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Contents

Multi-agent Online Optimization

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ABSTRACT

This monograph provides an overview of distributed online optimization in multi-agent systems. Online optimization approaches planning and decision problems from a robust learning perspective, where one learns through feedback from sequentially arriving costs, resembling a game between a learner (agent) and the environment. Recently, multi-agent systems have become important in diverse areas including smart power grids, communication networks, machine learning, and robotics, where agents work with decentralized data, costs, and decisions to collectively minimize a system-wide cost. In such settings, agents make distributed decisions and collaborate with neighboring agents through a communication network, leading to scalable solutions that often perform as well as centralized methods. The monograph offers a unified introduction, starting with fundamental algorithms for basic problems, and gradually covering state-of-the-art techniques for more complex settings. The interplay between individual agent learning rates, network structure, and communication complexity is highlighted in the overall system performance.

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1

Introduction

1.1 Online Optimization

Online optimization treats the optimization process as one where data and cost functions are introduced sequentially. This approach traces back to classical work on sequential decision-making, particularly in multi-armed bandit problems. Recently, online optimization has become an important tool in machine learning, addressing problems such as recommender systems and spam filtering [38]. In this framework, decisions are made to optimize time-varying cost functions, with the process evolving through feedback, allowing the learner to improve over time. Performance is typically evaluated against a static optimal decision that could have been made in hindsight. Formally, online optimization is modeled as a game between a learner and an adversary, played over a finite time horizon $t = 1, \ldots, T$.

Online Optimization Paradigm [77]

Initialize $\mathcal{X}$ as a convex subset of $\mathbb{R}^d$.
For $t = 1, \ldots, T$, DO

(i) The adversary selects a convex cost function $\ell_t(\cdot) : \mathcal{X} \subseteq \mathbb{R}^d \to \mathbb{R}$ and keeps it to itself;

(ii) The learner makes a decision $\mathbf{x}_t \in \mathcal{X}$;

(iii) The learner suffers a loss $\ell_t(\mathbf{x}_t)$, and receives the cost function $\ell_t(\cdot)$ (full information), or just the value of the loss $\ell_t(\mathbf{x}_t)$ (bandit information).

In classical optimization (i.e., a classical learner), the loss function $\ell_t(\cdot)$ at step t is revealed before the learner attempts to minimize it. In contrast, online optimization acknowledges the difficulty in knowing $\ell_t(\cdot)$ or even a model of it before decisions are made. The learner receives information about $\ell_t(\cdot)$ after she has taken a decision and this information can be the whole function, a scenario referred to as *full information*; or the learner only experiences losses at selected decisions, and in this case, we talk about *bandit information*. The loss functions $\ell_t(\cdot)$ are generally assumed to be arbitrary (but chosen from a given function class). Hence, it is impossible for the learner to infer $\ell_t(\cdot)$ before the decisions are made. As a result, it is sensible for the learner to identify $\mathbf{x}_1, \ldots, \mathbf{x}_T \in \mathcal{X}$ so that the *regret*, i.e.,

$$\texttt{Reg}(T) := \sum_{t=1}^{T} \ell_t(\mathbf{x}_t) - \min_{\mathbf{x} \in \mathcal{X}} \sum_{t=1}^{T} \ell_t(\mathbf{x}) \tag{1.1}$$

is minimized. In the above definition, $\min_{\mathbf{x} \in \mathcal{X}} \sum_{t=1}^{T} \ell_t(\mathbf{x})$ is the minimal accumulated loss of an oracle making a static decision to whom all $\ell_t(\cdot)$ are known before $t = 1$. Therefore, $\texttt{Reg}(T)$ represents the difference between the actual accumulated loss experienced by the learner compared to that of such an oracle.

The central premise of online optimization lies in the possibility of achieving an infinitesimal regret on average in the asymptotic sense, i.e.,

$$\texttt{Reg}(T)/T = o(1)$$

for carefully crafted algorithms as T grows large. Typically, one may achieve $\texttt{Reg}(T) = \mathcal{O}(T^\alpha)$ for some $0 < \alpha < 1$, implying a robust learning process.

1.2 Multi-agent Optimization

Multi-agent optimization arises from emerging applications in smart grids, machine learning, robotics, etc., where data and decisions are spread over physically separated subsystems represented by agents. The agents are interconnected through a communication network, aiming to minimize a system-level cost [12]. In the simplest form, the communication network is modeled as an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a finite set of indices representing agents, and $\{i, j\} \in \mathcal{E}$ indicates that nodes i and j can communicate.

Example 1.1. (Optimal Power Flow [30]) Consider an electrical network with n nodes indexed in $\mathcal{V} = \{1, \dots, n\}$. Let $\mathbf{v}_i \in \mathbb{C}$ and $\mathbf{i}_i \in \mathbb{C}$ be the voltage and inflow current at node i. The network structure is captured by an admittance matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$. Then $\mathbf{x}_i := \mathrm{Re}(\mathbf{v}_i\mathbf{i}_i^\dagger)$ defines the active power at node i, where † is the complex conjugate. Let $\ell_i(\mathbf{x}_i)$ denote the cost associated with the power allocation at node i. An optimal power flow problem seeks to minimize the cost of electric power generation while satisfying operating constraints:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \sum_{i=1}^{n} \ell_i(\mathbf{x}_i) \\ \text{s. t.} \quad & \mathbf{x}_i = \mathrm{Re}\left(\mathbf{v}_i\mathbf{i}_i^\dagger\right),\ i = 1, \dots, n \\ & \mathbf{v}_i\mathbf{i}_i^\dagger = \mathbf{v}_i \sum_{j=1}^{n} \mathbf{A}_{ij}^\dagger \mathbf{v}_j^\dagger,\ i = 1, \dots, n. \end{aligned} \tag{1.2}$$

Example 1.2. (Collaborative Learning [29]) Consider n data owners indexed in $\mathcal{V} = \{1, \dots, n\}$. Each data owner i holds a private data set $\{(\mathbf{y}_{ik}, \mathbf{z}_{ik}) : k = 1, \dots, K_i\}$, where $\mathbf{y}_{ik}$ represents input data, and $\mathbf{z}_{ik}$ is the corresponding label. When $\mathbf{x}_i$ is a local model for the learning representation, the local cost for agent i is

$$\ell_i(\mathbf{x}_i) = \sum_{k=1}^{K_i} g(\mathbf{x}_i; \mathbf{y}_{ik}, \mathbf{z}_{ik}),$$

where g is a loss function quantifying the accuracy of the local models. A collaborative learning problem is then described by

$$\begin{aligned} \min_{\mathbf{x}} \; & \sum_{i=1}^{n} \ell_i(\mathbf{x}_i) \\ \text{s. t.} \; & \mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_n \end{aligned} \tag{1.3}$$

where the agents collectively train a model from all data sets.

Example 1.3. (Multi-robot Localization and Mapping [89]) Consider a team of robots indexed in $\mathcal{V} = \{1, \ldots, n\}$. Each robot i is modeled as a 3D rigid body described by a pose $\mathbf{x}_i$. A conventional representation of $\mathbf{x}_i$ is via a 4×4 transformation matrix that combines the translation of the center of the mass $x_i \in \mathbb{R}^3$ and the rotation of the robot body $R_i \in \mathrm{SO}(3)$:

$$\mathbf{x}_i := \begin{bmatrix} R_i & x_i \\ 0_{3\times 1} & 1 \end{bmatrix}.$$

There are also k features $m_1, \ldots, m_k$ used as landmarks for the environment. Each robot i observes its relative pose y_{is} to landmark s. The robots aim to solve the following optimization problem

$$\min_{\mathbf{x}, m_1, \ldots, m_k} \sum_{i=1}^{n} \sum_{s=1}^{k} \|\ell_i(\mathbf{x}_i, m_s) - y_{is}\|_2^2 \tag{1.4}$$

to estimate the robot poses and map features in the ground frame. Here $\ell_i(\mathbf{x}_i, m_s)$ is the measurement model. For example, if the observation y_{is} is from a monocular camera [100], we have

$$\ell_i(\mathbf{x}_i, m_s) = R_i^\top (m_s - x_i) / \|R_i^\top (m_s - x_i)\|_2.$$

The premise of multi-agent optimization lies in the possibility that all agents in $\mathcal{V}$, only knowing their local cost functions, solve the network-level optimization problem by exchanging their decisions with neighbors (i.e., nodes that share a link) over the communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. For a large-scale network where nodes are physically separated, multi-agent optimization brings scalability while providing global optimality.

1.3 Multi-agent Online Optimization

The multi-agent online optimization attempts to combine the strengths of online optimization and multi-agent optimization, creating robust and scalable optimization frameworks for complex multi-agent systems. The agents in $\mathcal{V} = \{1, \ldots, n\}$ experience local and sequential losses $\ell_{i,t}(\cdot)$ for $t = 1, \ldots, T$, and they locally implement and exchange their local decisions $\mathbf{x}_i(t) \in \mathcal{X}$ with neighbors over the communication graph $\mathcal{G}$. Now the network-level goal is to minimize the accumulated system-wide loss, defined as the worst possible regret among agents:

$$\texttt{SReg}(T) := \max_{i \in \mathcal{V}} \left[\sum_{t=1}^{T} \sum_{j=1}^{n} \ell_{j,t}(\mathbf{x}_i(t)) - \sum_{t=1}^{T} \sum_{j=1}^{n} \ell_{j,t}(\mathbf{x}^\star) \right] \tag{1.5}$$

where $\mathbf{x}^\star = \arg\min_{\mathbf{x} \in \mathcal{X}} \sum_{t=1}^{T} \sum_{j=1}^{n} \ell_{j,t}(\mathbf{x})$ is the system-level decision taken by a static optimal oracle.

Multi-agent Online Convex Optimization

Initialize $\mathcal{X}$ as a convex subset of $\mathbb{R}^d$.

For $t = 1, \ldots, T$, agents in $\mathcal{V}$ DO

- Each agent $i \in \mathcal{V}$ selects $\mathbf{x}_i(t) \in \mathcal{X}$, and a local adversary chooses $\ell_{i,t}(\cdot) : \mathbb{R}^d \to \mathbb{R}$ as a convex cost function;
- Each agent experiences a loss $\ell_{i,t}(\mathbf{x}_i(t))$;
- The function $\ell_{i,t}$ is revealed to agent i;
- The decisions of the neighbors of the agent i are revealed to i from the communication network $\mathcal{G}$, i.e., $\mathbf{x}_j(t)$ for $j \in \mathcal{N}_i := \{j : \{i, j\} \in \mathcal{E}\}$.

The first challenge in multi-agent online optimization is whether and how the robustness of online optimization and the scalability of distributed multi-agent optimization comply with each other in the algorithm design. Multi-agent online optimization algorithms delicately adapt existing online optimization algorithms to the new decentralized settings, and the achievable global performance $\mathtt{SReg}(T)$ depends on the learning rate, network structure, and communication complexity, leading to new challenges in performance evaluations.

1.4 Scope and Organization

We present a unified introduction to the state-of-the-art distributed optimization algorithms for multi-agent systems under full information or bandit information feedback. We also provide a full and self-contained analysis for their achievable regret bounds, where distributed decisions planned and executed by agents over a communication graph provide scalable solutions with performances often matching their centralized counterparts. For the majority of the monograph, we adopt a simple problem setting, where the cost functions are continuously differentiable, and the communication graph does not change over time. This allows us to present the fundamental algorithms and their analysis as directly as possible. We then move to more complex settings and report the state-of-the-art results in the literature.

The remainder of the monograph is organized as follows. Section 2 presents the technical preliminaries on graph theory, distributed multi-agent optimization, and online optimization that will be used in subsequent discussions. Section 3 presents three fundamental distributed online gradient-based algorithms under full information feedback: the Distributed Online Gradient Descent, Distributed Online Mirror Descent, and Distributed Online Dual Averaging algorithms. Section 4 then moves to bandit feedback and presents the bandit variations of the three algorithms, where gradient estimators allow gradient descent without access to the true gradient information. Sections 5 and 6 explore two important extensions of the basic problem settings on long-term constraints and online linear regressions, respectively. Finally, Sections 7 and 8 move to recently reported results for compressed node-to-node communications and dynamic networks.

1.5 Notation and Basic Definitions

We denote scalars and vectors with lowercase letters (e.g., a) and boldface letters (e.g., $\mathbf{x}$), respectively. We use $\mathbb{R}$ to denote the set of real numbers, $\mathbb{R}_+$ to denote the set of nonnegative real numbers. The set of d-dimensional real vectors is denoted $\mathbb{R}^d$. We use $\mathbb{R}^d_+$ to denote the nonnegative orthant, i.e., $\mathbb{R}^d_+ = \{\mathbf{x} \in \mathbb{R}^d \mid [\mathbf{x}]_i \geq 0, i = 1, \ldots, d\}$. The i-th element of a vector $\mathbf{x}$ is denoted $[\mathbf{x}]_i$. The set of real $n \times n$ matrices is denoted $\mathbb{R}^{n\times n}$. We use $\|\mathbf{x}\|_2$ and $\|\mathbf{x}\|_1$ to denote the Euclidean (or ℓ_2) norm and ℓ_1 norm of a vector $\mathbf{x} \in \mathbb{R}^d$, respectively; for the Euclidean norm, we omit the subscript when it is clear from the context. A generic norm of a vector is denoted by $\|\mathbf{x}\|$ and its dual norm is defined by $\|\mathbf{x}\|_* = \sup_{\|\mathbf{y}\|=1} \mathbf{x}^\top \mathbf{y}$. The definition of the dual norm immediately implies $\mathbf{x}^\top \mathbf{y} \leq \|\mathbf{x}\|\|\mathbf{y}\|_*$. We use $[N]$ to denote the set $\{1, \ldots, N\}$ for any $N \geq 2$. We denote $\mathbb{B}_R = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_2 \leq R\}$ with $0 < R < \infty$. Denote $\mathbf{0}$, $\mathbf{1}$, and $\mathbf{I}$ as the all-zero vector, the all-one vector, and the identity matrix, respectively, where their dimensions are implied in the context.

A set $\mathcal{X}$ is convex if for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$ and any $\theta \in [0, 1]$, we have $\theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2 \in \mathcal{X}$. When $\mathcal{X}$ is closed, $\mathcal{P}_{\mathcal{X}}(\mathbf{x})$ denotes the Euclidean projection of point $\mathbf{x}$ onto a convex set $\mathcal{X}$, i.e., $\mathcal{P}_{\mathcal{X}}(\mathbf{x}) = \arg\min_{\mathbf{y}\in\mathcal{X}} \|\mathbf{y} - \mathbf{x}\|_2$. A function $\ell : \mathbb{R}^d \to \mathbb{R}$ is convex if its domain is a convex set and for any $\mathbf{x}_1$ and $\mathbf{x}_2$ in the domain and $\theta \in [0, 1]$, we have

$$\ell(\theta\mathbf{x}_1 + (1-\theta)\mathbf{x}_2) \leq \theta\ell(\mathbf{x}_1) + (1-\theta)\ell(\mathbf{x}_2).$$

When ℓ is differentiable, it is convex if and only if for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$,

$$\ell(\mathbf{x}_1) \geq \ell(\mathbf{x}_2) + \nabla\ell(\mathbf{x}_2)^\top(\mathbf{x}_1 - \mathbf{x}_2).$$

A function is called G-Lipschitz over $\mathcal{X}$ with respect to a norm $\|\cdot\|$ if for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$ we have

$$|\ell(\mathbf{x}_1) - \ell(\mathbf{x}_2)| \leq G\|\mathbf{x}_1 - \mathbf{x}_2\|.$$

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