Game Theory: Models, Numerical Methods and Applications

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Contents

Abstract

Game theory is the theory of "strategic thinking". Developed for military purposes and defense, in the past it has also been used as an alternative and complementary approach to deal with robustness in the presence of worst-case uncertainties or disturbances in many areas such as economics, engineering, computer science, just to name a few. However, game theory is recently gaining ground in systems and control engineering, mostly in engineered systems involving humans, where there is a trend to use game theoretic tools to design protocols that will provide incentives for people to cooperate. For instance, scientists tend to use game theoretic tools to design optimal traffic flows, or predicting or avoiding blackouts in power networks or congestion in cyber-physical networked controlled systems.

Incentives to cooperate are also crucial in dynamic resource allocation, multi-agent systems and social models (including social and economic networks). This paper assembles the material of two graduate courses given at the Department of Engineering Science of the University of Oxford in June-July 2013 and at the Department of Electrical and Electronic Engineering of Imperial College, in October-December 2013. The paper covers the foundations of the theory of noncooperative and cooperative games, both static and dynamic. It also highlights new trends in cooperative differential games, learning, approachability (games with vector payoffs) and mean–field games (large number of homogeneous players). The course emphasizes theoretical foundations, mathematical tools, modeling, and equilibrium notions in different environments.

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1

Introduction

This first chapter is introductory and streamlines the foundations of the theory together with seminal papers and applications. The chapter introduces different types of games, such as simultaneous and sequential games, and the corresponding representations. In addition, it makes a clear distinction between cooperative and noncooperative games. The introduction proceeds with the formalization of fundamental notions like pure and mixed strategy, Nash equilibrium and dominant strategy (strategic/normal representation and extensive/tree representation). In the second part of this chapter, we pinpoint seminal results on the existence of equilibria. The end of the chapter is devoted to the illustration of classical games such as the Cournot duopoly, as an example of infinite game, or other stylized games in strategic form known as the coordination game, the Hawk and Dove game or the Stag-Hunt game. We make use of the Cournot duopoly to briefly discuss the iterated dominance algorithm.

1.1 Historical note, definitions and applications

The foundations of game theory are in the book [von Neumann and Morgenstern, 1944] by the mathematician John Von Neumann and the economist Oskar Morgenstern,

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Theory of games and economic behavior,
Princeton University Press, 1944.
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The book builds on prior research by von Neumann published in German [von Neumann, 1928]: *Zur Theory der Gesellschaftsspiele, Mathematische Annalen, 1928.* Quoting from [Aumann, 1987], *Morgenstern was the first economist clearly and explicitly to recognize that economic agents must take the interactive nature of economics into account when making their decisions. He and von Neumann met at Princeton in the late Thirties, and started the collaboration that culminated in the Theory of Games.*

Forerunners of the theory are considered the french philosopher and mathematician Antoine Augustin Cournot, who first introduced the "duopoly model" in 1838, and the german economist Heinrich Freiherr von Stackelberg, who formulated the equilibrium concept named after him in 1934 [von Stackelberg, 1934].

Game theory intersects several disciplines, see e.g., Table 1.1, and conventionally involves multiple players each one endowed with its own payoff. Thus, game theory is different from optimization where one has one single player who optimizes its own payoff. Game theory also differs from multi-objective optimization, the latter characterized by one player and multiple payoffs. In the '60s another discipline was founded dealing with multiple decision makers with a common payoff, known as team theory [Ho, 1980, Marschak and Radner, 1972, Bauso and Pesenti, 2012].

The literature provides several formal definitions of game theory. For instance, Maschler, Solan, and Zamir say that game theory is a methodology using *mathematical tools to model and analyze situations involving several decision makers (DMs), called players* [Maschler et al., 2013]. According to Osborne and Rubinstein game theory is *a bag of analytical tools designed to help us understand the phenomena that we*

	1 payoff	n payoffs
1 player	Optimization	Multi-objective optimization
n players	Team theory	Game theory

Table 1.1: A scheme relating game theory to other disciplines.

observe when DMs interact, (DMs are rational and reason strategically) [Osborne and Rubinstein, 1994]. Here, (individual) rationality and strategic reasoning mean that every DM *is aware of his alternatives, forms expectations about any unknowns, has clear preferences, and chooses his action deliberately after some process of optimization* [Osborne and Rubinstein, 1994]. Tijs in his book defines game theory as *a mathematical theory dealing with models of conflict and cooperation* [Tijs, 2003].

Game theoretic models arise in numerous application domains including:

Board and field games [Bewersdorff, 2004]. Board games like chess or draughts or field games such as football or rugby may admit a mathematical description via game theory, where the players' actions are elements of a given set, called actions' set, and the probability of win is the payoff that every player seeks to maximize. In rugby, for instance, certain tactics are successful only if the opponent is playing a certain tactic and thus the tactic choice is assimilated to a play of the rock-paper-scissors game. Theoretical foundations are to be found in *algorithmic game theory*, a research area intersecting algorithm design, game theory and artificial intelligence [Noam et al., 2007].

Marketing and commercial operations [Osborne and Rubinstein, 1990, Gibbons, 1992]. Competitive firms operating on a same market must be able to predict the impact of a new product. This involves a strategic analysis of the current market demand and of the reactions of the potential competitors in consequence of the introduction of the new product.

Politics [Morrow, 1994]. Here game theory provides useful indices to measure the power of parties involved in a governing coalition. Voting methods can also be rigorously analyzed through game theory. Regarding social policy making, game theory offers guidelines to governmental agencies to predict and analyze the impact of specific social policy choices, such as pension rules, education or labor reforms.

Defense [**Hamilton and Mesic, 2004**]. Game theory has contributed the notion of "strategic thinking" consisting in putting ourselves in the place of the opponent before making a decision, which is a milestone in the field of defense. Military applications related to missile pursuing fighter airplanes are also usually addressed using game theoretic models.

Robotics and multi-agent systems [**Shoham and Leyton-Brown, 2009**]. Here game theory provides models for the movement of automated robot vehicles with distributed task assignment. Path planning for robotic manipulation in presence of moving obstacles is also a classical game theory application.

Networks [Di Mare and Latora, 2007, Saad et al., 2009]. Game theory can be used to analyze the spread of innovation, or the propagation of opinions in social networks. In communication networks game theory is frequently used to study band allocations, and in security problems.

1.2 Types of games and representations

There are different types of games and corresponding representations. In this section, after providing a formal description of a game in generic terms, we distinguish between *cooperative* and *noncooperative*, *simultaneous* and *sequential* games and introduce the *strategic* or *normal* representation for the former games and the *extensive* or *tree* representation for the latter games.

1.2.1 What is a game?

A (strategic form) game is a tuple $\langle N, (\mathcal{A}_i)_{i\in N}, (u_i)_{i\in N}\rangle$, where

- $N = \{1, 2, \ldots, n\}$ is the set of players (maximizers),
- $\mathcal{A}_i$ is the set of actions of player i,

- $A := \{a|\, a = (a_i)_{i\in N},\, a_i \in \mathcal{A}_i,\, \forall i \in N\}$ is the set of action profiles,
- $u_i : A \to \mathbb{R}$ is the payoff function of player i, i.e.,

$$(a_1, \dots, a_n) \mapsto u_i(a_1, \dots, a_n).$$

Note that the payoff u_i is a profit (to maximize) but can also be a cost (to minimize).

An equivalent way of writing the action profiles is

$$(a_j)_{j\in N} = (a_1, \dots, a_n) = (a_i, a_{-i}),$$

where $a_{-i} = (a_j)_{j\in N, j\neq i}$ is the action profile of all players except i.

1.2.2 Noncooperative vs. cooperative

A first major distinction is between noncooperative and cooperative game theory. In noncooperative games i) every player seeks its best response based on the available information and in order to maximize its own payoff, ii) there are no binding agreements on optimal joint actions, iii) pre-play communication is possibly allowed.

In cooperative games (which in turn divide into games with transferable (TU) and nontransferable (NTU) utilities) i) the players seek optimal joint actions (NTU), or reasonable cost/reward sharing rules (TU) that make the coalitions stable, ii) pre-play communication is allowed, and iii) side payments are also allowed (TU).

Note that while noncooperative game theory dominates almost every textbook in game theory, and is by far more widespread than cooperative game theory, there is a large consensus on the idea that cooperative game theory has a broader range of applications. Only recently cooperative game theory has attracted the attention of scientists from disciplines other than economics, and has become a major design tool in engineered systems [Saad et al., 2009].

Example 1.1. (***Prisoners' dilemma***) This is one of the most common and simple strategic models developed by Merrill Flood and Melvin Dresher of the RAND Corporation in 1950. The prison-sentence interpretation, and thus the corresponding name is due to Albert W. Tucker. The story is the following one: two criminals are arrested under the suspicion of having committed a crime for which the maximal sentence is four years. Each one may choose whether to cooperate with (C) or defect (D) the other fellow. If both defect (D,D), the sentence is mitigated to three years (each one gets one year of freedom). If both cooperate (C,C), the suspects are released after one year due to lack of evidence (each one gets three years of freedom). If only one cooperates, (C,D) or (D,C), the one who defects is released immediately (four years of freedom), while the other is sentenced to the maximal punishment (zero years of freedom). The game is represented in bimatrix form as displayed in Fig. 7.1. In a purely noncooperative context, every player

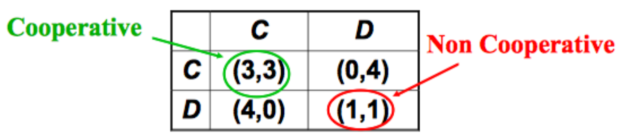

Figure 1.1: Prisoners' dilemma: cooperative vs. noncooperative solutions.

will choose to defect (D) considering that he has no guarantee on the other's choice and therefore the resulting solution is (D, D). Differently, in a cooperative scenario, where both players can collude and negotiate joint actions, it is likely that both will end up cooperating (C, C).

1.2.3 Simultaneous vs. sequential games

A second major distinction is between simultaneous games and sequential games. In simultaneous games i) decisions are made once and for all and at the same time by all players; ii) there is no state nor any concept of strategy; iii) these games admit a common *representation in normal form* (also called *strategic* or *bimatrix form*). The Prisoners' dilemma is a classical example of simultaneous games in strategic form, see Fig. 1.1. Here the rows and the columns are associated to the actions or decisions of the players and the entries of the bimatrix are the payoffs. The representation does not carry any inbuilt information structure.

On the other hand, sequential games are those where i) one has a specific order of events, ii) as a consequence, the game has a state variable that collects information on earlier decisions, iii) latter players may know perfectly or imperfectly the actual state (full or partial information), iv) decisions are made depending on the state from which the notion of *strategy*, namely a mapping from states to actions, v) such games are conveniently represented in *extensive* or *tree form*, see Fig. 1.2.

Here the nodes are the states and are labeled with the player who is to act; the branches are the actions; the payoffs are associated to leaf nodes and depend on the whole history of actions taken. The representation has an inbuilt information structure. Fig. 1.2 depicts a two-player extensive game where player 1 plays first (stage 1) and can select either left L or right R. Player 2 plays second (stage 2) and can in turn select left l or right r in both states 1 and 2, which yields four possible actions l_1, r_1, l_2, r_2.

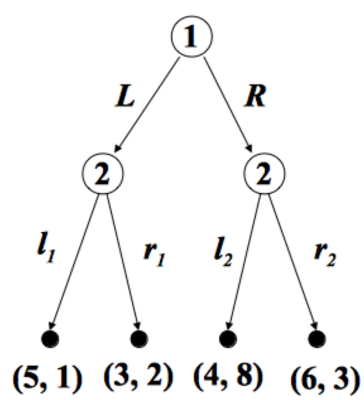

Figure 1.2: Example of extensive/tree form representation.

Nevertheless it is sometimes possible to derive an equivalent strategic form representation for an extensive game once we consider strategies rather than decisions. This is illustrated later on in Example 1.3 and Fig. 1.4. There we have four strategies for player 2, i.e., l_1l_2 (always left), l_1r_2 (left only in state 1, that is when player 1 picks L), l_2r_1 (left only in state 2, that is when player 1 picks R), and r_1r_2 (always right).

Thus, the set of "actions" for player 2 is $\mathcal{A}_2 = \{l_1 l_2, l_1 r_2, r_1 l_2, r_1 r_2\}$, while the one for player 1 is simply $\mathcal{A}_1 = \{L, R\}$.

Simultaneous games can also be played repeatedly over time in which case we address such games as *repeated games*. Repeated games admit an extensive form representation as shown below for the Prisoners' dilemma example in Fig. 1.3. Here payoffs or utilities are usually summed up over the rounds within a finite horizon or infinite horizon (discounted sum or long-term average) time window.

	C	**D**
C	**(3,3)**	**(0,4)**
D	**(4,0)**	**(1,1)**

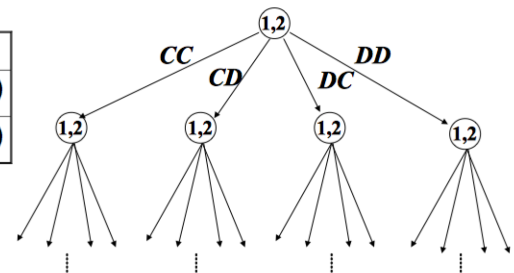

Figure 1.3: Extensive/tree form representation of the repeated Prisoners' dilemma.

1.3 Nash equilibrium and dominance

We review here basic solution concepts such as the Nash equilibrium and dominant strategy.

1.3.1 Nash equilibrium (NE)

In a Nash equilibrium "unilateral deviations" do not benefit any of the players. Unilateral deviations mean that only one player changes its own decision while the others stick to their current choices.

Definition 1.1. (Nash equilibrium [Nash Jr., 1950, 1951]) The action profile/outcome $(a_1^*, a_2^*, \ldots, a_n^*)$ is an NE if none of the players by deviating from it can gain anything, i.e.,

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*), \quad \forall a_i \in \mathcal{A}_i,\, \forall i \in N.$$

Let us introduce the best response set

$$\mathcal{B}_i(a_{-i}) := \{a_i^* \in \mathcal{A}_i |\, u_i(a_i^*, a_{-i}) = \max_{a_i \in \mathcal{A}_i} u_i(a_i, a_{-i})\}.$$

Then in an NE all players play a best response, namely

$$a_i^* \in \mathcal{B}_i(a_{-i}^*), \quad \forall i \in N.$$

Example 1.2. In the Prisoners' dilemma the solution (D, D) is a Nash equilibrium, as player 1 by deviating from it would get 0 years of freedom rather than 1 (stick to second column and move vertically to first row) and therefore would be worse off. Likewise for player 2.

	C	D
C	(3,3)	(0,4)
D	(4,0)	(1,1)

Example 1.3. In the extensive game of Fig. 1.4, player 2 has four strategies, i.e., l_1l_2 (always left), l_1r_2 (left only in state 1, that is when player 1 picks L), l_2r_1 (left only in state 2, that is when player 1 picks R), and r_1r_2 (always right). Thus, the set of "actions" for player 2 is $\mathcal{A}_2 = \{l_1l_2, l_1r_2, r_1l_2, r_1r_2\}$, while the one for player 1 is simply $\mathcal{A}_1 = \{L, R\}$. The game admits one Nash equilibrium (R, r_1l_2). This can be computed via dynamic programming backwardly. In state 1 (node down left), player 2's rational choice is r_1 (red line) as he gets 2 rather than 1 if he were to play l_1. In state 2, player 2 could play l_2 and get 8 or r_2 and get 3, then his rational choice is l_2 (red line). In stage 1 (top node), player one gets 4 by playing R, and 3 by playing L, so his best response is R (red line). The equilibrium payoffs are then $(4, 8)$.

- **E.g. Bimatrix Game**

	l_1l_2	l_1r_2	r_1l_2	r_1r_2
L	(5,1)	(5,1)	(3,2)	(3,2)
R	(4,8)	(6,3)	(4,8)*	(6,3)

Is this also a Nash equilibrium?

If Player 2 played l_1, L would be better than R for Player 1.

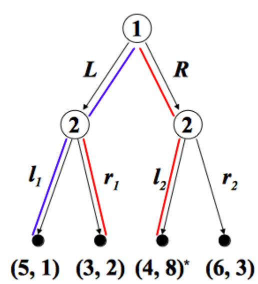

Figure 1.4: Nash equilibrium in an extensive tree game.

The representation in normal form of the game (left) shows another solution, $(R, l_1 l_2)$, returning the same payoffs as the equilibrium, which is not a Nash equilibrium as player 2 would benefit from changing from l_1 to r_1. So, in principle there may exist solutions that are not equilibria and which are equivalent to equilibria in terms of payoffs.

A weaker equilibrium solution concept is available in the literature, namely the so-called ϵ-Nash equilibrium.

Definition 1.2. **(ϵ-Nash equilibrium [Başar and Olsder, 1999, Chap. 4.2])** For a given $\epsilon \geq 0$, the action profile/outcome $(a_1^\epsilon, a_2^\epsilon, \ldots, a_n^\epsilon)$ is an ϵ-NE if none of the players by deviating from it can gain more than ϵ, i.e.,

$$u_i(a_i^\epsilon, a_{-i}^\epsilon) \geq u_i(a_i, a_{-i}^\epsilon) - \epsilon, \quad \forall a_i \in \mathcal{A}_i, \, \forall i \in N.$$

Needless to say, for $\epsilon = 0$, the ϵ-Nash equilibrium coincides with the Nash equilibrium.

1.3.2 Existence of equilibria and mixed strategies

The first seminal result on game theory is the minmax theorem by John Von Neumann, 1928, establishing the existence of equilibrium points for zero-sum games. These are games where the sum of the payoffs of the players is always zero. The result makes use of the notion of *mixed strategies*, namely strategies defined by a probability distribution over the finite set of the feasible strategies.

Theorem 1.1. **(Minmax theorem [von Neumann, 1928])** Each matrix game has a saddle point in the mixed strategies.

From a computational perspective, saddle points can be obtained via linear programming, which is the topic of Chap. 3 (see also Chap. 6, Tijs, 2003). The computation of NE is based on linear complementarity programming, which we will also discuss in Chap. 3 (see also Chap. 7, Tijs, 2003).

Existence of equilibria can be proven starting from the Kakutani's fixed point theorem 1941. The Kakutani's theorem analyzes sufficient conditions for a set-valued function, defined on a convex and compact subset of a Euclidean space, to have a *fixed point*, i.e. a point which is mapped to a set containing it.

Theorem 1.2. (**Kakutani's Fixed point theorem, 1941**) Let K be a non-empty subset of a finite dimensional Euclidean space. Let $f : K \to K$ be a correspondence, with $x \in K \mapsto f(x) \subseteq K$, satisfying the following conditions:

- K is a compact and convex set
- $f(x)$ is non-empty for all $x \in K$
- $f(x)$ is a convex-valued correspondence: for all $x \in K$, $f(x)$ is a convex set
- $f(x)$ has a closed graph: that is, if $\{x_n, y_n\} \to \{x, y\}$ with $y_n \in f(x_n)$, then $y \in f(x)$

Then, f has a fixed point, that is, there exists some $x \in K$, such that $x \in f(x)$.

Rather than the formal proof we provide a graphical illustration for a simple scalar case of the main ideas used in the proof. Let x be plotted in the horizontal axis, and $f(x)$ in the vertical axis as in Fig. 1.5. Fixed points, if exist, must solve $f(x) = x$ and therefore can be found at the intersection between the function $f(x)$ and the dotted line. On the left, the function $f(x)$ is not convex-valued and therefore it does not admit a fixed point. On the right, the function $f(x)$ does not have a closed graph which again implies that there exist no fixed point.

The Kakutani's theorem has been successively used by John Nash to prove the existence of a Nash equilibrium for nonzero-sum games. Essentially, the Nash's equilibrium theorem establishes the existence of at least one Nash equilibrium provided that i) the set of actions $\mathcal{A}_i$ are compact and convex subsets of $\mathbb{R}^n$, as it occurs in continuous (infinite) games, or games in mixed extension (we will expand more on it later); ii) payoffs $u_i(a_i, a_{-i})$ are continuous and concave in a_i for fixed strategy a_{-i} of the opponents.

Theorem 1.3. (Equilibrium point theorem [Nash Jr., 1950]) Each finite bimatrix game has an NE in the mixed strategies.

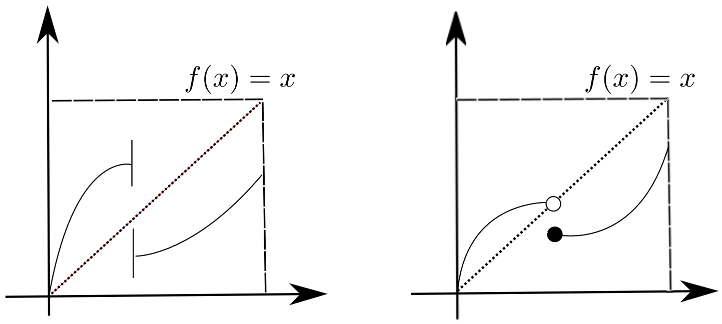

Figure 1.5: Graphical illustration of Kakutani's theorem. Function $f(x)$ is not convex valued (left), $f(x)$ has no closed graph (right). Courtesy by Asu Ozdaglar, slides of the course 6.254 Game Theory with Eng. Applications, MIT OpenCourseWare (2010).

Proof. We here provide only a sketch of the proof. Let us introduce the best response set,

$$\mathcal{B}_i(a_{-i}) := \{a_i^* \in \mathcal{A}_i |\, u_i(a_i^*, a_{-i}) = \max_{a_i \in \mathcal{A}_i} u_i(a_i, a_{-i})\}.$$

We can then apply the Kakutani's fixed point theorem to the best response correspondence $\mathcal{B} : \Delta \rightrightarrows \Delta$, $\Delta = \prod_{i \in N} \Delta_i$ (Δ_i is the simplex in the $\mathbb{R}^{|\mathcal{A}_i|}$)

$$\mathcal{B}(a) = \Big(\mathcal{B}_i(a_{-i})\Big)_{i \in N}.$$

□

An important property of mixed strategy Nash equilibria is that every action in the support of any player's equilibrium mixed strategy is a best response and yields that player the same payoff (cf. [Osborne and Rubinstein, 1994, Lemma 33.2]). We will henceforth refer to such a property as *Indifference Principle.*

Example 1.4. The example illustrated in Fig. 1.6 is borrowed from [Bressan, 2010] and describes a two-player continuous infinite game where the set of actions are segments in $\mathbb{R}$ (see horizontal and vertical axes). Level curves show that the maxima are attained at point P and Q for player 1 and 2 respectively. Note that the Nash equilibrium, which is point R, has horizontal and vertical tangents to the level curves

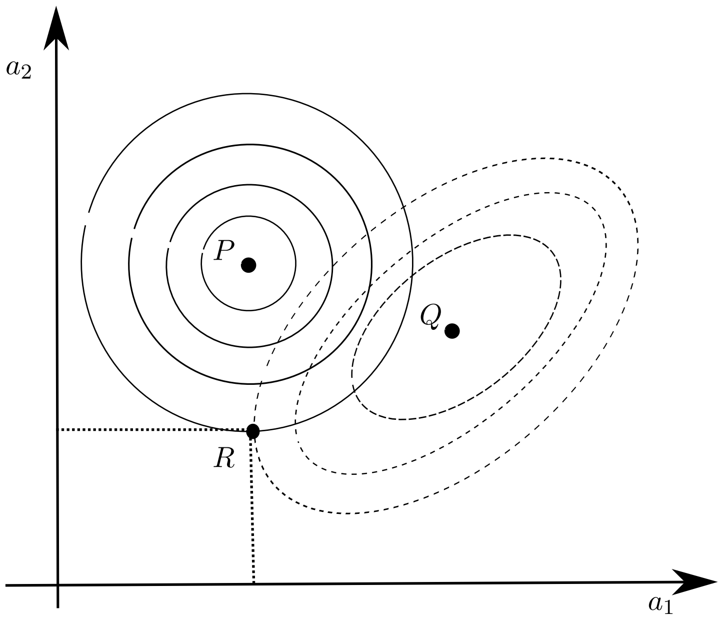

Figure 1.6: Two-player continuous infinite game. Level curves of player 1 (solid) and player 2 (dashed), action space of player 1 (horizontal axis), and of player 2 (vertical axis). Global maximum is P for player 1 and Q for player 2 while the NE is point R. Courtesy by Alberto Bressan, Noncooperative Differential Games. A Tutorial (2010) [Bressan, 2010].

of player 1 and 2 passing through it. From belonging to a horizontal tangent we know that the horizontal coordinate of point R is the best response of player 1 to player 2. Likewise, from belonging to a vertical tangent, the vertical coordinate of R is the best response of player 2 to player 1.

1.3.3 Dominant strategies

While the concept of equilibrium involves action profiles, the property of dominance is a characteristic related to a single action. Thus we say that an action profile is an NE, and that a given action is dominant. Dominance is a strong property, in that we know that an action profile made by dominant strategies is an NE but the converse is not true, i.e., we can have an NE that does not involve dominant strategies.

Definition 1.3. (Weak Dominance) Given two strategies, $a_i^*, a_i \in \mathcal{A}_i$, we say that a_i^* *weakly dominates* a_i if it is at least as good as a_i for all choices of the other players $a_{-i} \in \mathcal{A}_{-i}$,

$$u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i}), \quad \forall a_{-i} \in \mathcal{A}_{-i}.$$

If the above inequality holds strictly, then we say that a_i^* *(strictly) dominates* a_i.

Example 1.5. In the Prisoner's Dilemma, strategy D is a dominant strategy.

	C	*D*
C	(3,3)	(0,4)
D	(4,0)	(1,1)

Figure 1.7: D is a dominant strategy in the Prisoner's dilemma.

We say that a strategy is (weakly) dominant if it (weakly) dominates all other strategies. Note that a profile of dominant strategies is a Nash equilibrium. However, the converse is not true. Dominance is used in the renowned *iterated dominance algorithm*, which at each iteration eliminates subsets of dominated solutions by pruning the corresponding node in a so-called exploration tree. The term "pruning" is common in the parlance of combinatorial optimization, where the search for the optimum is graphically represented by an exploration tree. Here nodes describe families of solutions, and if, based on estimates, a family does not contain the optimum, then one says that the corresponding node is pruned. Back to the game theoretic setting, it is well known that a rationalizable/serially undominated strategy survives to the algorithm pruning. This is illustrated next for the Cournot duopoly.

1.4 Cournot duopoly and iterated dominance

Consider two manufacturers or firms $i = 1, 2$ competing on a same market. The production quantity of firm i is q_i. From the "law of demand", we assume that the sale price of firm i is $c_i = 30 - (q_1 + q_2)$.

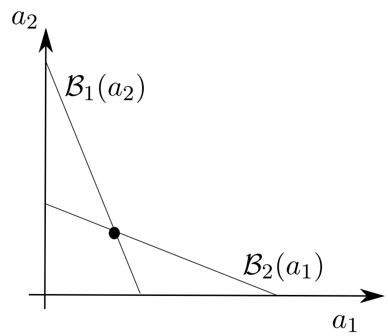

Figure 1.8: Reaction curves or best response for the Cournot duopoly.

Then, the payoff of firm i obtained by selling the produced quantity q_i at the price c_i is given by $u_i(q_i, q_j) = c_i q_i = 30q_i - q_i^2 - q_i q_j$. Note that this payoff is concave in q_i. Taking the derivative equal to zero, namely, $\frac{\partial u_i}{q_i} = 0$, it yields the best response $q_i^* = \mathcal{B}_i(q_j) = 15 - q_j/2$. The Nash equilibrium is $(10, 10)$, namely the intersection of the best response functions (see Fig. 1.8).

1.4.1 Iterated dominance algorithm

Figure 1.9 illustrates the iterated dominance algorithm on the Cournot duopoly model. To every iteration corresponds one round of elimination, which eliminates dominated strategies. Let us denote S_i^j the set of actions of player i that have survived to the elimination rounds up to iteration j. Then, one round of elimination yields $S_1^1 = [0, 1/2]$, $S_2^1 = [0, 1/2]$ (left). Indeed, player 1 knows that any production rate greater than $1/2$ is a dominated action for player 2 as $\mathcal{B}_2(a_1)$ lives in the range $[0, 1/2]$. We can repeat the same reasoning for player 2. Thus the search for equilibria can be restricted to the new domain $[0, 1/2]$ for both players (dotted square on the left plot). A second round yields $S_1^2 = [1/4, 1/2]$, $S_2^2 = [1/4, 1/2]$ (right). Actually, after restricting the best responses to the dotted square (left), every player knows that the best response of its opponent lives in the range $[1/4, 1/2]$. Thus the new search domain is $[1/4, 1/2]$ for both players which corresponds to the square (solid line) on the right plot. Repeating the same reasoning iteratively, the algorithm is proven to converge to the Nash equilibrium.

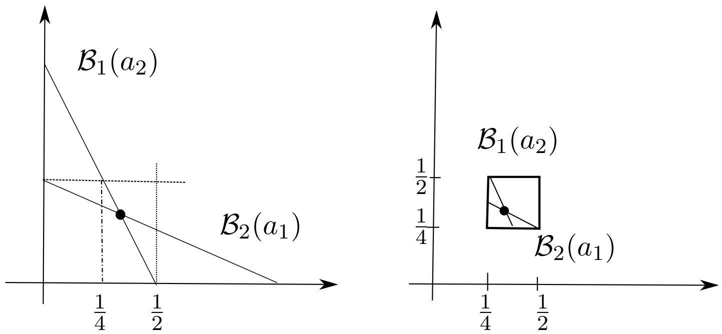

Figure 1.9: Iterated dominance algorithm illustrated on the Cournot duopoly model. Courtesy by Asu Ozdaglar, slides of the course 6.254 Game Theory with Eng. Applications, MIT OpenCourseWare (2010).

1.5 Examples

The rest of this chapter illustrates classical examples of strategic games such as: the *battle of the sexes*, the *coordination or typewriter game*, the *Hawk and dove or chicken game*, and the *Stag-Hunt game*.

Example 1.6. (Battle of the sexes) A couple agrees to meet in the evening either to go shopping S or to attend a cricket match C. The husband (column player) prefers to go to the cricket game while the wife (row player) would like to go shopping. In any case, both wish to go to the same place. Payoffs measure the happiness of the two. If they both go shopping, i.e. (SS), the woman is happy 2 and the husband is happy 1, while if they both go to the cricket game the happiness levels swap, 2 for the husband and 1 for the woman. If they end up in different places the level of happiness of both is 0.

	S	C
S	(2,1)	(0,0)
C	(0,0)	(1,2)

Example 1.7. (Coordination game or Typewriter game) Usually presented as a stylized model for diffusion of innovation, when and where is it convenient to adopt a new technology, this game considers a couple which agrees to meet in the evening either to go to a Mozart's or to a Mahler's concert. Both players have a small preference for Mozart

and if they both select $(Mozart, Mozart)$, then each level of happiness is 2. The levels of happiness are a bit lower, say 1, if they both go to a Mahler concert, i.e., $(Mahler, Mahler)$. Going to two different concerts returns a level of happiness equal to 0 to both. The action profiles $(Mozart, Mozart)$ and $(Mahler, Mahler)$ are both NE solutions.

	$Mozart$	$Mahler$
$Mozart$	(2,2)	(0,0)
$Mahler$	(0,0)	(1,1)

Example 1.8. (Hawk and dove or chicken game) The underlying idea is that while each player prefers not to give in to the other, the worst possible outcome occurs when both players do not yield. The game simulates a situation where two drivers drive towards each other and the one who swerves at the last moment is addressed "chicken". The same game under the name "Hawk and Dove" describes a scenario where two contestants can choose a nonaggressive or aggressive attitude. The game was useful to illustrate the strategic scenario during the cold war and in particular in occasion of the Cuban Missile Crisis. The game is mostly meaningful in the case where the cost of fighting exceeds the prize of victory, i.e., $C > V > 0$. If both player decide for a nonaggressive behavior and share the prey, i.e. they opt for $(Dove, Dove)$, their mutual reward is half of the prize of victory, $V/2$. If one yields, $(Hawk, Dove)$ or $(Dove, Hawk)$ the winner gets the entire prey, V and the loser is left with zero reward. If both players are aggressive and end up fighting, $(Hawk, Hawk)$, each will pay a cost equal to half of the prize of victory subtracted to the cost of fight. The game admits two NE solutions, $(Hawk, Dove)$ and $(Dove, Hawk)$.

	$Hawk$	$Dove$
$Hawk$	$\left(\frac{V-C}{2}, \frac{V-C}{2}\right)$	$(V, 0)$
$Dove$	$(0, V)$	$\left(\frac{V}{2}, \frac{V}{2}\right)$

Example 1.9. (Stag-Hunt game) Used to analyze and predict social cooperation, this game illustrates situations where two individuals can go out on a hunt and collaborate or not collaborate. Each hunter can decide to hunt a stag or hunt a hare without knowing what the other

is going to do. If both cooperate and go for a stag, $(Stag, Stag)$, they will share a large prey and each revenue is $3/2$. If both go for a hare $(Hare, Hare)$ the revenue to share is lower and equal to 1. If they go for different preys, $(Stag, Hare)$ or $(Hare, Stag)$ the one who goes for the smaller prey (the hare) gets the entire prey for himself, while the other is left with nothing as hunting a stag alone is not possible. He must have the cooperation of his partner in order to succeed. An individual can get a hare by himself, but a hare is worth less than a stag.

	$Stag$	$Hare$
$Stag$	$\left(\frac{3}{2}, \frac{3}{2}\right)$	(0,1)
$Hare$	(1,0)	$\left(1, 1\right)$

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