Multi-Period Trading via Convex Optimization

Stephen Boyd
Stanford University

Enzo Busseti
Stanford University

Steven Diamond
Stanford University

Ronald N. Kahn
Blackrock

Kwangmoo Koh
Blackrock

Peter Nystrup
Technical University of Denmark

Jan Speth
Blackrock

now
the essence of knowledge
Boston — Delft

Foundations and Trends® in Optimization

Published, sold and distributed by:
now Publishers Inc.
PO Box 1024
Hanover, MA 02339
United States
Tel. +1-781-985-4510
www.nowpublishers.com
sales@nowpublishers.com

Outside North America:
now Publishers Inc.
PO Box 179
2600 AD Delft
The Netherlands
Tel. +31-6-51115274

The preferred citation for this publication is

S. Boyd, E. Busseti, S. Diamond, R. N. Kahn, K. Koh, P. Nystrup, J. Speth. *Multi-Period Trading via Convex Optimization*. Foundations and Trends® in Optimization, vol. 3, no. 1, pp. 1–76, 2016.

This Foundations and Trends® issue was typeset in LaTeX using a class file designed by Neal Parikh. Printed on acid-free paper.

ISBN: 978-1-68083-328-7

Foundations and Trends® in Optimization
Volume 3, Issue 1, 2016

Editorial Scope

Topics

Foundations and Trends® in Optimization publishes survey and tutorial articles on methods for and applications of mathematical optimization, including the following topics:

- Algorithm design, analysis, and implementation (especially on modern computing platforms)
- Models and modeling systems
- New optimization formulations for practical problems
- Applications of optimization in:
  - Machine learning
  - Statistics
  - Data analysis
  - Signal and image processing
  - Computational economics and finance
  - Engineering design
  - Scheduling and resource allocation
  - and other areas

Information for Librarians

Foundations and Trends® in Optimization, 2016, Volume 3, 4 issues. ISSN paper version 2167-3888. ISSN online version 2167-3918. Also available as a combined paper and online subscription.

Foundations and Trends® in Optimization
Vol. 3, No. 1 (2016) 1–76

DOI: 10.1561/2400000023

Multi-Period Trading via Convex Optimization

Stephen Boyd
Stanford University
boyd@stanford.edu

Enzo Busseti
Stanford University
ebusseti@stanford.edu

Steven Diamond
Stanford University
stevend2@stanford.edu

Ronald N. Kahn
Blackrock
ron.kahn@blackrock.com

Kwangmoo Koh
Blackrock
kwangmoo.koh@blackrock.com

Peter Nystrup
Technical University of Denmark
pnys@dtu.dk

Jan Speth
Blackrock
jan.speth@blackrock.com

Contents

Abstract

We consider a basic model of multi-period trading, which can be used to evaluate the performance of a trading strategy. We describe a framework for single-period optimization, where the trades in each period are found by solving a convex optimization problem that trades off expected return, risk, transaction cost and holding cost such as the borrowing cost for shorting assets. We then describe a multi-period version of the trading method, where optimization is used to plan a sequence of trades, with only the first one executed, using estimates of future quantities that are unknown when the trades are chosen. The single-period method traces back to Markowitz; the multi-period methods trace back to model predictive control. Our contribution is to describe the single-period and multi-period methods in one simple framework, giving a clear description of the development and the approximations made. In this paper we do not address a critical component in a trading algorithm, the predictions or forecasts of future quantities. The methods we describe in this paper can be thought of as good ways to exploit predictions, no matter how they are made. We have also developed a companion open-source software library that implements many of the ideas and methods described in the paper.

S. Boyd, E. Busseti, S. Diamond, R. N. Kahn, K. Koh, P. Nystrup, J. Speth. *Multi-Period Trading via Convex Optimization*. Foundations and Trends® in Optimization, vol. 3, no. 1, pp. 1–76, 2016.
DOI: 10.1561/2400000023.

1

Introduction

Single and multi-period portfolio selection. Markowitz [54] was the first to formulate the choice of an investment portfolio as an optimization problem trading off risk and return. Traditionally, this was done independently of the cost associated with trading, which can be significant when trades are made over multiple periods [49]. Goldsmith [38] was among the first to consider the effect of transaction cost on portfolio selection in a single-period setting. It is possible to include many other costs and constraints in a single-period optimization formulation for portfolio selection [53, 63].

In multi-period portfolio selection, the portfolio selection problem is to choose a sequence of trades to carry out over a set of periods. There has been much research on this topic since the work of Samuelson [74] and Merton [58, 59]. Constantinides [22] extended Samuelson's discrete-time formulation to problems with proportional transaction costs. Davis and Norman [24] and Dumas and Lucian [30] derived similar results for the continuous-time formulation. Transaction costs, constraints, and time-varying forecasts are more naturally dealt with in a multi-period setting. Following Samuelson and Merton, the literature on multi-period portfolio selection is predominantly based on dynamic programming [5, 9], which properly takes into account the idea of recourse and updated information available as the sequence of trades are chosen (see [37] and references therein). Unfortunately, actually carrying out dynamic programming for trade selection is impractical, except for some very special or small cases, due to the 'curse of dimensionality' [72, 11]. As a consequence, most studies include only a very limited number of assets and simple objectives and constraints. A large literature studies multi-period portfolio selection in the absence of transaction cost (see, *e.g.*, [18] and references therein); in this special case, dynamic programming is tractable.

For practical implementation, various approximations of the dynamic programming approach are often used, such as approximate dynamic programming, or even simpler formulations that generalize the single-period formulations to multi-period optimization problems [11]. We will focus on these simple multi-period methods in this paper. While these simplified approaches can be criticized for only approximating the full dynamic programming trading policy, the performance loss is likely very small in practical problems, for several reasons. In [11], the authors developed a numerical bounding method that quantifies the loss of optimality when using a simplified approach, and found it to be very small in numerical examples. But in fact, the dynamic programming formulation is itself an approximation, based on assumptions (like independent or identically distributed returns) that need not hold well in practice, so the idea of an 'optimal strategy' itself should be regarded with some suspicion.

Why now? What is different now, compared to 10, 20, or 30 years ago, is vastly more powerful computers, better algorithms, specification languages for optimization, and access to much more data. These developments have changed how we can use optimization in multi-period investing. In particular, we can now quickly run full-blown optimization, run multi-period optimization, and search over hyper-parameters in back-tests. We can run end-to-end analyses, indeed many at a time in parallel. Earlier generations of investment researchers, relying on computers much less powerful than today, relied more on separate models and analyses to estimate parameter values, and tested signals using simplified (usually unconstrained) optimization.

Goal. In this tutorial paper we consider multi-period investment and trading. Our goal is to describe a simple model that takes into account the main practical issues that arise, and several simple and practical frameworks based on solving convex optimization problems [13] that determine the trades to make. We describe the approximations made, and briefly discuss how the methods can be used in practice. Our methods do not give a complete trading system, since we leave a critical part unspecified: Forecasting future returns, volumes, volatilities, and other important quantities (see, *e.g.*, [42]). This paper describes good practical methods that can be used to trade, given forecasts.

The optimization-based trading methods we describe are practical and reliable when the problems to be solved are convex. Real-world single-period convex problems with thousands of assets can be solved using generic algorithms in well under a second, which is critical for evaluating a proposed algorithm with historical or simulated data, for many values of the parameters in the method.

Outline. We start in chapter 2 by describing a simple model of multi-period trading, taking into account returns, trading costs, holding costs, and (some) corporate actions. This model allows us to carry out simulation, used for what-if analyses, to see what would have happened under different conditions, or with a different trading strategy. The data in simulation can be realized past data (in a *back-test*) or simulated data that did not occur, but could have occurred (in a *what-if simulation*), or data chosen to be particularly challenging (in a *stress-test*). In chapter 3 we review several common metrics used to evaluate (realized or simulated) trading performance, such as active return and risk with respect to a benchmark.

We then turn to optimization-based trading strategies. In chapter 4 we describe *single-period optimization* (SPO), a simple but effective framework for trading based on optimizing the portfolio performance over a single period. In chapter 5 we consider *multi-period optimiza-*

tion (MPO), where the trades are chosen by solving an optimization problem that covers multiple periods in the future.

Contribution. Most of the material that appears in this paper has appeared before, in other papers, books, or EE364A, the Stanford course on convex optimization. Our contribution is to collect in one place the basic definitions, a careful description of the model, and discussion of how convex optimization can be used in multi-period trading, all in a common notation and framework. Our goal is not to survey all the work done in this and related areas, but rather to give a unified, self-contained treatment. Our focus is not on theoretical issues, but on practical ones that arise in multi-period trading. To further this goal, we have developed an accompanying open-source software library implemented in Python, and available at

`https://github.com/cvxgrp/cvxportfolio`.

Target audience. We assume that the reader has a background in the basic ideas of quantitative portfolio selection, trading, and finance, as described for example in the books by Grinold & Kahn [42], Meucci [60], or Narang [65]. We also assume that the reader has seen some basic mathematical optimization, specifically convex optimization [13]. The reader certainly does not need to know more than the very basic ideas of convex optimization, for example the overview material covered in chapter 1 of [13]. In a nutshell, our target reader is a quantitative trader, or someone who works with or for, or employs, one.

References

[1] R. Almgren. High frequency volatility. Available at `http://cims.nyu.edu/~almgren/timeseries/notes7.pdf`, 2009.

[2] R. Almgren and N. Chriss. Optimal execution of portfolio transactions. *Journal of Risk*, 3(2):5–39, 2001.

[3] C. Bacon. *Practical portfolio performance measurement and attribution*. Wiley, 2nd edition, 2008.

[4] D. Bailey, J. Borwein, M. Lopez de Prado, and Q. Zhu. The probability of backtest overfitting. *Journal of Computational Finance*, 20(4):39–69, 2017.

[5] R. Bellman. Dynamic programming and Lagrange multipliers. *Proceedings of the National Academy of Sciences*, 42(10):767–769, 1956.

[6] A. Bemporad. Model predictive control design: New trends and tools. In *Proceedings of the 45th IEEE Conference on Decision and Control*, pages 6678–6683. IEEE, 2006.

[7] A. Bemporad, L. Bellucci, and T. Gabbriellini. Dynamic option hedging via stochastic model predictive control based on scenario simulation. *Quantitative Finance*, 14(10):1739–1751, 2014.

[8] N. Bershova and D. Rakhlin. The non-linear market impact of large trades: Evidence from buy-side order flow. *Quantitative Finance*, 13(11):1759–1778, 2013.

[9] D. Bertsekas. *Dynamic Programming and Optimal Control*. Athena Scientific, 1995.

[10] F. Black. Studies of stock price volatility changes. In *Proceedings of the 1976 Meetings of the American Statistical Association, Business and Economic Section*, pages 177–181. American Statistical Association, 1976.

[11] S. Boyd, M. Mueller, B. O'Donoghue, and Y. Wang. Performance bounds and suboptimal policies for multi-period investment. *Foundations and Trends in Optimization*, 1(1):1–72, 2014.

[12] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein. Distributed optimization and statistical learning via the alternating direction method of multipliers. *Foundations and Trends in Machine Learning*, 3(1):1–122, 2011.

[13] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[14] E. Busseti and S. Boyd. Volume weighted average price optimal execution. *arXiv preprint arXiv:1509.08503*, 2015.

[15] E. Busseti, S. Diamond, S. Boyd, and Blackrock. `CVXPortfolio`, 2017. Available at `https://github.com/cvxgrp/cvxportfolio`.

[16] E. Busseti, E. Ryu, and S. Boyd. Risk-constrained Kelly gambling. *Journal of Investing*, 25(3), 2016.

[17] J. Campbell, A. Lo, and A. MacKinlay. *The Econometrics of Financial Markets*, volume 1. Princeton University Press, 1997.

[18] J. Campbell and L. Viceira. *Strategic Asset Allocation: Portfolio Choice for Long-Term Investors*. Oxford University Press, 2002.

[19] L. Chan, J. Karceski, and J. Lakonishok. On portfolio optimization: Forecasting covariances and choosing the risk model. *The Review of Financial Studies*, 12(5):937–974, 1999.

[20] E. Cho, K. Thoney, T. Hodgson, and R. King. Supply chain planning: Rolling horizon scheduling of multi-factory supply chains. In *Proceedings of the 35th conference on Winter simulation: driving innovation*, pages 1409–1416. Winter Simulation Conference, 2003.

[21] V. Chopra and W. Ziemba. The effect of errors in means, variances, and covariances on optimal portfolio choice. *Journal of Portfolio Management*, 19(2):6–11, 1993.

[22] G. Constantinides. Multiperiod consumption and investment behavior with convex transactions costs. *Management Science*, 25(11):1127–1137, 1979.

[23] G. Cornuejols and R. Tütüncü. *Optimization Methods in Finance*. Cambridge University Press, 2006.

[24] M. Davis and A. Norman. Portfolio selection with transaction costs. *Mathematics of Operations Research*, 15(4):676–713, 1990.

[25] V. DeMiguel, L. Garlappi, F. Nogales, and R. Uppal. A generalized approach to portfolio optimization: Improving performance by constraining portfolio norms. *Management Science*, 55(5):798–812, 2009.

[26] V. DeMiguel, L. Garlappi, and R. Uppal. Optimal versus naive diversification: How inefficient is the $1/N$ portfolio strategy? *Review of Financial Studies*, 22(5):1915–1953, 2009.

[27] S. Diamond and S. Boyd. CVXPY: A Python-embedded modeling language for convex optimization. *Journal of Machine Learning Research*, 17(83):1–5, 2016.

[28] S. Diamond, R. Takapoui, and S. Boyd. A general system for heuristic solution of convex problems over nonconvex sets. *Optimization Methods and Software*, to appear, 2017.

[29] A. Domahidi, E. Chu, and S. Boyd. ECOS: An SOCP solver for embedded systems. In *European Control Conference (ECC)*, pages 3071–3076, 2013.

[30] B. Dumas and E. Luciano. An exact solution to a dynamic portfolio choice problem under transactions costs. *Journal of Finance*, 46(2):577–595, 1991.

[31] E. Elton, M. Gruber, and C. Blake. Survivor bias and mutual fund performance. *The Review of Financial Studies*, 9(4):1097–1120, 1996.

[32] F. Fabozzi, D. Huang, and G. Zhou. Robust portfolios: Contributions from operations research and finance. *Annals of Operations Research*, 176(1):191–220, 2010.

[33] B. Fastrich, S. Paterlini, and P. Winker. Constructing optimal sparse portfolios using regularization methods. *Computational Management Science*, 12(3):417–434, 2015.

[34] C. Fougner and S. Boyd. Parameter selection and pre-conditioning for a graph form solver. In R. Tempo, S. Yurkovich, and P. Misra, editors, *Emerging Applications of Control and System Theory*. Springer, 2017. To appear.

[35] J. Friedman, T. Hastie, and R. Tibshirani. *The Elements of Statistical Learning*. Springer, 2001.

[36] M. Frittelli and E. Gianin. Putting order in risk measures. *Journal of Banking & Finance*, 26(7):1473–1486, 2002.

[37] N. Gârleanu and L. Pedersen. Dynamic trading with predictable returns and transaction costs. *Journal of Finance*, 68(6):2309–2340, 2013.

[38] D. Goldsmith. Transactions costs and the theory of portfolio selection. *Journal of Finance*, 31(4):1127–1139, 1976.

[39] C. Gomes and H. Waelbroeck. Is market impact a measure of the information value of trades? Market response to liquidity vs. informed metaorders. *Quantitative Finance*, 15(5):773–793, 2015.

[40] M. Grant, S. Boyd, and Y. Ye. Disciplined convex programming. In *Global Optimization: From Theory to Implementation*, Nonconvex Optimization and its Applications, pages 155–210. Springer, 2006.

[41] R. Grinold. A dynamic model of portfolio management. *Journal of Investment Management*, 4(2):5–22, 2006.

[42] R. Grinold and R. Kahn. *Active Portfolio Management: A Quantitative Approach for Providing Superior Returns and Controlling Risk*. McGraw-Hill, 2nd edition, 1999.

[43] F. Herzog, G. Dondi, and H. Geering. Stochastic model predictive control and portfolio optimization. *International Journal of Theoretical and Applied Finance*, 10(2):203–233, 2007.

[44] M. Ho, Z. Sun, and J. Xin. Weighted elastic net penalized mean–variance portfolio design and computation. *SIAM Journal on Financial Mathematics*, 6(1):1220–1244, 2015.

[45] R. Jagannathan and T. Ma. Risk reduction in large portfolios: Why imposing the wrong constraints helps. *Journal of Finance*, 58(4):1651–1683, 2003.

[46] P. Jorion. International portfolio diversification with estimation risk. *Journal of Business*, 58(3):259–278, 1985.

[47] R. Kan and G. Zhou. Optimal portfolio choice with parameter uncertainty. *Journal of Financial and Quantitative Analysis*, 42(3):621–656, 2007.

[48] J. Kelly, Jr. A new interpretation of information rate. *IRE Transactions on Information Theory*, 2(3):185–189, 1956.

[49] P. Kolm, R. Tütüncü, and F. Fabozzi. 60 years of portfolio optimization: Practical challenges and current trends. *European Journal of Operational Research*, 234(2):356–371, 2014.

[50] W. Kwon and S. Han. *Receding Horizon Control: Model Predictive Control for State Models*. Springer-Verlag London, 2006.

[51] J. Li. Sparse and stable portfolio selection with parameter uncertainty. *Journal of Business & Economic Statistics*, 33(3):381–392, 2015.

[52] F. Lillo, J. Farmer, and R. Mantegna. Econophysics: Master curve for price-impact function. *Nature*, 421(6919):129–130, 2003.

[53] M. Lobo, M. Fazel, and S. Boyd. Portfolio optimization with linear and fixed transaction costs. *Annals of Operations Research*, 152(1):341–365, 2007.

[54] H. Markowitz. Portfolio selection. *Journal of Finance*, 7(1):77–91, 1952.

[55] J. Mattingley and S. Boyd. CVXGEN: A code generator for embedded convex optimization. *Optimization and Engineering*, 13(1):1–27, 2012.

[56] J. Mattingley, Y. Wang, and S. Boyd. Receding horizon control: Automatic generation of high-speed solvers. *IEEE Control Systems Magazine*, 31(3):52–65, 2011.

[57] W. McKinney. *Python for Data Analysis: Data Wrangling with Pandas, NumPy, and IPython.* O'Reilly Media, 2012.

[58] R. Merton. Lifetime portfolio selection under uncertainty: The continuous-time case. *Review of Economics and Statistics*, 51(3):247–257, 1969.

[59] R. Merton. Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3(4):373–413, 1971.

[60] A. Meucci. *Risk and Asset Allocation.* Springer, 2005.

[61] A. Meucci. Historical scenarios with fully flexible probabilities. *GARP Risk Professional*, pages 47–51, 2010.

[62] R. Michaud. The Markowitz optimization Enigma: Is 'optimized' optimal? *Financial Analysts Journal*, 45(1):31–42, 1989.

[63] C. Moallemi and M. Sağlam. Dynamic portfolio choice with linear rebalancing rules. *Journal of Financial and Quantitative Analysis*, 52:1247–1278, 2017.

[64] E. Moro, J. Vicente, L. Moyano, A. Gerig, J. Farmer, G. Vaglica, F. Lillo, and R. Mantegna. Market impact and trading profile of hidden orders in stock markets. *Physical Review E*, 80(6):066102, 2009.

[65] R Narang. *Inside the Black Box: A Simple Guide to Quantitative and High Frequency Trading.* John Wiley & Sons, 2013.

[66] Y. Nesterov and A. Nemirovskii. *Interior-point Polynomial Algorithms in Convex Programming.* SIAM, 1994.

[67] P. Nystrup, H. Madsen, and E. Lindström. Dynamic portfolio optimization across hidden market regimes. Working paper, Technical University of Denmark, 2016.

[68] A. Obizhaeva and J. Wang. Optimal trading strategy and supply/demand dynamics. *Journal of Financial Markets*, 16(1):1–32, 2013.

[69] B. O'Donoghue, E. Chu, N. Parikh, and S. Boyd. Conic optimization via operator splitting and homogeneous self-dual embedding. *Journal of Optimization Theory and Applications*, 169(3):1042–1068, June 2016.

[70] A. Perold. Large-scale portfolio optimization. *Management Science*, 30(10):1143–1160, 1984.

[71] A. Perold. The implementation shortfall: Paper versus reality. *Journal of Portfolio Management*, 31:106, 1998.

[72] W. Powell. *Approximate Dynamic Programming: Solving the Curses of Dimensionality*. John Wiley & Sons, 2007.

[73] Quandl. WIKI end-of-day data, 2016. Available at `https://www.quandl.com/data/WIKI`.

[74] P. Samuelson. Lifetime portfolio selection by dynamic stochastic programming. *Review of Economics and Statistics*, 51(3):239–246, 1969.

[75] W. Sharpe. Mutual fund performance. *Journal of Business*, 39(1):119–138, 1966.

[76] W. Sharpe. The arithmetic of active management. *Financial Analysts Journal*, 47(1):7–9, 1991.

[77] W. Sharpe. The Sharpe ratio. *The Journal of Portfolio Management*, 21(1):49–58, 1994.

[78] R. Tibshirani. Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society*, pages 267–288, 1996.

[79] M. Udell, K. Mohan, D. Zeng, J. Hong, S. Diamond, and S. Boyd. Convex optimization in Julia. *SC14 Workshop on High Performance Technical Computing in Dynamic Languages*, 2014.