Solving Free-boundary Problems with Applications in Finance

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Foundations and Trends® in Stochastic Systems

Published, sold and distributed by:
now Publishers Inc.
PO Box 1024
Hanover, MA 02339
USA
Tel. +1-781-985-4510
www.nowpublishers.com
sales@nowpublishers.com

Outside North America:
now Publishers Inc.
PO Box 179
2600 AD Delft
The Netherlands
Tel. +31-6-51115274

The preferred citation for this publication is K. Muthuraman and S. Kumar, Solving Free-boundary Problems with Applications in Finance, Foundation and Trends® in Stochastic Systems, vol 1, no 4, pp 259–341, 2006

ISBN: 978-1-60198-168-4

Foundations and Trends® in Stochastic Systems
Volume 1 Issue 4, 2006

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Foundations and Trends® in Stochastic Systems, 2006, Volume 1, 4 issues. ISSN paper version 1551-3106. ISSN online version 1551-3092. Also available as a combined paper and online subscription.

Foundations and Trends® in
Stochastic Systems
Vol. 1, No. 4 (2006) 259–341

DOI: 10.1561/0900000006

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Abstract

Stochastic control problems in which there are no bounds on the rate of control reduce to so-called free-boundary problems in partial differential equations (PDEs). In a free-boundary problem the solution of the PDE and the domain over which the PDE must be solved need to be determined simultaneously. Examples of such stochastic control problems are singular control, optimal stopping, and impulse control problems. Application areas of these problems are diverse and include finance, economics, queuing, healthcare, and public policy. In most cases, the free-boundary problem needs to be solved numerically.

In this survey, we present a recent computational method that solves these free-boundary problems. The method finds the free-boundary by solving a sequence of fixed-boundary problems. These fixed-boundary problems are relatively easy to solve numerically. We summarize and unify recent results on this *moving boundary* method, illustrating its

application on a set of classical problems, of increasing difficulty, in finance. This survey is intended for those are primarily interested in computing numerical solutions to these problems. To this end, we include actual Matlab code for one of the problems studied, namely, American option pricing.

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1

Introduction

Singular stochastic control problems are those problems in which control effort can effect instantaneous displacement in state. A wide variety of problems can be modeled as singular stochastic control problems. A representative list of applications include economics [16], portfolio optimization in finance [15], dynamic control of queueing networks [22], revenue management [11], and environmental clean-up issues in public policy [28].

Despite their wide applicability, singular control problems are not analytically tractable except in very special cases. Therefore, one is forced to solve these problems numerically. Based on the application, various numerical methods have been proposed for solving such problems.

Our goal in this survey is to describe a general yet efficient numerical method for solving such problems. As a consequence we rely quite heavily on our own past work in this area, and borrow heavily from our papers. Our goal is to provide a unified treatment of this method emphasizing application in Finance.

Our method makes use of the special structure of these problems, namely, that optimal policies are characterized by so-called *regions of* *inaction*. So we reduce the problem of finding the optimal policies to searching for the right region of inaction. This is still a difficult problem because there is no explicit characterization of the optimal region. Rather it is implicitly specified by the solution of a partial differential equation, whose domain it is. Thus, we are faced with a so-called *free-boundary problem* where the solution of a PDE and the domain of which it must be solved need to be simultaneously determined. Our method solves these free-boundary problems by reducing them into a sequence of fixed-boundary problems which are relatively easy to solve numerically. The key to our method is a boundary update procedure that allows us to construct the next fixed-boundary problem from the solution of the previous one.

A valuable by-product of our method is that it is capable of solving other stochastic control problems that can be cast as free-boundary problems but are not singular control problems *per se.* An important class of problems that fall in this category are optimal stopping problems. A very important example of an optimal stopping problem is the American option pricing problem, and our method is applicable to these problems as well, as we will discuss in a later section. In the same vein, our method is also applicable to impulse control problems [18], but we do not discuss these problems in this survey.

Rather than describe the method in vague words, we provide a simple, concrete illustration using a one-dimensional singular control problem. Restricting attention initially to one-dimensional problems allows us to achieve two objectives. We can provide a description of the procedure that is easier to comprehend, and we illustrate the kind of theoretical guarantees on the behavior of the procedure that can be obtained.

1.1 Motivation: Controlled Brownian Motion

Consider the problem of using two non-negative, nondecreasing, RCLL processes to control a given continuous path $w(\cdot)$ as

$$x(t) = x + \mu t + \sigma w(t) + L(t) - U(t), \quad t \geq 0, \tag{1.1}$$

where μ and $\sigma > 0$ are constants, and the initial state $x \in [0,1]$. Suppose that we needed $x(t) \in [0,1]$ for all t, and furthermore, we were interested in maintaining $x(t)$ as close to $\bar{x}$ in the sense of minimizing $\int_0^\infty e^{-\lambda t}h(x(t))dt$, where $\lambda > 0$ is a discount rate, and $h(y) = (y - \bar{x})^2$, where $\bar{x} \in (0,1)$. For now, we impose no probabilistic structure on the problem. This problem is trivially solved. If $x \neq \bar{x}$ we simply make an initial jump using either L or R to move $x(0+)$ to $\bar{x}$ and maintain $x(t) = \bar{x}$ for all $t > 0$ using L and U. Note that we may have jumps in L or U even though w is continuous. In particular, there are no rate constraints on L and U. Hence the name *singular control*.

Now consider costs of control as well. Suppose, we have cost rates $c > 0$ and $r > 0$ such that the overall cost of using controls L and U is $\int_0^\infty e^{-\lambda t}h(x(t))dt + \int_0^\infty e^{-\lambda t}c dL(t) + \int_0^\infty e^{-\lambda t} r dU(t)$, where the last two terms are interpreted as Reimann–Steiljes integrals. Now the choice of controls is no longer obvious. Attempting to make $h(x(t))$ small comes at the price of incurring control costs. If the path w is not of bounded variation, then costs of control incurred by attempting to maintain $x(t) \equiv \bar{x}$ is prohibitive. So in order to trade-off the holding cost h against control costs, one pick controls that do nothing as long as $x(t)$ is close to $\bar{x}$ but intervene only when it has deviated sufficiently, i.e., it has hit the boundary of an interval around $\bar{x}$. It is not hard to see that the same interval cannot be appropriate for all paths. So we can no longer hope for path-wise solutions. To solve this problem we need to impose additional probabilistic structure on it.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\{\mathcal{F}_t,\ t \in \mathbf{R}_+\}$ be a right continuous filtration on this space and let w be a standard ($\mathbf{R}$-valued) Brownian motion with respect to this filtration. Let L and U be RCLL, non-negative, nondecreasing and adapted to $\mathcal{F}_t$. Now suppose that we are again interested in the controlled process $x(\cdot)$ specified by (1.1), and in minimizing the expected infinite horizon discounted cost (also called a value function) among all admissible policies L and U.

$$J(x,L,U) = E_x\left[\int_0^\infty e^{-\lambda t}h(x(t))dt + \int_0^\infty e^{-\lambda t}c dL(t) + \int_0^\infty e^{-\lambda t} r dU(t)\right], \qquad (1.2)$$

For this problem to make sense we restrict attention to only those controls for which $E_x\left[\int_0^\infty e^{-\lambda t}dL(t)\right] < \infty$ and $E_x\left[\int_0^\infty e^{-\lambda t}dU(t)\right] < \infty$.

As before we restrict attention to those policies that maintain $x(t) \in B^0 \equiv [0,1]$ for all $t > 0$ and to initial states $x \in B^0$.

The first thing we do is to reduce this singular, stochastic control problem into a problem in differential equations. The following result is fairly standard, and follows from an application of Itô's formula. See, for example, [30].

Lemma 1.1. If we can find a twice continuously differentiable function f^*: $B^0 \to \mathbf{R}$ such that $f^*(x) = J(x,L,U)$ for all $x \in B^0$ for some admissible (L,U), and satisfies

$$\min\left(\frac{\sigma^2}{2}f^*_{xx}(x) + bf^*_x(x) - \lambda f^*(x) + h(x), f^*_x(x) + c, -f^*_x(x) + r\right) = 0 \tag{1.3}$$

in B^0, then the (L,U) must be optimal. (Here and elsewhere in this volume f_x denotes the derivative of f and f_{xx} the second derivative.)

Although this theorem has allowed us to translate the problem to one in ODE's it has not yet given us a clue as to its solution. Now suppose we can find a function f in the following class F. Each function $f \in F$ is specified by an interval $B = [b_l, b_u] \subset B^0$, is continuously differentiable in B^0, twice continuously differentiable in the interior of B, and satisfies the following ordinary differential equation (ODE) in B.

$$\frac{1}{2}\sigma^2 f_{xx}(x) + bf_x(x) - \lambda f(x) + h(x) = 0, \tag{1.4}$$

$$f_x(b_l) = -c, \quad \text{and} \tag{1.5}$$

$$f_x(b_u) = r. \tag{1.6}$$

Furthermore, the function is defined in $B^0 - B$ by the linear extension

$$f(x) = f(b_l) + c(b_l - x) \text{ for } b^0_l \leq x < b_l \quad \text{and} \tag{1.7}$$

$$f(x) = f(b_u) + r(x - b_u) \text{ for } b^0_u \geq x > b_u. \tag{1.8}$$

Note that every $f \in F$ is almost a candidate for f^*. The differentiator among these functions $f \in F$, that is, among the intervals B is the need for f^* to be twice continuously differentiable over B^0 and not just B. This means that the optimal choice of interval B^* will have to be made so as to ensure *smooth pasting* [4] of the solution of the ODE inside B^* and the linear extension outside B^*. Thus we are faced with the problem of finding the solution of an ODE and the domain over which it must be solved simultaneously, resulting in the so-called *free-boundary* problem. The second issue that we need to tackle is whether f^* is a value function under some admissible controls.

Before we go into a procedure for finding B^*, we first note the following connection between functions in F and the so-called *regulated Brownian motions*, which provides us with an interpretation of $f \in F$. Consider policies that maintain x in B with the minimum amount of pushing required to do so. That is, for any interval $B \subset B^0$, let L_B and U_B be the (unique) non-negative, nondecreasing, RCLL processes adapted to $\mathcal{F}_t$ such that

$$\begin{gathered} x(t) \in B \quad \text{for all } t > 0 \\ \int_0^t (b_l - x(s))^+ dL_B(s) = \int_0^t (x(s) - b_u)^+ dU_B(s) = 0 \quad \text{for each } t > 0 \\ U_B(0+) = (x(0) - b_u)^+ \quad \text{and} \quad L_B(0+) = (b_l - x(0))^+. \end{gathered} \tag{1.9}$$

The x that results from the use of such L_B and U_B is called a regulated Brownian motion or *two-sided regulator* applied to Brownian motion. The following result, which is also standard, gives the connection between regulated Brownian motions and $f \in F$.

Lemma 1.2. Consider a $B \subseteq B^0$ and the $f \in F$ corresponding to B that satisfies (1.4–1.6). Then

$$f = E_x\left[\int_0^\infty e^{-\lambda t} h(x(t))dt + \int_0^\infty e^{-\lambda t} c \cdot dL_B(t) + \int_0^\infty e^{-\lambda t} r \cdot dU_B(t)\right],$$

where L_B and U_B are the unique admissible controls that satisfy (1.9).

So we can search for an f^* by searching for a B^*. Once we find a B^* such that the solution to (1.4–1.6) is twice continuously differentiable on B^0, we are done because the optimal policy is specified by the regulator (L_{B^*}, U_{B^*}). An interpretation that follows from the definition of regulated Brownian motion is that B^* can be though of as the *region of inaction*. As long as x is inside B^* no controls are applied. Controls are only applied on the boundary of B^*. Thus our search reduces to the search for a region of inaction. We now describe an iterative procedure for finding B^*.

We begin the iterative procedure with B^0 as the initial choice for the region of inaction. We solve the set of Equations (1.4)–(1.6) to find the value function corresponding to a regulated Brownian motion whose region of inaction is B^0. Then we iterate as follows to obtain successive regions of inaction $B^1, B^2, \ldots$ and the corresponding value functions $f^1, f^2, \ldots$. The key to getting this procedure to work to find an update rule that allows to efficiently determine B^{k+1} given B^k and will converge to B^*. With that in mind, we impose two desiderata on the update procedure.

D1 **(The Superset Condition).** We want the regions of inaction to be monotone decreasing; we need $B^{k+1} \subseteq B^k$. As is evident from (1.4)–(1.8), we only obtain "real" information about f^k inside B^k. So we have no way of telling how far to back-out if we did not have a monotone sequence of regions and needed to back out. So the superset condition is a requirement for efficient search.

D2 **(Policy Improvement).** We would like that $f^{k+1}(x) \leq f^k(x)$ for all $x \in B^0$. That is, the policy obtained in the next iteration is an improvement on the current policy. This ensures that the f^k will converge.

Upfront, it is not clear that a procedure that meets both D1 and D2 exists. In what follows we construct such a procedure.

The crucial step to constructing the procedure is deciding on the update rule. Given B^k and f^k, define the right and left second derivatives as $f^{k+}_{xx}(x) := \lim_{\delta\downarrow 0}(f^k_x(x+\delta) - f^k_x(x))/\delta$ and $f^{k-}_{xx}(x) := \lim_{\delta\downarrow 0}(f^k_x(x) - f^k_x(x-\delta))/\delta$. At $x = {b_l}^k$, $f^{k+}_{x_i x_i}(x)$ need not equal $f^{k-}_{x_i x_i}(x)$. (Note that $f^{k-}_{x_i x_i}(x) = 0$ because of our construction.) Consider the case when $f^{k+}_{xx}({b_l}^k) < 0$. The situation is as shown in Figure 1.1.

In this case updating the boundary inwards, i.e., setting ${b_l}^{k+1} > {b_l}^k$ helps us achieve D1 of course. But it also helps us achieve D2.

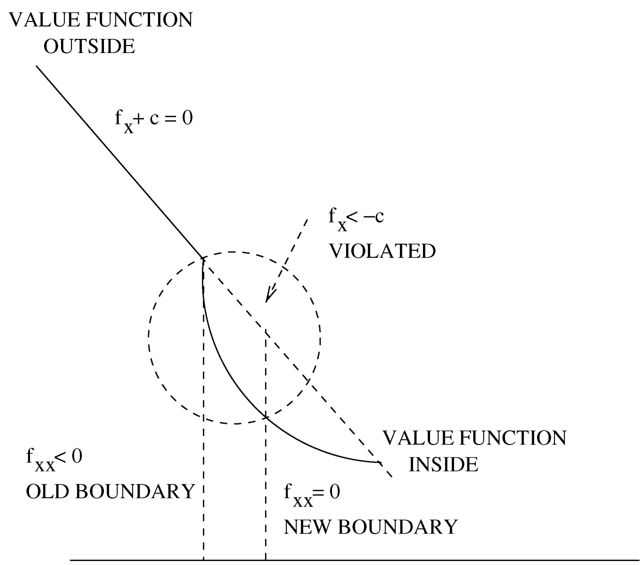

Fig. 1.1 Illustrating the smooth paste update procedure.

To see this, consider a modification ℓ_{B^k} of the control L_{B^k}. Under the control ℓ_{B^k}, if the initial state x is such that $x < {b_l}^k$, it is translated instantaneously, not to ${b_l}^k$ as by L_{B^k}, but to a point x' in the immediate vicinity of ${b_l}^k$ in the interior of B^k. Thereafter, L_{B^k} is mimicked by ℓ_{B^k}. Then

$$J(\ell_{B^k}) - J(L_{B^k}) = \int_{{b_l}^k}^{x'} (f_x^k + c)dx < 0.$$

So ℓ_{B^k} is an improved policy, and by repeating this argument, we suspect that a policy that maintains $x(t)$ in a smaller interval than B^k will be an improvement. (All of this will be formalized shortly.)

Although the heuristic argument above tells us that we should move inwards, it does not tell us by how much. Since we are striving to find a function in $C^2(B^0)$, a natural candidate to update the boundary inwards is the point where $f_{xx}^k(x) = 0$. That is, we move the boundary to a point where, if the resulting region of inaction was indeed the fixed point of the iterations, then the function inside the region (obtained by solving (1.4–1.6)) and its linear extension outside the region would be smoothly pasted, a requirement for converging to B^* as discussed earlier. Therefore, we choose the local minimizer of f_x^k nearest ${b_l}^k$ as the updated ${b_l}^{k+1}$. That is,

$$
\begin{aligned}
{b_l}^{k+1} &= \min\{x^* \geq b_l^k | x^* \text{ is a local minimizer of } f_x^k\}, \text{ or equivalently} \\
{b_l}^{k+1} &= \min\left\{x^* \geq b_l^k | \exists\, \epsilon > 0 \text{ s.t. } f_x^k(x^*) = \min_{-\epsilon \leq \delta \leq \epsilon} f_x^k(x^* + \delta)\right\}.
\end{aligned}
\tag{1.10}
$$

Arguing similarly at b_u^k, if $f_{xx}^{k-} < 0$ the natural candidate for the update is

$$
\begin{aligned}
{b_u}^{k+1} &= \max\{x^* \leq b_u^k | x^* \text{ is a local maximizer of } f_x^k\}, \text{ or equivalently} \\
{b_u}^{k+1} &= \max\left\{x^* \leq b_u^k | \exists\, \epsilon > 0 \text{ s.t. } f_x^k(x^*) = \max_{-\epsilon \leq \delta \leq \epsilon} f_x^k(x^* + \delta)\right\}.
\end{aligned}
\tag{1.11}
$$

For completeness we need to consider the case when $f_{xx}^{k+}({b_l}^k) > 0$. If $f_{xx}^{0+}(0) > 0$ then we suspect that the interval $B^0 = [0,1]$ is not big enough. But our requirement that $x(t) \in [0,1]$ implies that we must simply retain $b_l^1 = 0$. The same holds when $f_{xx}^{0-}(1) > 0$, we must retain $b_u^1 = 1$. Now if these two conditions are ruled out by assumption (by assuming that $c + r$ is sufficiently small, for example) then our update procedure *guarantees* that $f_{xx}^{k+}({b_l}^k) < 0$ implies $f_{xx}^{(k+1)+}({b_l}^k) < 0$. That is, if the procedure works at the first step, it will work at every subsequent step. We illustrate this first via a numerical example taken from [30]. We then quote a result from [30] that the procedure is well-defined and that it converges. The numerical example uses the parameter choices $\lambda = 0.01, \sigma^2 = 2, b = 1, \bar{x} = 0.6, c = 0.02$, $r = 0.01$. Figure 1.2 plots $f^k(x), f_x^k(x), f_{xx}^k(x)$ for $k = 0,1,2$.

We start with the initial region of inaction $B^0 = [0,1]$. We then solve the resulting ODE (1.4) in $[0,1]$ with the boundary conditions $f_x^0(0) = -c, f_x^0(1) = r$. Although the resulting expressions are messy, this ODE can be solved analytically. The top set of plots in Figure 1.2 show the resulting f^0 and its first and second derivatives, f_x^0 and f_{xx}^0. As can be seen from the plots, $f_x^0(x) < -c$ for all $x < 0.52$. Thus, (1.3) is violated and therefore we need to move the lower barrier from 0.

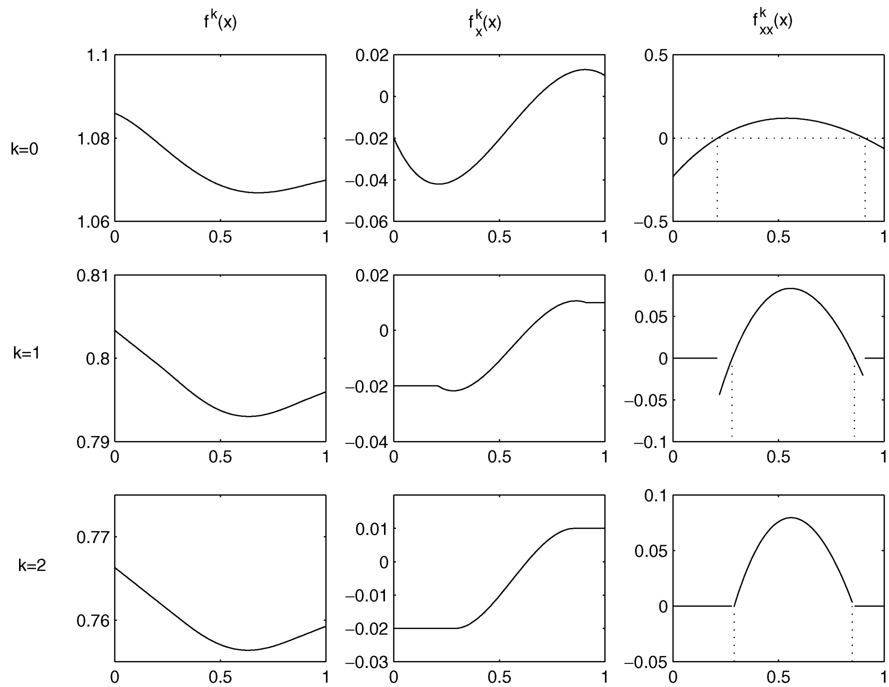

Fig. 1.2 One-dimensional example.

We move the lower boundary to b_l^1 such that $f_x^0(b_l^1)$ is the minimum of f_x^0. Of course, $f_{xx}^0(b_l^1) = 0$. Similarly, we move the right boundary to b_u^1 such that $f_x^0(b_u^1)$ is the maximum of f_x^0. Once again we analytically solve ODE (1.4) for f^1 in $[b_l^1, b_u^1]$ with the boundary conditions $f_x^1(b_l^1) = -c, f_x^0(b_u^1) = r$. The linear extension gives $f^1(x) = f^1(b_l^1) + c(b_l^1 - x)$ for $x \in [b_l^0, b_l^1]$ and $f^1(x) = f^1(b_u^1) + r(x - b_u^1)$ for $x \in [b_u^1, b_u^0]$. The second row of plots in Figure 1.2 show the resulting f^1, f_x^1, and f_{xx}^1. We still have regions where $f_x^1(x) < -c$ and $f_x^1(x) > r$. So we repeat the procedure, arriving at the plots shown in the third row in Figure 1.2. Now we note that $f_x^2(x) \geq -c$ and $f_x^2(x) \leq r$ for all $x \in [0,1]$ and that $f_{xx}^2(b_l^2) = f_{xx}^2(b_u^2) = 0$ upto the resolution of our code, and we terminate the process.

We now provide theoretical justification for the procedure in Proposition 1.3 below, taken from [30].

Proposition 1.3. If $B^k \equiv [b_l^k, b_u^k]$ is such that $f_{xx}^{k+}(b_l^k) < 0$ and $f_{xx}^{k-}(b_u^k) < 0$ for a given k, then

(1) $b_l^{k+1} < b_u^{k+1}$,
(2) $B^{k+1} \subset B^k$, i.e., $b_l^{k+1} > b_l^k$ and $b_u^{k+1} < b_u^k$,
(3) $f^{k+1}(x) < f^k(x)$, for all $x \in B^0$, and
(4) $f_{xx}^{(k+1)+}(b_l^{k+1}) < 0$ and $f_{xx}^{(k+1)-}(b_u^{k+1}) < 0$.

Therefore, if $B^0 \equiv [b_l^0, b_u^0]$ is such that $f_{xx}^{0+}(b_l^0) < 0$ and $f_{xx}^{0-}(b_u^0) < 0$,

(5) $b_l^k \to b_l^*$ and $b_u^k \to b_u^*$, as $k \to \infty$, where $b_l^* > b_l^k$ and $b_u^* < b_u^k$ for any $k \geq 0$,
(6) $f^k \to f^*$ uniformly on B^0 as $k \to \infty$,
(7) $f^* \in C^2(B^0)$ satisfies $\Gamma f^* - \lambda f^* + h = 0$ in $[b_l^*, b_u^*]$, with $f_{xx}^*(b_l^*) = 0$ and $f_{xx}^*(b_u^*) = 0$,
(8) f^* solves (1.3).

Proposition 1.3 establishes several properties of the algorithm. First, it establishes that the update procedure is well-defined, and the region of inactions obtained in each iteration is non-empty. Second, it establishes that update improves the value function D2, and that the superset condition D1 is satisfied. Further, it establishes convergence of the regions of inaction as well as the value function. Finally, it proves that the value function to which the procedure converges is indeed the optimal value function. Moreover, the converged value function can be interpreted as the value function under the policy that instantaneously translates the initial state to $[b_l^*, b_u^*]$ and maintains it thereafter in this interval using the two-sided regulator.

1.2 The General Method

The example in the previous section is quite restrictive. The problem is one dimensional and the stochastic process being controlled is Brownian motion. As a consequence, both identifying the update rule and verifying its properties is quite easy. Yet, it suffices to provide us with the outline of the general procedure, enumerated below.

(1) Write out the stochastic control problem, as in (1.1), (1.2). In the sections that follow, the control problems we study will be different from the one studied here, but they either will involve singular controls or will be closely related problems such as optimal stopping problems.
(2) Translate the stochastic control problem into a problem in PDEs as in Lemma 1.1. This PDE is commonly called the Hamilton–Jacobi–Bellman (HJB) equation.
(3) Look for solutions to the HJB equation that are specified by free-boundary problems as in (1.4)–(1.8). These correspond to policies that are specified by regions of inaction as in Lemma 1.2. A key issue that arises in higher dimensions is whether classical (smooth) solutions exist for these free-boundary problems. That is, can we find an $f^* \in F$ which is smooth across the boundary of B^*? In some cases, we will quote results that establish existence of smooth solutions. In other cases, we will simply assume existence and proceed. A related issue here is the existence of solutions to the so-called Skorohod problem as in (1.9), which we also simply assume.
(4) We will use an iterative procedure that solves the free-boundary problem by solving a sequence of fixed-boundary problems over domains $B^0, B^1, \ldots$. The key to constructing this procedure is the way to update the boundaries ∂B^k. In every case that we consider, we will insist on D1 and D2 for the reasons mentioned.
(5) At each step in the procedure we will need to solve a PDE over B^k analogous to (1.4) with boundary conditions analogous to (1.5)–(1.6). We need to take into account the direction in which control is exercised in higher dimensions and so the boundary conditions can be nonstandard. The way we set up our method this PDE is a linear elliptic PDE and thus amenable to numerical solution. Although the user is

free to choose the numerical method used to solve this PDE, the ability to handle nonstandard boundary conditions and arbitrary domains B^k suggests the use of Finite Element Methods, which we describe.

(6) In all the cases, we will use the idea of smooth pasting to come up with our update procedure, updating each point on ∂B^k by moving along the direction of control to the point where smooth pasting would hold if the value function did not change, just as in (1.10)–(1.11).

(7) Finally, we need to provide a justification of the method as in Proposition 1.3. In two of the problems we study, namely portfolio optimization with one stock in Section 2 and American option pricing in Section 3, we quote results that provide such guarantees. For the higher dimensional portfolio optimization problems, we offer no guarantees beyond extensive numerical studies.

1.3 Structure and Intended Audience

The intended reader of this survey is one who is conversant with the basic formulations in Mathematical Finance. We provide no modeling justification for the control problems that we study. We assume that the reader knows why these problems are worth solving numerically. We also assume that the reader has at least a passing familiarity with PDE approach to solving these control problems. While we do cite work that carries out the translation from the stochastic control problem to the free-boundary problem in each of our settings, we do not provide any details of this translation. To summarize, this volume is intended for the reader who knows how to set up a stochastic control problem as a free-boundary problem, and wants to find an efficient numerical procedure for its solution. Although the applications we study are in Finance, the actual domain of applicability of our method is much larger. Finally, this survey is intended for those who will actually compute solutions, rather than prove theorems about the control problems or computational schemes. So the level of rigor is not as high as one would find in [20, 27].

One of the shortcomings of this volume is that no general procedure is outlined. The primary reason for this is that it is nearly impossible to justify the method, let alone guarantee its performance, in a sufficiently general set-up that encompasses all the applications we cover. So we choose a case-by-case approach. The procedure is carefully constructed and justified in each of the settings we consider. And in two of these settings performance guarantees are provided. It is the authors' belief that it is easier for the reader to tailor the procedure for an application of interest, and to justify it using the special structure of the problem at hand, by understanding how the moving-boundary method works in a variety of settings.

Finally, we do provide actual MATLAB code for one of the settings we consider, the pricing of American options. This is intended to help the potential user get oriented with implementation of our method.

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