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Integer Programming Games

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Integer Programming Games

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ABSTRACT

We provide a comprehensive survey of Integer Programming Games (IPGs), focusing on both simultaneous games and bilevel programs. These games are characterized by integral constraints within the players' strategy sets. We start from the fundamental definitions of these games and various solution concepts associated with them, and derive the properties of the games and the solution concepts. For each of the two types of games – simultaneous and bilevel – we have one section dedicated to the analysis of the games and another section dedicated to the development and analyses of algorithms to solve them. The analyses sections present results on the computational complexity of the general game as well as various other restricted versions. These sections also discuss the structural properties of the games and the equilibrium concepts associated with them. The algorithm sections, in contrast, present some of the state-of-the-art algorithms developed to solve these games, either exactly, approximately or fast under fixed-parameter assumptions.

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These sections also contain proofs of the correctness of these algorithms and an assessment of their theoretical run times in the worst-case scenario.

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Preface

Game theory has been a powerful tool to model the strategic interaction between multiple players when their objectives are in conflict. The domain has a long history in the economics literature, where this has been used to model the behavior of firms. One of the foundational results in this area is the existence of Nash equilibria in finite games, which was proved by Nash in the 1950s. Since then, a variety of games have been studied in the literature, including games where players make moves sequentially, games with uncertainty in payoffs, games with asymmetry in information between players, and so on.

In the traditional sense, a handful of strategies are considered, with the payoffs described for every combination of strategies picked by the followers. Many standard references [37, 51, for example] provide a comprehensive overview of such games and the solution concepts associated with them. This is where mathematical programming games, in particular, integer programming games, take a different approach. We do not restrict ourselves to a handful of strategies or even set of strategies that can be *nicely described* (for example, an interval). Instead, we allow each player's set of actions to be described by a set of constraints – for example, as mixed-integer points in a polyhedron. They warrant attention in many applications, where strategic decision makers are *actually* solving optimization problems to identify their decisions, and their payoffs are also determined by the actions of other similar

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strategic players. Thus, it becomes important to identify various types of equilibria in these games, and to develop algorithms to compute them.

This survey focuses on optimization-based approaches to study such games and the solution concepts associated with them. In particular, the focus is towards two classes of games – simultaneous games and bilevel programs. For each of these families of games, which has developed a rich literature over the past decade, we provide a set of analytical results, which helps the reader better understand the structure of the problem. These results, where possible, are also interspersed with a collection of examples that assist in understanding the concepts better. We also comment, where possible, on the computational complexity of determining various solution concepts in these games. Following such analysis, we also provide a set of algorithms that have been developed to solve these games, and provide a theoretical analysis of their performance.

We also note that we have been selective about certain algorithms, (i) based on their simplicity and (ii) with a motivation to capture a large variety of ideas, rather than a complete deep dive on a single family of algorithms. We believe that a more advanced reader can use this survey as a stepping stone to dive deeper into the literature on this topic.

Finally, we note that some topics, which could fall under the broad ambit of games are not considered in this survey. We have not provided any result in the context of cooperative games. We have also not analyzed multi-level games, or even bilevel games where there are more than one "leader" or more than one "follower." We have not considered games with any uncertainty or asymmetric information between players. The survey focuses exclusively on deterministic games with complete information. We have also not considered special families of simultaneous or bilevel games. For example, bilevel knapsack games define a very active area of research, but we do not provide much attention to such special cases, and restrict ourselves to the general setting.

With that, we believe that this survey will help the reader understand the landscape of integer programming games, and provide a stepping stone for further research in this area.

Margarida, Gabriele, Andrea and Sriram
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