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Graziano Chesi
The University of Hong Kong
chesi@eee.hku.hk

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Contents

LMI-Based Robustness Analysis in Uncertain Systems

Graziano Chesi

Department of Electrical and Electronic Engineering, The University of Hong Kong, Hong Kong; chesi@eee.hku.hk

ABSTRACT

The study of uncertain systems has undoubtedly played a primary role in the history of control engineering as unknown quantities are often present in the available mathematical model of a plant. This monograph aims to provide the reader with a unified framework for the fundamental and challenging area of robustness analysis of uncertain systems, where even the most basic problem of establishing robust stability may be still open due to complexity and conservatism even for a third order system linearly affected by a scalar parameter. The described framework is based on linear matrix inequalities (LMIs) and exploits polynomials that can be expressed as sums of squares of polynomials (SOS). The interest for this framework is motivated by several reasons, such as allowing to consider various types of uncertainties, providing guarantees for robust stability and robust performance, requiring the solution of convex optimization problems, allowing for trade-off between conservatism and complexity, and including a number of methods in the literature as special cases. Several numerical examples are also provided to illustrate the use and potentialities

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of the presented framework, shedding some light on what can be achieved and what cannot.

1

Introduction

1.1 A Brief History

Robustness analysis via LMIs started in the 1970s when the problem of simultaneous stability was addressed by looking for a common quadratic Lyapunov function through LMIs, see, e.g., Araki (1976), Barker *et al.* (1978), Khalil and Kokotovic (1979), Boyd and Yang (1989), and Nemirovski (1993). In the 1990s, the celebrated book by Boyd *et al.* (1994), not only generalized this framework to systems with different types of uncertainties such as polytopic and norm bounded, for both robust stability analysis and robust performance analysis, but especially introduced LMIs and their potentialities to the research community, opening the doors to the huge development of various and sophisticated LMI methods that the last three decades have witnessed. In the area of robustness analysis, this includes the development of LMI methods based on nonquadratic or parameter dependent Lyapunov functions, which have been introduced for reducing the conservatism of the classic common quadratic Lyapunov functions, see, e.g., Zelentsovsky (1994), Jarvis-Wloszek and Packard (2002), and Chesi *et al.* (2003) for contributions based on common polynomial Lyapunov functions, Gahinet *et al.* (1996), Neto (1999), Oliveira *et al.* (1999), and

Dettori and Scherer (2000) for contributions based on linearly parameter dependent quadratic Lyapunov functions, Zhang *et al.* (2003), Bliman (2004), Chesi *et al.* (2005b), and Ebihara and Hagiwara (2005) for contributions based on polynomially parameter dependent quadratic Lyapunov function, and Chesi *et al.* (2007) for contributions based on polynomially parameter dependent polynomial Lyapunov functions. LMIs have been exploited for robustness analysis using not only Lyapunov functions but also other tools such as determinants, eigenvalue combinations, Hermite matrices and stability tables, see, e.g., Chesi *et al.* (1999) and Henrion *et al.* (2004). Last but not least, LMIs have been exploited for robustness analysis not only in the case of uncertainties defined in the time domain, but also in the case of uncertainties defined in the frequency domain, in particular through the Kalman-Yakubovich-Popov lemma that establishes an equivalence between a frequency condition and the existence of a quadratic Lyapunov function, see, e.g., Rantzer (1996) and Iwasaki and Hara (2005).

The scope of the monograph is to present a unified framework for robustness analysis of uncertain systems via LMIs. The considered model for uncertain systems includes both parametric and nonparametric uncertainties, the former represented by a vector of time varying parameters supposed constrained with their time variations into semialgebraic sets (which include, e.g., polytopes, multi intervals, hyper ellipsoids), the latter by a generic block connected in closed loop supposed constrained by polynomial constraints on the input and output (that allow to consider, e.g., norm bounded, positive real, and sector bounded uncertainties). LMI conditions are presented for establishing robust stability and various robust performance indexes (i.e., decay rate, L_2 gain, dissipation, impulse response energy, impulse response peak and D-stability) of this model of uncertain systems by exploiting the theory of positive polynomials, in particular, SOS polynomials. The interest for these conditions is fivefold: 1) allow to consider various type of uncertainties; 2) provide guarantees for robust stability and various robust performance indexes; 3) require the solution of convex optimization problems; 4) allow for trade-off between conservatism and complexity; 5) include a number of methods in the literature (such as the classical methods based on quadratic Lyapunov functions) as special cases.

Several numerical examples are also provided to illustrate the use and potentialities of the presented framework. These examples study uncertain systems with various orders and various numbers and types of uncertainties, and aim to shed some light on what can be achieved by the presented framework and what cannot.

It is worth remarking that even the problem of establishing robust stability, which is the most basic one in robustness analysis, may be still open even for a third order system linearly affected by a scalar parameter as shown in the numerical examples in Section 5.3. This is due to two main factors. The first factor is complexity, as the computational burden quickly grows with the system dimensions (i.e., order of the system, number of uncertainties, and degree of the system matrices on the uncertainties) and with the degree of the polynomials used for the investigation (e.g., Lyapunov functions). The second factor is conservatism, as the available conditions for establishing robust stability are generally only sufficient, and, though some of them may be also necessary by suitably increasing the computational burden (e.g., by increasing the degree of the Lyapunov functions), it is generally unknown a priori the increment required for achieving necessity.

A final note concerns our previous work Chesi *et al.* (2009), where robustness analysis of uncertain systems with polytopic uncertainty is addressed by exploiting a class of SOS polynomials, specifically, SOS homogeneous polynomials. This monograph aims to provide a more general framework, in particular by considering nonparametric uncertainties in addition to the parametric uncertainty, by allowing the set of admissible parametric uncertainty to be a generic semialgebraic set rather than a polytope only, and by addressing the investigation of various robust performance indexes that are not considered in our previous work.

1.2 Section Content

The monograph is organized in seven sections. Section 1 introduces some historical notes, a summary of the contributions of the various sections, and the notation.

Section 2 introduces LMIs and three main optimization problems with LMIs, specifically, the LMI problem (LMIP), which considers the task of establishing feasibility of a system of LMIs, the eigenvalue problem (EVP) or semidefinite program (SDP), which consider the minimization of the maximum eigenvalue of an affine matrix function subject to LMIs or equivalently the minimization of a linear function subject to LMIs, and the generalized eigenvalue problem (GEVP), which considers the minimization of the maximum generalized eigenvalue of a pair of affine matrix functions subject to LMIs.

Section 3 investigates positive matrix polynomials, in particular by introducing the class of SOS matrix polynomials, i.e., matrix polynomials that can be expressed as sums of matrix polynomials multiplied by their transposes, and the Gram matrix method, which allows to establish if a matrix polynomial is SOS via an LMIP. This leads to the introduction of three optimization problems, namely, SOS-LMIP, SOS-SDP and SOS-GEVP, that extend the optimization problems with LMIs defined in Section 2 by including constraints that impose that some matrix polynomials with coefficients depending linearly on some decision variables are SOS. Conditions for establishing positive semidefiniteness or definiteness of a matrix polynomial are formulated as an SOS-LMIP by recalling that nonnegative polynomials can be expressed as sums of squares of rational functions. Analogous conditions are presented for establishing double positive semidefiniteness or definiteness, and for establishing positive semidefiniteness or definiteness over a semialgebraic set or over the simplex.

Section 4 introduces the model considered in this monograph for uncertain systems, namely, the mixed parametric nonparametric representation (MPNR), which is a state space model that includes parametric and nonparametric uncertainties. This model can be seen as a generalization of three basic models for uncertain systems, specifically, the parametric direct representation (PDR), where the system matrices are rational functions of the parametric uncertainty, the parametric linear fractional representation (PLFR), where a closed loop is built through auxiliary inputs and outputs connected via a matrix gain that depends rationally on the parametric uncertainty, and the nonparametric linear fractional representation (NLFR), where an analogous closed loop is built via a generic block regarded as nonparametric uncertainty. In all models, the parametric uncertainty and its time variation are supposed constrained into semialgebraic sets, while the nonparametric uncertainty is supposed constrained by polynomial constraints that can model typical uncertainties in the literature such as norm bounded, positive real, and sector bounded uncertainties. Analysis conditions for the MPNR are hence formulated through SOS-LMIPs by exploiting the methods presented in Section 3.

Section 5 investigates robust stability of the MPNR. A sufficient condition based on the search for a polynomial Lyapunov function and testable with an SOS-LMIP is presented. The specialization of this condition to the three basic models for uncertain systems included in the MPNR, as well as the necessity of the resulting conditions, are discussed. It is also shown how classical LMI methods for quadratic stability are covered as special cases. Hence, alternative LMI conditions for establishing robust stability of the MPNR in the TI parametric mode (i.e., when the parametric uncertainty is TI and the nonparametric uncertainties are absent) are presented based on the use of tables and determinants.

Section 6 investigates robust performance of the MPNR. Specifically, the worst-case value of the decay rate, $\mathcal{L}_2$ gain, dissipation, impulse response energy and impulse response peak, are investigated, showing that bounds for these quantities can be established or searched for via SOS-LMIPs, SOS-SDPs or SOS-GEVPs. A physical system, in particular, an electric circuit with variable system order and variable number of uncertainties, is introduced to illustrate the use of these methods. Lastly, the problem of establishing robust D-stability of the MPNR in TI parametric mode is considered.

Lastly, Section 7 concludes the monograph with some final remarks.

1.3 Notation

The notation adopted in this monograph is as follows:

- $\mathbb{C}, \mathbb{R}, \mathbb{N}$: sets of complex numbers, real numbers, nonnegative integers;
- $\mathbb{S}^n$: set of symmetric matrices in $\mathbb{R}^{n\times n}$;

- **0**: null matrix of size specified by the context;
- I_n, I: identity matrices of size $n \times n$ and size specified by the context;
- $\mathrm{Re}(x), \mathrm{Im}(x), |x|$: real part, imaginary part, magnitude of x;
- X^T, X^H: transpose and conjugate transpose of matrix X;
- $\mathrm{He}(X)$: $X + X^H$;
- $\det(X), \mathrm{spec}(X), \mathrm{tr}(X), \lambda_{min}(X), \lambda_{max}(X)$: determinant, spectrum, trace, minimum eigenvalue, maximum eigenvalue of matrix X;
- $\mathrm{diag}(X_1, X_2, \ldots)$: block diagonal matrix constructed with ordered blocks $X_1, X_2, \ldots$;
- $\mathrm{co}(X_1, X_2, \ldots)$: convex hull of $X_1, X_2, \ldots$;
- $X > 0$, $X \geq 0$, $X = 0$: entrywise positive, nonnegative, zero matrix X;
- $X \succ 0$, $X \succeq 0$: positive definite, positive semidefinite matrix X;
- $X \otimes Y$: Kronecker product of matrices X, Y;
- $\|X\|$: 2-norm of matrix X;
- $\|x\|_p$: p-norm of vector x;
- x^y (with $x, y \in \mathbb{R}^n$): $x_1^{y_1} x_2^{y_2} \cdots x_n^{y_n}$;
- x^z (with $x \in \mathbb{R}^n$, $z \in \mathbb{R}$): $x_1^z x_2^z \cdots x_n^z$;
- $\mathrm{sum}(x)$: sum of the entries of vector x;
- $\mathrm{vec}(X)$: column vector obtained by stacking the columns of matrix X from the first to the last;
- $\mathrm{ver}(\mathcal{X})$: set of vertices of polytope $\mathcal{X}$;
- $\dot{x}(t)$: derivative of $x(t)$ with respect to t;

- $\star$: corresponding block in symmetric matrices;
- s.t.: subject to.

1.4 Acronyms

The following acronyms are used in this monograph:

- ATV: arbitrarily time varying;
- BRTV: bounded rate time varying;
- CT: continuous time;
- DT: discrete time;
- EVP: eigenvalue problem;
- GEVP: generalized eigenvalue problem;
- LMI: linear matrix inequality;
- LMIP: linear matrix inequality problem;
- LTI: linear time invariant;
- LTV: linear time varying;
- MPNR: mixed parametric nonparametric representation;
- NB: norm bounded;
- NLFR: nonparametric linear fractional representation;
- NR: number of rows;
- NV: number of variables;
- OOM: out of memory;
- PDR: parametric direct representation;
- PLFR: parametric linear fractional representation;

- PNS: positive not sum of squares of polynomials;
- PR: positive real;
- SB: sector bounded;
- SDP: semidefinite program;
- SOS: sum of squares of polynomials;
- TI: time invariant.

1.5 Software

The SOS problems mentioned in this monograph (i.e., SOS-LMIP, SOS-SDP and SOS-GEVP) can be solved with existing SOS program solvers such as SOSTools, Yalmip, GloptiPoly, etc., which convert the SOS problems into LMI problems (i.e., LMIP, SDP, GEVP) and pass the obtained LMI problems to an LMI solver such as the LMI toolbox, SeDuMi, Sdpt3, Mosek, etc.

The solutions of the numerical examples reported in this monograph are computed with Matlab on a standard computer with Windows 11, Intel Core i7, 3.2 GHz, 16 GB RAM, and approximated to the third fractional digit unless reported otherwise. The SOS problems are converted into LMI problems by generating the Gram matrices with the Matlab code reported in Appendix G, and the obtained LMI problems are solved with the LMI solver SeDuMi (Sturm, 1999). Some examples on the use of this Matlab code are reported in Appendixes A–F. The Matlab code reported in Appendixes A–G can be directly used by simple copy and paste.

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