Information Theory, Inference, and Learning Algorithms*. Cambridge University Press, 2003.

[163] N. V. R. Mahadev and U. N. Peled. *Threshold Graphs and Related Topics*. North-Holland, 1995.

[164] M. Mézard and A. Montanari. *Information, Physics, and Computation*. Oxford University Press, 2009.

[165] D. K. Mohlzahn, J. T. Holzer, B. C. Lesieutre, and C. L. DeMarco. Implementation of a large-scale optimal power flow solver based on semidefinite programming. *IEEE Transactions on Power Systems*, 28(4):3987–3998, 2013.

[166] R. D. C. Monteiro. Primal-dual path-following algorithms for semidefinite programming. *SIAM Journal on Optimization*, 7:663–678, 1997.

[167] J. J. Moreau. Proximité et dualité dans un espace hilbertien. *Bull. Math. Soc. France*, 93:273–299, 1965.

[168] MOSEK ApS. *The MOSEK Optimization Tools Manual. Version 6.0.*, 2010. Available from `www.mosek.com`.

[169] K. Nakata, K. Fujitsawa, M. Fukuda, M. Kojima, and K. Murota. Exploiting sparsity in semidefinite programming via matrix completion II: implementation and numerical details. *Mathematical Programming Series B*, 95:303–327, 2003.

[170] H. Nelis, E. Deprettere, and P. Dewilde. Approximate inversion of positive definite matrices, specified on a multiple band. In *Proceedings SPIE*, volume 975, pages 48–58, 1988.

[171] A. S. Nemirovski and M. J. Todd. Interior-point methods for optimization. *Acta Numerica*, 17:191–234, 5 2008.

[172] Y. Nesterov. Squared functional systems and optimization problems. In J. Frenk, C. Roos, T. Terlaky, and S. Zhang, editors, *High Performance Optimization Techniques*, pages 405–440. Kluwer Academic Publishers, 2000.

[173] Y. Nesterov. Towards non-symmetric conic optimization. *Optimization Methods and Software*, 27(4-5):893–917, 2012.

[174] Yu. Nesterov. Nonsymmetric potential-reduction methods for general cones. Technical Report 2006/34, CORE Discussion Paper, Université catholique de Louvain, 2006.

[175] Yu. Nesterov and A. Nemirovskii. *Interior-point polynomial methods in convex programming*, volume 13 of *Studies in Applied Mathematics*. SIAM, Philadelphia, PA, 1994.

[176] Yu. E. Nesterov and M. J. Todd. Self-scaled barriers and interior-point methods for convex programming. *Mathematics of Operations Research*, 22(1):1–42, 1997.

[177] J. Nocedal and S. J. Wright. *Numerical Optimization*. Springer, 2nd edition, 2006.

[178] B. O'Donoghue, E. Chu, N. Parikh, and S. Boyd. Operater splitting for conic optimization via homogeneous self-dual embedding, 2013. `arxiv.org/abs/1312.3039`.

[179] T. Ohtsuki. A fast algorithm for finding an optimal ordering for vertex elimination on a graph. *SIAM Journal on Computing*, 5(1):133–145, 1976.

[180] T. Ohtsuki, L. K. Cheung, and T. Fujisawa. Minimal triangulation of a graph and optimal pivoting order in a sparse matrix. *Journal of Mathematical Analysis and Applications*, 54:622–633, 1976.

[181] N. Parikh and S. Boyd. Proximal algorithms. *Foundations and Trends in Optimization*, 1(3):123–231, 2013.

[182] N. Parikh and S. Boyd. Block splitting for distributed optimization. *Mathematical Programming Computation*, 6(1):77–102, 2014.

[183] S. Parter. The use of linear graphs in Gauss elimination. *SIAM Review*, 3(2):119–130, 1961.

[184] J. Pearl. *Probabilistic Reasoning in Intelligent Systems*. Morgan Kaufmann, 1988.

[185] F. M. Q. Pereira and J. Palsberg. Register allocation via coloring of chordal graphs. In K. Yi, editor, *Programming Languages and Systems*, pages 315–329. Springer, 2005.

[186] B. W. Peyton. Minimal orderings revisited. *SIAM Journal on Matrix Analysis and Applications*, 23(1):271–294, 2001.

[187] A. Pothen. *Sparse Null Bases and Marriage Theorems*. PhD thesis, Cornell University, 1984.

[188] A. Pothen and C. Sun. Compact clique tree data structures in sparse matrix factorizations. In T. F. Coleman and Y. Li, editors, *Large-Scale Numerical Optimization*, pages 180–204. SIAM, 1990.

[189] J. L. Ramírez Alfonsín and B. A. Reed, editors. *Perfect Graphs*. Wiley, 2001.

[190] J. Renegar. *A Mathematical View of Interior-Point Methods in Convex Optimization*. SIAM, 2001.

[191] N. Robertson and P. D. Seymour. Graph minors. II. Algorithmic aspects of tree-width. *Journal of Algorithms*, 7:309–322, 1986.

[192] R. T. Rockafellar. *Convex Analysis.* Princeton University Press, second edition, 1970.

[193] D. J. Rose. Triangulated graphs and the elimination process. *Journal of Mathematical Analysis and Applications*, 32:597–609, 1970.

[194] D. J. Rose. On simple characterizations of $k$-trees. *Discrete Mathematics*, 7:317–322, 1974.

[195] D. J. Rose, R. E. Tarjan, and G. S. Lueker. Algorithmic aspects of vertex elimination on graphs. *SIAM Journal on Computing*, 5(2):266–283, 1976.

[196] S. H. Schmieta and F. Alizadeh. Extension of primal-dual interior point algorithms to symmetric cones. *Mathematical Programming*, 96:409–438, 2003.

[197] I. J. Schoenberg. Remarks to Maurice Fréchet's article "Sur la définition axiomatique d'une classe d'espaces vectoriels distanciés applicables vectoriellement sur l'espace de Hilbert". *Annals of Mathematics*, 36(3):724–732, 1935.

[198] I. J. Schoenberg. Metric spaces and positive definite functions. *Transactions of the American Mathematical Society*, 44(3):522–536, 1938.

[199] A. Schrijver. *Combinatorial Optimization. Polyhedra and Efficiency.* Springer, 2003.

[200] L. K. Schubert. Modification of a quasi-Newton method for nonlinear equations with a sparse Jacobian. *Mathematics of Computation*, 24:27–30, 1970.

[201] G. Shafer and P. P. Shenoy. Local computation in hypertrees. Technical report, School of Business, University of Kansas, 1988.

[202] D. F. Shanno. On variable-metric methods for sparse Hessians. *Mathematics of Computation*, 34(150):499–514, 1980.

[203] C. E. Shannon. The zero error capacity of a noisy channel. *IEEE Transactions on Information Theory*, 2(3):8–19, 1956.

[204] D. R. Shier. Some aspects of perfect elimination orderings in chordal graphs. *Discrete Applied Mathematics*, 7:325–331, 1984.

[205] A. Skajaa and Y. Ye. A homogeneous interior-point algorithm for nonsymmetric convex conic optimization. *Mathematical Programming*, pages 1–32, 2014.

[206] R. L. Smith. The positive definite completion problem revisited. *Linear Algebra and Its Applications*, 429:1442–1452, 2008.

[207] J. E. Spingarn. Partial inverse of a monotone operator. *Applied Mathematics and Optimization*, 10:247–265, 1983.

[208] J. E. Spingarn. Applications of the method of partial inverses to convex programming: decomposition. *Mathematical Programming*, 32:199–223, 1985.

[209] G. Srijuntongsiri and S. Vavasis. A fully sparse implementation of a primal-dual interior-point potential reduction method for semidefinite programming. 2004. `arXiv:cs/0412009`.

[210] J. F. Sturm. Using SEDUMI 1.02, a Matlab toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11-12:625–653, 1999.

[211] J. F. Sturm. Similarity and other spectral relations for symmetric cones. *Linear Algebra and Its Applications*, 312:135–154, 2000.

[212] Y. Sun, M. S. Andersen, and L. Vandenberghe. Decomposition in conic optimization with partially separable structure. *SIAM Journal on Optimization*, 24:873–897, 2014.

[213] Y. Sun and L. Vandenberghe. Decomposition methods for sparse matrix nearness problems. 2015.

[214] R. E. Tarjan. Applications of path compression on balanced trees. *Journal of the Association for Computing Machinery*, 26(4):690–715, 1979.

[215] R. E. Tarjan and M. Yannakakis. Simple linear-time algorithms to test chordality of graphs, test acyclicity of hypergraphs, and selectively reduce acyclic hypergraphs. *SIAM Journal on Computing*, 13(3):566–579, 1984.

[216] M. J. Todd. A study of search directions in primal-dual interior-point methods for semidefinite programming. *Optimization Methods and Software*, 11(1-4):1–46, 1999.

[217] M. J. Todd. Semidefinite optimization. *Acta Numerica*, 10:515–560, 2001.

[218] K.-C. Toh. Solving large scale semidefinite programs via an iterative solver on the augmented systems. *SIAM Journal on Optimization*, 14(3):670–698, 2003.

[219] P. L. Toint. On sparse and symmetric matrix updating subject to a linear equation. *Mathematics of Computation*, 31(140):954–961, 1977.

[220] P. Tseng. Further applications of a splitting algorithm to decomposition in variational inequalities and convex programming. *Mathematical Programming*, 48:249–263, 1990.

[221] P. Tseng. Applications of a splitting algorithm to decomposition in convex programming and variational inequalities. *SIAM Journal on Control and Optimization*, 29(1):119–138, 1991.

[222] P. Tseng. Dual coordinate ascent methods for non-strictly convex minimization. *Mathematical Programming*, 59:231–247, 1993.

[223] L. Tunçel. *Polyhedral and Semidefinite Programming Methods in Combinatorial Optimization.* The American Mathematical Society, 2010.

[224] R. H. Tütüncü, K. C. Toh, and M. J. Todd. Solving semidefinite-quadratic-linear programs using SDPT3. *Mathematical Programming Series B*, 95:189–217, 2003.

[225] L. Vandenberghe and S. Boyd. A primal-dual potential reduction method for problems involving matrix inequalities. *Mathematical Programming*, 69(1):205–236, July 1995.

[226] L. Vandenberghe and S. Boyd. Semidefinite programming. *SIAM Review*, 38(1):49–95, 1996.

[227] M. J. Wainwright and M. I. Jordan. Graphical models, exponential families, and variational inferencing. *Foundations and Trends in Machine Learning*, 1(1-2):1–305, 2008.

[228] H. Waki, S. Kim, M. Kojima, and M. Muramatsu. Sums of squares and semidefinite program relaxations for polynomial optimization problems with structured sparsity. *SIAM Journal on Optimization*, 17(1):218–241, 2006.

[229] H. Waki, S. Kim, M. Kojima, M. Muramatsu, and H. Sugimoto. Algoritm 883: SparsePOP—a sparse semidefinite programming relaxation of polynomial optimization problems. *ACM Transactions on Mathematical Software*, 35(2):15:1–15:13, 2008.

[230] K. Q. Weinberger and L. K. Saul. Unsupervised learning of image manifolds by semidefinite programming. *International Journal of Computer Vision*, 70:77–90, 2006.

[231] K. Q. Weinberger, F. Sha, Q. Zhu, and L. K. Saul. Graph Laplacian regularization for large-scale semidefinite programming. In B. Schölkopf, J. C. Platt, and T. Hoffman, editors, *Advances in Neural Information Processing Systems 19*, pages 1489–1496, Cambridge, MA, 2007. MIT Press.

[232] Z. Wen. *First-Order Methods for Semidefinite Programming*. PhD thesis, Columbia University, 2009.

[233] Z. Wen, D. Goldfarb, and W. Yin. Alternating direction augmented Lagrangian methods for semidefinite programming. *Mathematical Programming Computation*, 2(3-4):203–230, 2010.

[234] A. S. Willsky. Multiresolution Markov models for signal and image processing. *Proceedings of the IEEE*, 90(8):1396–1458, 2002.

[235] H. Wolkowicz, R. Saigal, and L. Vandenberghe, editors. *Handbook of Semidefinite Programming*, volume 27 of *International Series in Operations Research and Management Science*. Kluwer Academic Publishers, Boston, MA, 2000.

[236] L. Xiao and S. Boyd. Fast linear iterations for distributed averaging. *Systems & Control Letters*, 53(1):65–78, 2004.

[237] L. Xiao, J. Sun, and S. Boyd. A duality view of spectral methods for dimensionality reduction. In *Proceedings of the 23rd International Conference on Machine Learning (ICML 2006)*, 2006.

[238] M. Yamashita, K. Fujisawa, and M. Kojima. Implementation and evaluation of SDPA 6.0 (Semidefinite Programming Algorithm 6.0). *Optimization Methods and Software*, 18(4):491–505, 2003.

[239] N. Yamashita. Sparse quasi-Newton updates with positive definite matrix completion. *Mathematical Programming, Series A*, 115(1):1–30, 2008.

[240] M. Yannakakis. Computing the minimum fill-in is NP-complete. *SIAM Journal on Algebraic and Discrete Methods*, 2(1):77–79, 1981.

[241] Y. Zhang. On extending some primal-dual interior-point algorithms from linear programming to semidefinite programming. *SIAM Journal on Optimization*, 8:365–386, 1998.

[242] X. Zhao, D. Sun, and K.-C. Toh. A Newton-CG augmented Lagrangian method for semidefinite programming. *SIAM Journal on Optimization*, 20(4):1737–1765, 2010.

[243] H. Zhu and G. B. Giannakis. Power system nonlinear state estimation using distributed semidefinite programming. *IEEE Journal of Selected Topics in Signal Processing*, 8(6):1039–1050, 2014.