Quantum Hamiltonian Complexity

Sevag Gharibian
Simons Institute for the Theory of Computing,
University of California, Berkeley

Yichen Huang
University of California, Berkeley

Zeph Landau
Simons Institute for the Theory of Computing,
University of California, Berkeley

Seung Woo Shin
University of California, Berkeley

the essence of knowledge
Boston — Delft

Foundations and Trends® in Theoretical Computer Science

Published, sold and distributed by:
now Publishers Inc.
PO Box 1024
Hanover, MA 02339
United States
Tel. +1-781-985-4510
www.nowpublishers.com
sales@nowpublishers.com

Outside North America:
now Publishers Inc.
PO Box 179
2600 AD Delft
The Netherlands
Tel. +31-6-51115274

The preferred citation for this publication is

S. Gharibian, Y. Huang, Z. Landau and S. W. Shin. *Quantum Hamiltonian Complexity*. Foundations and Trends® in Theoretical Computer Science, vol. 10, no. 3, pp. 159–282, 2014.

This Foundations and Trends® issue was typeset in LaTeX using a class file designed by Neal Parikh. Printed on acid-free paper.

ISBN: 978-1-68083-007-1

Editorial Scope

Topics

Foundations and Trends® in Theoretical Computer Science publishes surveys and tutorials on the foundations of computer science. The scope of the series is broad. Articles in this series focus on mathematical approaches to topics revolving around the theme of efficiency in computing. The list of topics below is meant to illustrate some of the coverage, and is not intended to be an exhaustive list.

- Algorithmic game theory
- Computational algebra
- Computational aspects of combinatorics and graph theory
- Computational aspects of communication
- Computational biology
- Computational complexity
- Computational geometry
- Computational learning
- Computational Models and Complexity
- Computational Number Theory
- Cryptography and information security
- Data structures
- Database theory
- Design and analysis of algorithms
- Distributed computing
- Information retrieval
- Operations research
- Parallel algorithms
- Quantum computation
- Randomness in computation

Information for Librarians

Foundations and Trends® in Theoretical Computer Science, 2014, Volume 10, 4 issues. ISSN paper version 1551-305X. ISSN online version 1551-3068. Also available as a combined paper and online subscription.

Foundations and Trends® in
Theoretical Computer Science
Vol. 10, No. 3 (2014) 159–282

DOI: 10.1561/0400000066

the essence of knowledge

Quantum Hamiltonian Complexity

Sevag Gharibian
Simons Institute for the Theory of Computing,
University of California, Berkeley

Yichen Huang
University of California, Berkeley

Zeph Landau
Simons Institute for the Theory of Computing,
University of California, Berkeley

Seung Woo Shin
University of California, Berkeley

Contents

1 Introduction **2**

2 Preliminaries **6**
2.1 Basics of quantum information 6
2.2 Basics of Quantum Hamiltonian Complexity 17

3 Roadmap and Organization **22**

4 A Brief History **24**
4.1 The computer science perspective 24
4.2 The physics perspective 29
4.3 Selected recent developments 31

5 Motivations From Physics **33**
5.1 Setting the stage 33
5.2 Time evolution versus thermal equilibrium 34
5.3 Where do Hamiltonians come from? 36
5.4 The study of Hamiltonians: Techniques and tools 37

6 Physics Concepts in Greater Depth **48**
6.1 A glossary of physics terms 48
6.2 Mean-field theory 56

6.3 Tensor networks 60
6.4 Density Matrix Renormalization Group 63
6.5 Multi-Scale Entanglement Renormalization Ansatz 67
6.6 Area laws 70

7 Reviews of Selected Results **75**
7.1 5-local Hamiltonian is QMA-complete 75
7.2 2-local Hamiltonian is QMA-complete 82
7.3 Commuting k-local Hamiltonians and the Structure Lemma 89
7.4 Quantum 2-SAT is in P 94
7.5 Area laws for one-dimensional gapped quantum systems . . 100

Acknowledgements **109**

References **111**

Abstract

Constraint satisfaction problems are a central pillar of modern computational complexity theory. This survey provides an introduction to the rapidly growing field of Quantum Hamiltonian Complexity, which includes the study of quantum constraint satisfaction problems. Over the past decade and a half, this field has witnessed fundamental breakthroughs, ranging from the establishment of a "Quantum Cook-Levin Theorem" to deep insights into the structure of 1D low-temperature quantum systems via so-called area laws. Our aim here is to provide a computer science-oriented introduction to the subject in order to help bridge the language barrier between computer scientists and physicists in the field. As such, we include the following in this survey: (1) The motivations and history of the field, (2) a glossary of condensed matter physics terms explained in computer-science friendly language, (3) overviews of central ideas from condensed matter physics, such as indistinguishable particles, mean field theory, tensor networks, and area laws, and (4) brief expositions of selected computer science-based results in the area. For example, as part of the latter, we provide a novel information theoretic presentation of Bravyi's polynomial time algorithm for Quantum 2-SAT.

S. Gharibian, Y. Huang, Z. Landau and S. W. Shin. *Quantum Hamiltonian Complexity*. Foundations and Trends® in Theoretical Computer Science, vol. 10, no. 3, pp. 159–282, 2014.
DOI: 10.1561/0400000066.

1

Introduction

> *"Computers are physical objects, and computations are physical processes. What computers can or cannot compute is determined by the laws of physics alone..."*
> *— David Deutsch [125]*

The Cook-Levin Theorem [53, 111], which states that the SATISFIABILITY problem is NP-complete, is one of the cornerstones of modern computational complexity theory [22]. One of its implications is the following simple, yet powerful, statement: Computation is, in a well-defined sense, *local.* Yet, as David Deutsch's quote above perhaps foreshadows, this is not the end of the story, but rather its beginning. Indeed, just as a sequence of computational steps on a Turing machine can be encoded into local classical constraints (as in the Cook-Levin theorem), the quantum world around us also evolves "locally", and this quantum evolution can be encoded into an analogous notion of local *quantum* constraints. The study of such quantum constraint systems underpins an emerging field at the intersection of condensed matter physics, computer science, and mathematics, known as *Quantum Hamiltonian Complexity (QHC).*

At the heart of QHC lies a central object of study: The notion of a *local Hamiltonian* H, which can intuitively be thought of as a quantum constraint system (in this introduction, we will keep our discussion informal in order to convey high-level ideas; all formal definitions, including an introduction to quantum information, are given in Chapter 2). To introduce local Hamiltonians, we begin with the fact that the state of a quantum system S on n qudits is described by some d^n-dimensional complex unit vector $|\psi\rangle \in (\mathbb{C}^d)^{\otimes n}$. How can we describe the evolution of the state $|\psi\rangle$ of S as time elapses? This is given by the Schrödinger equation, which says that after time t, the new state of our system is $e^{-iHt}|\psi\rangle$, where H is a $d^n \times d^n$-dimensional complex (more precisely, Hermitian) operator called a *Hamiltonian*. Here, the precise definition of the matrix exponential e^{iHt} is irrelevant; what *is* important is the dependence of the Schrödinger equation on H. In other words, Hamiltonians are intricately tied to the evolution of quantum systems. We thus arrive at a natural question: Which classes of Hamiltonians correspond to *actual* quantum evolutions for systems occurring in nature? It turns out that typically, only a special class of Hamiltonians is physically relevant: These are known as *local* Hamiltonians.

Roughly, a *k-local* Hamiltonian is a Hermitian matrix which has a succinct representation of the form

$$H = \sum_i H_i,$$

where each H_i acts "non-trivially" only on some subset of k qudits. Here, each H_i should be thought of as a "quantum constraint" or "clause", analogous to the notion of a k-local clause in classical constraint satisfaction problems. For example, just as a classical clause such as $(x_i \vee x_j \vee x_k)$ for $x_i, x_j, x_k \in \{0,1\}$ forces its bits to lie in set $x_i x_j x_k \in \{001, 010, 011, 100, 101, 110, 111\}$ (where $\vee$ denotes logical OR), a quantum clause H_i restricts the state of the k qudits it acts on to lie in a certain *subspace* of $(\mathbb{C}^d)^{\otimes n}$. Moreover, each clause H_i requires $O(k)$ bits express (assuming all matrix entries are specified to constant precision). This is because each H_i is given as a $d^k \times d^k$ complex matrix (this is made formal in Section 2.2). As a result, although H itself is a matrix of dimension $d^n \times d^n$, i.e. H has dimension exponential in n the number of qudits, the description of H in terms of local clauses $\{H_i\}$ has size *polynomial* in n.

Since local Hamiltonians are intricately tied to the time evolution of quantum systems in nature, the goal of QHC is to study properties of local Hamiltonians H. Common computational tasks include estimating the *ground state energy* (smallest eigenvalue) of H, or computing features of H's *ground state* (eigenvector corresponding to the smallest eigenvalue). Intuitively, the ground state can be thought of as the vector $|\psi\rangle$ which "maximally satisfies" the constraints $\{H_i\}$ (i.e. the "optimal solution" to the quantum constraint system), and is of particular interest as it encodes the state of the corresponding quantum system when cooled to low temperature. In fact, any classical Constraint Satisfaction Problem (CSP) of arity k can be embedded into a k-local Hamiltonian, such that determining the ground state of the Hamiltonian yields the optimal solution to the CSP. (This connection is made explicit in §2.2.) Thus, ground states are interesting from a complexity theoretic perspective.

Let us also motivate ground states from a physics perspective. Consider the case of helium-4: When cooled to near absolute zero, helium-4 relaxes to a state $|\psi\rangle$ which is the ground state of some local Hamiltonian H (the precise form of H is beyond the scope of this introduction). This ground state exhibits an exotic phase of matter known as *superfluidity* — it acts like a fluid with zero viscosity. (See [1] for a video demonstrating this remarkable phenomenon.) Ideally, we would like to understand the properties of the superfluid phase demonstrated by $|\psi\rangle$, so that, for example, we can in turn use this knowledge to design new, advanced materials. In this direction, QHC might ask questions such as: Which quantum systems in nature have a ground state with a succinct classical representation? Can we run efficient classical simulations to predict when a quantum system will exhibit interesting phenomena, such as a phase transition? Can we quantify the hardness of determining certain properties of local Hamiltonians by establishing connections to computational complexity theory? In the context of helium-4, for example, the first of these questions is particularly relevant — to the best of our knowledge, a closed form for the ground state energy or the ground state of helium-4 remain elusive. (Heuristic approximations based on variational methods, however, have long been known; see, e.g. [139].)

This state of affairs illustrates the formidable challenge facing QHC: Namely, we are interested in computing properties of k-local Hamiltonians H, which are matrices of dimension $d^n \times d^n$, whereas an efficient algorithm must run in time polynomial in n, the number of qudits H acts on. Despite this challenge, QHC has proven a very fruitful area of research. For example, in 1999 Kitaev established [106] a quantum version of the celebrated Cook-Levin theorem [53, 111] for local Hamiltonian systems. In 2006, Bravyi gave a polynomial time algorithm for solving the quantum analogue of 2-SATISFIABILITY, known as Quantum 2-SAT [38]. And though the heuristic approach of White [168, 169] (known as "Density Matrix Renormalization Group") was known to solve 1-dimensional (gapped) Hamiltonians in practice efficiently, Hastings' 1D area law in 2007 [88] helped explain the efficacy of this heuristic by strongly characterizing the entanglement structure of such 1-dimensional systems. This survey aims to review a select subset of such fundamental results in QHC.

To help make this survey accessible to computer scientists with little or no background in quantum information, we begin in §2.1 with a review of basic quantum information. We next establish some of the fundamental definitions of QHC in §2.2, including an explicit sketch of how an instance of 3-CSP can be encoded into a local Hamiltonian. With this basic background in place, we finally proceed in Chapter 3 to give a roadmap for the remainder of this survey.

References

[1] Superfluid helium. https://www.youtube.com/watch?v=2Z6UJbwxBZI.

[2] S. Aaronson. The quantum PCP manifesto, 2006. http://scottaaronson.com/blog/?p=139.

[3] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover Publications, New York, 1964.

[4] K. Van Acoleyen, M. Mariën, and F. Verstraete. Entanglement rates and area laws. *Physical Review Letters*, 111:170501, 2013.

[5] G. Adesso, S. Ragy, and A. R. Lee. Continuous variable quantum information: Gaussian states and beyond. *Open Systems & Information Dynamics*, 21:1440001, 2014.

[6] I. Affleck, T. Kennedy, E. Lieb, and H. Tasaki. Rigorous results on valence-bond ground states in antiferromagnets. *Physical Review Letters*, 59:799–802, 1987.

[7] I. Affleck, T. Kennedy, E. Lieb, and H. Tasaki. Valence bond ground states in isotropic quantum antiferromagnets. *Communications in Mathematical Physics*, 115(3):477–528, 1988.

[8] I. Affleck and E. Lieb. A proof of part of Haldane's conjecture on spin chains. *Letters in Mathematical Physics*, 12(1):57–69, 1986.

[9] D. Aharonov, I. Arad, and S. Irani. Efficient algorithm for approximating one-dimensional ground states. *Physical Review A*, 82:012315, 2010.

[10] D. Aharonov, I. Arad, Z. Landau, and U. Vazirani. The detectability lemma and quantum gap amplification. In *Proceedings of 41st ACM Symposium on Theory of Computing (STOC 2009)*, pages 417–426, 2009.

[11] D. Aharonov, I. Arad, Z. Landau, and U. Vazirani. The 1D area law and the complexity of quantum states: A combinatorial approach. In *Proceedings of the IEEE 52nd Annual Symposium on Foundations of Computer Science (FOCS 2011)*, pages 324–333, 2011.

[12] D. Aharonov, I. Arad, and T. Vidick. The quantum PCP conjecture. In *ACM SIGACT News*, volume 44, pages 47–79. 2013.

[13] D. Aharonov and L. Eldar. On the complexity of commuting local Hamiltonians, and tight conditions for Topological Order in such systems. In *Proceedings of the 52nd IEEE Symposium on Foundations of Computer Science (FOCS 2011)*, pages 334–343, 2011.

[14] D. Aharonov and L. Eldar. Commuting local Hamiltonians on expanders, locally testable quantum codes, and the qPCP conjecture. Available at arXiv.org e-Print quant-ph/1301.3407, 2013.

[15] D. Aharonov and L. Eldar. The commuting local Hamiltonian problem on locally expanding graphs is approximable in NP. *Quantum Information Processing*, 14(1):83–101, 2015.

[16] D. Aharonov, D. Gottesman, S. Irani, and J. Kempe. The power of quantum systems on a line. *Communications in Mathematical Physics*, 287:41–65, 2009.

[17] D. Aharonov, A. W. Harrow, Z. Landau, D. Nagaj, M. Szegedy, and U. Vazirani. Local tests of global entanglement and a counterexample to the generalized area law. In *Proceedings of the 55th IEEE Symposium on Foundations of Computer Science (FOCS 2014)*, pages 246–255, 2014.

[18] D. Aharonov and T. Naveh. Quantum NP - A survey. Available at arXiv.org e-Print quant-ph/0210077v1, 2002.

[19] I. Arad, A. Kitaev, Z. Landau, and U. Vazirani. An area law and subexponential algorithm for 1D systems. Available at arXiv.org e-Print quant-ph/1301.1162, 2013.

[20] I. Arad, Z. Landau, and U. Vazirani. Improved one-dimensional area law for frustration-free systems. *Physical Review B*, 85:195145, 2012.

[21] I. Arad, M. Santha, A. Sundaram, and S. Zhang. Linear time algorithm for quantum 2SAT. arXiv:1508.06340, 2015.

[22] S. Arora and B. Barak. *Computational Complexity: A Modern Approach*. Cambridge University Press, 2009.

[23] S. Arora, C. Lund, R. Motwani, M. Sudan, and M. Szegedy. Proof verification and the hardness of approximation problems. *Journal of the ACM*, 45(3):501–555, 1998. Prelim. version FOCS '92.

[24] S. Arora and S. Safra. Probabilistic checking of proofs: A new characterization of NP. *Journal of the ACM*, 45(1):70–122, 1998. Prelim. version FOCS '92.

[25] B. Aspvall, M. F. Plass, and R. E. Tarjan. A linear-time algorithm for testing the truth of certain quantified boolean formulas. *Information Processing Letters*, 8(3):121–123, 1979.

[26] F. Baharona. On the computational complexity of ising spin glass models. *Journal of Physics A - Mathematical and General*, 15:3241, 1982.

[27] N. Bansal, S. Bravyi, and B. M. Terhal. Classical approximation schemes for the ground-state energy of quantum and classical Ising spin Hamiltonians on planar graphs. *Quantum Information & Computation*, 9(7&8):701–720, 2009.

[28] J. D. Bekenstein. Black holes and entropy. *Physical Review D*, 7:2333, 1973.

[29] P. Benioff. The computer as a physical system: A microscopic quantum mechanical Hamiltonian model of computers as represented by Turing machines. *Journal of Statistical Physics*, 22:563–591, 1980.

[30] P. Benioff. Quantum mechanical Hamiltonian models of Turing machines. *Journal of Statistical Physics*, 29:515–546, 1982.

[31] P. Benioff. Quantum mechanical Hamiltonian models of Turing machines that dissipate no energy. *Physical Review Letters*, 48:1581–1585, 1982.

[32] H. Bethe. Zur Theorie der Metalle. *Zeitschrift für Physik*, 71(3–4):205–226, 1931.

[33] J. D. Biamonte and P. J. Love. Realizable Hamiltonians for universal adiabatic quantum computers. *Physical Review A*, 78:012352, 2008.

[34] L. Bombelli, R. K. Koul, J. Lee, and R. D. Sorkin. Quantum source of entropy for black holes. *Physical Review D*, 34:373, 1986.

[35] F. G. S. L. Brandão and M. Horodecki. Exponential decay of correlations implies area law. *Communications in Mathematical Physics*, 333(2):761–798, 2015.

[36] F. Brandão and M. Cramer. Entanglement area law from specific heat capacity. *Physical Review B*, 92:115134, 2015.

[37] F. Brandão and A. Harrow. Product-state approximations to quantum ground states. In *Proceedings of the 45th ACM Symposium on the Theory of Computing (STOC 2013)*, pages 871–880, 2013.

[38] S. Bravyi. Efficient algorithm for a quantum analogue of 2-SAT. Available at arXiv.org e-Print quant-ph/0602108v1, 2006.

[39] S. Bravyi. Monte Carlo simulation of stoquastic hamiltonians. Available at arXiv.org e-Print quant-ph/1402.2295, 2014.

[40] S. Bravyi, A. Bessen, and B. Terhal. Merlin-Arthur games and stoquastic complexity. Available at arXiv.org e-Print quant-ph/0611021v2, 2006.

[41] S. Bravyi, L. Caha, R. Movassagh, D. Nagaj, and P. W. Shor. Criticality without frustration for quantum spin-1 chains. *Physical Review Letters*, 109:207202, 2012.

[42] S. Bravyi, D. DiVincenzo, R. Oliveira, and B. Terhal. The complexity of stoquastic local Hamiltonian problems. *Quantum Information & Computation*, 8(5):0361–0385, 2008.

[43] S. Bravyi and M. Hastings. On complexity of the quantum Ising model. Available at arXiv.org e-Print quant-ph/1410.0703, 2014.

[44] S. Bravyi and B. Terhal. Complexity of stoquastic frustration-free Hamiltonians. *SIAM Journal on Computing*, 39(4):1462, 2009.

[45] S. Bravyi and M. Vyalyi. Commutative version of the local Hamiltonian problem and common eigenspace problem. *Quantum Information & Computation*, 5(3):187–215, 2005.

[46] W. J. L. Buyers, R. M. Morra, R. L. Armstrong, M. J. Hogan, P. Gerlach, and K. Hirakawa. Experimental evidence for the Haldane gap in a spin-1 nearly isotropic, antiferromagnetic chain. *Physical Review Letters*, 56:371–374, Jan 1986.

[47] P. Calabrese and J. Cardy. Entanglement entropy and quantum field theory. *Journal of Statistical Mechanics: Theory and Experiment*, 2004(06):P06002, 2004.

[48] P. Calabrese and J. Cardy. Entanglement entropy and conformal field theory. *Journal of Physics A: Mathematical and Theoretical*, 42(50):504005, 2009.

[49] A. M. Childs, D. Gosset, and Z. Webb. The Bose-Hubbard model is QMA-complete. In *Proceedings of the 41st International Colloquium on Automata, Languages, and Programming (ICALP 2014)*, pages 308–319, 2013.

[50] C. T. Chubb and S. T. Flammia. Computing the degenerate ground space of gapped spin chains in polynomial time. Available at arXiv.org e-Print quant-ph/1502.06967, 2015.

[51] I. Cirac and F. Verstraete. Renormalization and tensor product states in spin chains and lattices. *Journal of Physics A: Mathematical and Theoretical*, 42(50):504004, 2009.

[52] J. I. Cirac and P. Zoller. Goals and opportunities in quantum simulation. *Nature Physics*, 8:264–266, 2012.

[53] S. Cook. The complexity of theorem proving procedures. In *Proceedings of the 3rd ACM Symposium on Theory of Computing (STOC 1972)*, pages 151–158, 1972.

[54] T. Cubitt and A. Montanaro. Complexity classification of local Hamiltonian problems. In *Proceedings of the 55nd IEEE Symposium on Foundations of Computer Science (FOCS 2014)*, pages 120–129, 2014.

[55] T. Cubitt, D. Perez-Garcia, and M. M. Wolf. Undecidability of the spectral gap (full version). Available at arXiv.org e-Print quant-ph/1502.04573, 2015.

[56] N. de Beaudrap and S. Gharibian. A linear time algorithm for quantum 2-SAT. Available at arXiv.org e-Print quant-ph/1508.07338, 2015.

[57] B. DeMarco and D. S. Jin. Onset of Fermi degeneracy in a trapped atomic gas. *Science*, 285(5434):1703–1706, 1999.

[58] I. Dinur. The PCP theorem by gap amplification. *Journal of the ACM*, 54(3), 2007.

[59] J. Dubail and N. Read. Tensor network trial states for chiral topological phases in two dimensions. Available at arXiv.org e-Print cond-mat.mes-hall/1307.7726, 2013.

[60] Carl Eckart and Gale Young. The approximation of one matrix by another of lower rank. *Psychometrika*, 1(3):211, 1936.

[61] J. Eisert, M. Cramer, and M. B. Plenio. Area laws for the entanglement entropy. *Reviews of Modern Physics*, 82:277–306, Feb 2010.

[62] S. Even, A. Itai, and A. Shamir. On the complexity of time table and multi-commodity flow problems. *SIAM Journal on Computing*, 5(4):691–703, 1976.

[63] H. Fan, V. Korepin, and V. Roychowdhury. Entanglement in a valence-bond solid state. *Physical Review Letters*, 93:227203, 2004.

[64] M. Fannes, B. Nachtergaele, and R.F. Werner. Finitely correlated states on quantum spin chains. *Communications in Mathematical Physics*, 144(3):443–490, 1992.

[65] R. Feynman. Forces in molecules. *Physical Review*, 56:340–343, Aug 1939.

[66] R. Feynman. Simulating physics with computers. *International Journal of Theoretical Physics*, 21(6–7):467–488, 1982.

[67] R. Feynman. Quantum mechanical computers. *Optics News*, 11:11, 1985.

[68] J. Fitzsimons and T. Vidick. A multiprover interactive proof system for the local hamiltonian problem. In *Proceedings of the 2015 Conference on Innovations in Theoretical Computer Science (ITCS 2015)*, pages 103–112, 2015.

[69] Y. Ge and J. Eisert. Area laws and efficient descriptions of quantum many-body states. Available at arXiv.org e-Print quant-ph/1411.2995, 2014.

[70] S. Gharibian. Strong NP-hardness of the quantum separability problem. *Quantum Information and Computation*, 10(3&4):343–360, 2010.

[71] S. Gharibian. *Approximation, proof systems, and correlations in a quantum world.* PhD thesis, University of Waterloo, 2013. Available at arXiv.org e-Print quant-ph/1301.2632.

[72] S. Gharibian and J. Kempe. Approximation algorithms for QMA-complete problems. In *Proceedings of 26th IEEE Conference on Computational Complexity (CCC 2011)*, pages 178–188, 2011.

[73] S. Gharibian and J. Kempe. Hardness of approximation for quantum problems. In *Proceedings of 39th International Colloquium on Automata, Languages and Programming (ICALP 2012)*, pages 387–398, 2012.

[74] S. Gharibian, Z. Landau, S. W. Shin, and G. Wang. Tensor network non-zero testing. *Quantum Information & Computation*, 15(9&10):885–899, 2015.

[75] S. Gharibian and J. Sikora. Ground state connectivity of local Hamiltonians. In *Proceedings of the 42nd International Colloquium on Automata, Languages, and Programming (ICALP 2015)*, pages 617–628, 2015.

[76] N. Gisin. Hidden quantum nonlocality revealed by local filters. *Physics Letters A*, 210(8):151–156, 1996.

[77] D. Gosset and D. Nagaj. Quantum 3-SAT is QMA1-complete. In *Proceedings of the 54th IEEE Symposium on Foundations of Computer Science (FOCS 2013)*, pages 756–765, 2013.

[78] D. Gosset, B. M. Terhal, and A. Vershynina. Universal adiabatic quantum computation via the space-time circuit-to-Hamiltonian construction. *Physical Review Letters*, 114:140501, 2015.

[79] D. Gottesman and S. Irani. The quantum and classical complexity of translationally invariant tiling and Hamiltonian problems. In *Proceedings of the 50th IEEE Symposium on Foundations of Computer Science (FOCS 2009)*, pages 95–104, 2009.

[80] D. J. Griffiths. *Introduction to Quantum Mechanics*. Pearson Prentice Hall, 2nd edition, 2004.

[81] Z.-C. Gu and X.-G. Wen. Tensor-entanglement-filtering renormalization approach and symmetry-protected topological order. *Physical Review B*, 80:155131, Oct 2009.

[82] L. Gurvits. Classical deterministic complexity of Edmond's problem and quantum entanglement. In *Proceedings of the 35th Symposium on Theory of computing (STOC 2003)*, pages 10–19. ACM Press, 2003.

[83] F. Haldane. Continuum dynamics of the 1-D Heisenberg antiferromagnet: Identification with the O(3) nonlinear sigma model. *Physics Letters A*, 93(9):464–468, 1983.

[84] F. Haldane. Nonlinear field theory of large-spin Heisenberg antiferromagnets: Semiclassically quantized solitons of the one-dimensional easy-axis Néel state. *Physical Review Letters*, 50:1153–1156, 1983.

[85] S. Hallgren, D. Nagaj, and S. Narayanaswami. The Local Hamiltonian problem on a line with eight states is QMA-complete. *Quantum Information & Computation*, 13(9&10):0721–0750, 2013.

[86] M. B. Hastings. Lieb-Schultz-Mattis in higher dimensions. *Physical Review B*, 69:104431, 2004.

[87] M. B. Hastings. Solving gapped Hamiltonians locally. *Physical Review B*, 73:085115, 2006.

[88] M. B. Hastings. An area law for one-dimensional quantum systems. *Journal of Statistical Mechanics*, P08024(08), 2007.

[89] M. B. Hastings. Matrix product operators and central elements: Classical description of a quantum state. *Geometry & Topology Monographs*, 18:115–160, 2012.

[90] P. Hayden, D. Leung, and A. Winter. Aspects of generic entanglement. *Communications in Mathematical Physics*, 265(1):95–117, 2006.

[91] Y. Huang. Area law in one dimension: Degenerate ground states and Renyi entanglement entropy. Available at arXiv.org e-Print cond-mat.str-el/1403.0327, 2014.

[92] J. Hubbard. Electron correlations in narrow energy bands. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 276(1365):238–257, 1963.

[93] S. Irani. Ground state entanglement in one-dimensional translationally invariant quantum systems. *Journal of Mathematical Physics*, 51:022101, 2010.

[94] A. R. Its, B.-Q. Jin, and V. E. Korepin. Entanglement in the XY spin chain. *Journal of Physics A: Mathematical and General*, 38:2975, 2005.

[95] M. Jerrum and A. Sinclair. Polynomial-time approximation algorithms for the Ising model. *SIAM Journal on Computing*, 22(5):1087–1116, 1993.

[96] S. P. Jordan, D. Gosset, and P. J. Love. Quantum-Merlin-Arthur-complete problems for stoquastic Hamiltonians and Markov matrices. *Physical Review A*, 81:032331, 2010.

[97] R. Jördens, N. Strohmaier, K. Günter, H. Moritz, and T. Esslinger. A Mott insulator of fermionic atoms in an optical lattice. *Nature*, 455:204–207, 2008.

[98] R. Jozsa and N. Linden. On the role of entanglement in quantum-computational speed-up. *Proceedings of the Royal Society of London; Series A, Mathematical and Physical Sciences*, 459:2011–2032, 2003.

[99] R. Karp. Reducibility among combinatorial problems. In *Complexity of Computer Computations*, pages 85–103. New York: Plenum, 1972.

[100] A. Kay. Quantum-Merlin-Arthur-complete translationally invariant Hamiltonian problem and the complexity of finding ground-state energies in physical systems. *Physical Review A*, 76(3):030307, 2007.

[101] P. Kaye, R. Laflamme, and M. Mosca. *An Introduction to Quantum Computing*. Oxford University Press, 2007.

[102] J. Kempe, A. Kitaev, and O. Regev. The complexity of the local Hamiltonian problem. *SIAM Journal on Computing*, 35(5):1070–1097, 2006.

[103] J. Kempe and O. Regev. 3-local Hamiltonian is QMA-complete. *Quantum Information & Computation*, 3(3):258–264, 2003.

[104] A. Kitaev. Quantum NP, 1999. Talk at Second Workshop on Algorithms in Quantum Information Processing (AQIP 1999), DePaul University.

[105] A. Kitaev. Fault-tolerant quantum computation by anyons. *Annals of Physics*, 303(1):2–30, 2003.

[106] A. Kitaev, A. Shen, and M. Vyalyi. *Classical and Quantum Computation*. American Mathematical Society, 2002.

[107] E. Knill, R. Laflamme, and L. Viola. Theory of quantum error correction for general noise. *Physical Review Letters*, 84:2525, 2000.

[108] H. A. Kramers and G. H. Wannier. Statistics of the two-dimensional ferromagnet. Part II. *Physical Review*, 60:263–276, 1941.

[109] M. R. Krom. The decision problem for a class of first-order formulas in which all disjunctions are binary. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 13:15–20, 1967.

[110] Z. Landau, U. Vazirani, and T. Vidick. A polynomial-time algorithm for the ground state of 1D gapped local Hamiltonians. *Nature Physics*, 11:566–569, 2015.

[111] L. Levin. Universal sequential search problems. *Problems of Information Transmission*, 9(3):265–266, 1973.

[112] E. Lieb, T. Schultz, and D. Mattis. Two soluble models of an antiferromagnetic chain. *Annals of Physics*, 16(3):407–466, 1961.

[113] E. H. Lieb and F. Y. Wu. Absence of Mott transition in an exact solution of the short-range, one-band model in one dimension. *Physical Review Letters*, 20(25):1445–1448, 1968.

[114] N. Linden, S. Popescu, A. J. Short, and A. Winter. Quantum mechanical evolution towards thermal equilibrium. *Physical Review E*, 79:061103, 2009.

[115] Y.-K. Liu, M. Christandl, and F. Verstraete. Quantum computational complexity of the N-representability problem: QMA complete. *Physical Review Letters*, 98:110503, 2007.

[116] M. Mariën, K. M. R. Audenaert, K. Van Acoleyen, and F. Verstraete. Entanglement rates and the stability of the area law for the entanglement entropy. Available at arXiv.org e-Print quant-ph/1411.0680, 2014.

[117] C. Marriott and J. Watrous. Quantum Arthur-Merlin games. *Computational Complexity*, 14(2):122–152, 2005.

[118] L. Masanes. An area law for the entropy of low-energy states. *Physical Review A*, 80:052104, 2009.

[119] A. Meyer and L. Stockmeyer. The equivalence problem for regular expressions with squaring requires exponential time. In *Proceedings of the 13th Symposium on Foundations of Computer Science*, pages 125–129, 1972.

[120] A. Molnar, N. Schuch, F. Verstraete, and J. I. Cirac. Approximating Gibbs states of local Hamiltonians efficiently with projected entangled pair states. *Physical Review B*, 91(4):045138, 2015.

[121] R. M. Morra, W. J. L. Buyers, R. L. Armstrong, and K. Hirakawa. Spin dynamics and the Haldane gap in the spin-1 quasi-one-dimensional antiferromagnet $CsNiCl_3$. *Physical Review B*, 38:543–555, 1988.

[122] R. Movassagh and P. W. Shor. Power law violation of the area law in quantum spin chains. Available at arXiv.org e-Print quant-ph/1408.1657, 2014.

[123] D. Nagaj. *Local Hamiltonians in Quantum Computation*. PhD thesis, Massachusetts Institute of Technology, Boston, 2008. Available at arXiv.org e-Print quant-ph/0808.2117v1.

[124] D. Nagaj and S. Mozes. A new construction for a QMA complete 3-local Hamiltonian. *Journal of Mathematical Physics*, 48(7):072104, 2007.

[125] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.

[126] T. Nishino, T. Hikihara, K. Okunishi, and Y. Hieida. Density matrix renormalization group: Introduction from a variational point of view. *International Journal of Modern Physics B*, 13(1):1–24, 1999.

[127] R. Oliveira and B. M. Terhal. The complexity of quantum spin systems on a two-dimensional square lattice. *Quantum Information & Computation*, 8(10):0900–0924, 2008.

[128] L. Onsager. Crystal statistics. I. A two-dimensional model with an order-disorder transition. *Physical Review*, 65:117–149, 1944.

[129] T. J. Osborne. Hamiltonian complexity. *Reports on Progress in Physics*, 75(2):022001, 2012.

[130] S. Östlund and S. Rommer. Thermodynamic limit of density matrix renormalization. *Physical Review Letters*, 75:3537–3540, 1995.

[131] D. Perez-Garcia, F. Verstraete, M. M. Wolf, and J. I. Cirac. Matrix product state representations. *Quantum Information & Computation*, 7(5):401–430, 2007.

[132] P. Pfeuty. The one-dimensional Ising model with a transverse field. *Annals of Physics*, 57(1):79–90, 1970.

[133] F. Pollmann, E. Berg, A. M. Turner, and M. Oshikawa. Symmetry protection of topological phases in one-dimensional quantum spin systems. *Physical Review B*, 85:075125, 2012.

[134] J. Quintanilla and C. Hooley. The strong-correlations puzzle. *Physics World*, 22(6):32–37, 2009.

[135] J. P. Renard, M. Verdaguer, L. P. Regnault, W. A. C. Erkelens, J. Rossat-Mignod, and W. G. Stirling. Presumption for a quantum energy gap in the quasi-one-dimensional s = 1 Heisenberg antiferromagnet $Ni(C_2H_8N_2)_2NO_2(ClO_4)$. *Europhysics Letters*, 3(8):945, 1987.

[136] S. Rommer and S. Östlund. Class of ansatz wave functions for one-dimensional spin systems and their relation to the density matrix renormalization group. *Physical Review B*, 55:2164–2181, 1997.

[137] Subir Sachdev. *Quantum Phase Transitions*. Cambridge University Press, 2011.

[138] T. J. Schaefer. The complexity of satisfiability problems. In *Proceedings of the 10th Symposium on Theory of computing*, pages 216–226, 1978.

[139] D. Schiff and L. Verlet. Ground state of liquid helium-4 and helium-3. *Physical Review*, 160:208, 1967.

[140] U. Schneider, L. Hackermüller, S. Will, Th. Best, I. Bloch, T. A. Costi, R. W. Helmes, D. Rasch, and A. Rosch. Metallic and insulating phases of repulsively interacting fermions in a 3D optical lattice. *Science*, 322(5907):1520–1525, 2008.

[141] U. Schollwöck. The density-matrix renormalization group. *Reviews of Modern Physics*, 77:259–315, 2005.

[142] U. Schollwöck. The density-matrix renormalization group in the age of matrix product states. *Annals of Physics*, 326(1):96–192, 2011.

[143] E. Schrödinger. Die gegenwärtige Situation in der Quantenmechanik. *Naturwissenschaften*, 23(48):807–812, 1935.

[144] N. Schuch. Complexity of commuting Hamiltonians on a square lattice of qubits. *Quantum Information & Computation*, 11:901–912, 2011.

[145] N. Schuch, I. Cirac, and F. Verstraete. Computational difficulty of finding matrix product ground states. *Physical Review Letters*, 100:250501, 2008.

[146] N. Schuch and J. I. Cirac. Matrix product state and mean-field solutions for one-dimensional systems can be found efficiently. *Physical Review A*, 82:012314, 2010.

[147] N. Schuch and F. Verstraete. Computational complexity of interacting electrons and fundamental limitations of density functional theory. *Nature Physics*, 5:732–735, 2009.

[148] N. Schuch, M. Wolf, F. Verstraete, and J. I. Cirac. Computational complexity of projected entangled pair states. *Physical Review Letters*, 98:140506, 2007.

[149] N. Schuch, M. M. Wolf, F. Verstraete, and J. I. Cirac. Entropy scaling and simulability by matrix product states. *Physical Review Letters*, 100:030504, 2008.

[150] T. D. Schultz, D. C. Mattis, and E. H. Lieb. Two-dimensional ising model as a soluble problem of many fermions. *Reviews of Modern Physics*, 36:856–871, 1964.

[151] M. Srednicki. Entropy and area. *Physical Review Letters*, 71:666, 1993.

[152] F. Verstraete and J. I. Cirac. Renormalization algorithms for quantum-many body systems in two and higher dimensions. Available at arXiv.org e-Print cond-mat/0407066, 2004.

[153] F. Verstraete and J. I. Cirac. Matrix product states represent ground states faithfully. *Physical Review B*, 73(9):094423, 2006.

[154] F. Verstraete, V. Murg, and I. Cirac. Matrix product states, projected entangled pair states, and variational renormalization group methods for quantum spin systems. *Advances in Physics*, 57(2):143–224, 2008.

[155] F. Verstraete, D. Porras, and J. I. Cirac. Density matrix renormalization group and periodic boundary conditions: A quantum information perspective. *Physical Review Letters*, 93:227205, 2004.

[156] F. Verstraete, M. Wolf, D. Pérez-García, and J. I. Cirac. Projected entangled states: Properties and applications. *International Journal of Modern Physics B*, 20:5142, 2006.

[157] G. Vidal. Efficient classical simulation of slightly entangled quantum computations. *Physical Review Letters*, 91:147902, 2003.

[158] G. Vidal. Efficient simulation of one-dimensional quantum many-body systems. *Physical Review Letters*, 93:040502, 2004.

[159] G. Vidal. Entanglement renormalization. *Physical Review Letters*, 99:220405, 2007.

[160] G. Vidal. Class of quantum many-body states that can be efficiently simulated. *Physical Review Letters*, 101:110501, 2008.

[161] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev. Entanglement in quantum critical phenomena. *Physical Review Letters*, 90:227902, 2003.

[162] T. B. Wahl, H.-H. Tu, N. Schuch, and J. I. Cirac. Projected entangled-pair states can describe chiral topological states. *Physical Review Letters*, 111:236805, 2013.

[163] X.-B. Wang, T. Hiroshima, A. Tomita, and M. Hayashi. Quantum information with Gaussian states. *Physics Reports*, 448(1–4):1–111, 2007.

[164] J. Watrous. Lecture 2: Mathematical Preliminaries (Part 2), 2008. Latest version available at: `www.cs.uwaterloo.ca/~watrous/CS766/`.

[165] J. Watrous. *Encyclopedia of Complexity and System Science*, chapter Quantum Computational Complexity. Springer, 2009.

[166] T.-C. Wei, M. Mosca, and A. Nayak. Interacting boson problems can be QMA hard. *Physical Review Letters*, 104:040501, 2010.

[167] A. Weichselbaum, F Verstraete, U Schollwöck, J. I. Cirac, and J. von Delft. Variational matrix product state approach to quantum impurity models. *Physical Review B*, 80:165117, 2009.

[168] S. R. White. Density matrix formulation for quantum renormalization groups. *Physical Review Letters*, 69:2863–2866, 1992.

[169] S. R. White. Density-matrix algorithms for quantum renormalization groups. *Physical Review B*, 48:10345–10356, 1993.

[170] S. R. White and R. M. Noack. Real-space quantum renormalization groups. *Physical Review Letters*, 68:3487–3490, 1992.

[171] K. G. Wilson. Renormalization group and critical phenomena. I. Renormalization group and the kadanoff scaling picture. *Physical Review B*, 4:3174–3183, 1971.

[172] K. G. Wilson. Renormalization group and critical phenomena. II. Phase-space cell analysis of critical behavior. *Physical Review B*, 4:3184–3205, 1971.

[173] K. G. Wilson. Feynman-graph expansion for critical exponents. *Physical Review Letters*, 28:548–551, 1972.

[174] K. G. Wilson. The renormalization group: Critical phenomena and the kondo problem. *Reviews of Modern Physics*, 47:773–840, 1975.

[175] K. G. Wilson and M. E. Fisher. Critical exponents in 3.99 dimensions. *Physical Review Letters*, 28:240–243, 1972.

[176] M. M. Wolf. Violation of the entropic area law for fermions. *Physical Review Letters*, 96:010404, 2006.

[177] M. M. Wolf, F. Verstraete, M. B. Hastings, and J. I. Cirac. Area laws in quantum systems: Mutual information and correlations. *Physical Review Letters*, 100:070502, 2008.

[178] J. Yan and D. Bacon. The k-local Pauli commuting Hamiltonians problem is in P. Available at arXiv.org e-Print quant-ph/1203.3906, 2012.

[179] C. N. Yang and C. P. Yang. One-dimensional chain of anisotropic spin-spin interactions. I. Proof of Bethe's hypothesis for ground state in a finite system. *Physical Review*, 150:321–327, 1966.

[180] C. N. Yang and C. P. Yang. One-dimensional chain of anisotropic spin-spin interactions. II. Properties of the ground-state energy per lattice site for an infinite system. *Physical Review*, 150:327–339, 1966.