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Original Paper

MELODY: Analyzing the Message-Opinion Coevolution and the Messages' Influence on Opinion Dynamics in Social Networks

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ABSTRACT

For effective guidance of agents' opinions in social networks, it is important to understand how messages evolve and analyze their impact on agents' opinions. Opinion dynamics model how agents influence each other's opinions and how the entire network's opinions evolve. In the literature, many works have used opinion dynamics to study the influence of messages on agents' opinions. However, most works assume static messages or independence among messages at different times. Studies in mass media theory show that the message evolution process exhibits temporal continuity, randomness, and polarization features. In this work, we first propose the Bounded Brownian Message (BBM) model to describe the message evolution process, jointly considering the above features. We then combine the BBM model with the classic DeGroot opinion dynamics model and propose the Message EvoLution and Opinion DYnamics (MELODY) model to study the impact of message evolution on opinion dynamics. We theoretically analyze the probability distributions and statistics of messages and opinions and study how messages influence the agents' steady-state opinions. Simulations and real user tests validate our analyses. This study is critical to a

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better understanding of how messages shape agents' opinions in social networks and design effective mechanisms to guide agents' opinions.

Keywords: Message evolution, opinion dynamics, social network, statistical analysis.

1 Introduction

Social media platforms, such as newspapers and broadcasting, serve as channels for disseminating messages and wielding substantial influence over the opinions of agents, e.g., users, agencies, and organizations [24]. However, the dissemination of fake messages on social media platforms poses a serious threat to the integrity and accuracy of information. These fake messages, often designed to be persuasive, can easily influence agents' opinions [1]. For example, during political elections, fake messages about candidates' policies or personal lives can sway voters' opinions and potentially impact the election results [11]. In the realm of finance, fake messages related to stocks can exacerbate investors' underreaction to legitimate messages, thereby influencing investors' opinions of market trends and precipitating substantial negative fluctuations in stock prices [18]. Without proper verification and scrutiny, agents may be misled into adopting harmful or uninformed opinions. Therefore, it is critical to understand how messages shape agents' opinions in social networks and design effective mechanisms to guide agents' opinions [10].

1.1 Literature Review

From mass media theory, *message* refers to specific information or content produced by the *message source*, while *opinion* pertains to the subjective views and attitudes of agents towards the messages [42]. The seminal work in [33] provided a qualitative analysis that demonstrated the intricate interplay between messages and agents' opinions. Their findings revealed that agents' beliefs are not solely shaped by the messages they receive but are also significantly influenced by the opinions of their neighbors. Furthermore, this study highlighted the dynamic nature of both messages and agents' opinions, indicating that they coevolve over time.

In the following, we will review related works in opinion dynamics models and message evolution in mass media theory.

1.1.1 Opinion dynamics

Opinion dynamics model how agents influence each other's opinions and how the entire network's opinions evolve [55, 45, 61, 34, 62]. Over the past years, numerous opinion dynamics models have been proposed. The classic works include the DeGroot model [17], the Friedkin-Johnsen model [20], the Bounded Confidence models [16, 26], etc. The DeGroot model assumed that agents weigh the opinions of others based on their perceived expertise and update their opinions in a linear manner [17]. When the network is connected, opinion consensus can be reached where all agents hold the same opinion. To model the social behavior of stubborn agents, the Friedkin-Johnsen model extended the DeGroot model and introduced agents' intrinsic beliefs [20]. The Bounded Confidence models, including the Deffuant-Weisbuch model [16] and the Hegselmann-Krause model [26, 4], etc., described how agents update their opinions based on the opinions of others, but only to a certain degree of similarity, reflecting their limited willingness to accept different opinions. This phenomenon is referred to as selective exposure [52]. Based on the Hegselmann-Krause model, the Asynchronous Hegselmann-Krause model in Bernardo *et al.* [6] further considers the scenario where agents update their opinions one after another. Apart from the Deffuant-Weisbuch and Hegselmann-Krause models, many other works investigated the phenomenon of selective exposure in opinion dynamics. The Biased Opinion Formation model in Dandekar *et al.* [15] introduced a biased opinion term into the DeGroot model to model selective exposure. Building upon the Hegselmann-Krause model, the Stochastic Bounded Confidence model in [3] employed a function of agents' opinion distances as the weighting factor for opinion updates. All the above models come in the discrete-time and the continuous-time versions, utilizing difference and differential equations to describe the update of agents' opinions, respectively [37, 46].

1.1.2 Opinion dynamics models with static and dynamic messages

There have been many works using opinion dynamics models to study the influence of messages on agents' opinions. In Li and Zhu [36] and Yang *et al.* [59] and the Cyber-Social Network model in Mao *et al.* [38, 39], an agent maintaining a constant opinion, i.e., a stubborn agent, was introduced into the DeGroot model or the Friedkin-Johnsen model, serving as a time-invariant message that shapes the opinion updates of other agents. A similar approach was used in the Deffuant-Weisbuch model in Carletti *et al.* [12], Gargiulo *et al.* [22], Martins *et al.* [41], Sîrbu *et al.* [53, 54], and Pineda and Buendía [49]. In Hegselmann and Krause [27] and Kurz and Rambau [32], an agent first updates his/her opinion according to the Hegselmann-Krause model, and then takes the average of the updated opinion and a time-invariant message as the modified opinion, iterating this process continuously. Additionally, Crokidakis [14], Fotouhi and Rabbat [19], Colaiori and Castellano [13], Boudin and Salvarani [9], and Muslim *et al.* [44] employed other opinion dynamics models, such as the Sznajd model [58] and the voter model [57, 56], to analyze the impact of the time-invariant message on opinions. Nevertheless, the above works predominantly assumed that messages remain static and do not evolve, and we call them opinion models with static messages.

To our knowledge, there is limited literature studying the influence of time-varying messages on opinion dynamics. We call these models opinion models with dynamic messages. Mirtabatabaei *et al.* [43] and Gündüç [25] assumed that the message at each time step follows a predetermined probability distribution, such as a binomial distribution or a truncated normal distribution. They integrated the message with the Hegselmann-Krause model and the Deffuant-Weisbuch model to study the impact of time-varying messages on opinion dynamics, respectively. However, these models assume the independence of message probability distributions across different time steps and fail to capture the features of message evolution, as messages across different time steps are correlated. Quattrociocchi *et al.* [50] introduced a computational model of opinion dynamics that considers the coexistence of media as separated mechanisms and their feedback loops. In this model, agents consider both the messages received and the opinions of other agents as inputs for the Deffuant-Weisbuch model. At each time step, a dominant message source emerges based on the proximity between the message and agents' opinions. Subsequently, other message sources adjust their messages to align with the dominant message source. Due to the complexity of this model, it is challenging to theoretically analyze the impact of message evolution on opinion dynamics.

1.1.3 Message evolution in mass media theory

From mass media theory, the message evolution exhibits three features: temporal continuity [21], randomness [23], and polarization [2, 7]. When a valuable topic is disseminated in social networks, it continues to attract agents' attention. Consequently, the messages related to this topic form a sequentially evolving sequence over time exhibiting *temporal continuity*, and messages at different times are correlated [21]. The *randomness* refers to the uncertainty in the topic progression and the noise introduced by message sources when they edit the message contents [23]. Finally, when a message source adopts an extreme stance, it persistently emphasizes or reinforces that position in subsequent coverage, rather than shifting towards the opposite direction, which is referred to as *polarization* [2, 7]. To our knowledge, few works have quantitatively modeled the message evolution considering all these three features and analyzed the impact of message evolution on opinion dynamics.

1.2 Our Contributions

Taking into account the above features of message evolution, in this paper, we quantitatively model the message evolution and incorporate it into the opinion dynamics model to study the impact of messages on agents' opinions. As an example, we study the impact of messages about stock prices on agents' opinions regarding the trend in the financial market. From the work in Osborne [47], the temporal continuity and randomness features of stock prices align with those of messages. Additionally, the circuit breaker, the mechanism that ensures the stock price remains stationary after hitting the upper or lower limits, is akin to polarization [30]. In finance theory, Brownian motion has been extensively used to model the dynamic changes in stock prices and has been validated using real-world data [51]. Following these prior works, we use the Brownian motion to model the evolution of messages about stock prices. However, they did not consider the polarization feature. In this work, we introduce two absorbing bounds into the Brownian motion model to describe the polarization, which we call the **B**ounded **B**rownian **M**essage (*BBM*) model. Following the works by Mao *et al.* [38, 39], we model the messages and agents' opinions in social networks as a dynamic linear system, where agents' opinions are the state variables, and messages are the input variables. We incorporate the BBM model into the classic DeGroot opinion dynamics model [17] and propose the **M**essage **E**vo**L**ution and **O**pinion **DY**namics (*MELODY*) model to analyze the message and opinion coevolution and study the impact of messages on agents' opinions.

Our contributions can be summarized as follows.

- We propose the BBM model to model the message evolution process, which jointly considers the temporal continuity, randomness, and polarization features, and we quantitatively analyze the statistics of messages.
- We propose the MELODY model to model the opinion dynamics with consideration of message evolution and study the impact of message evolution on opinion dynamics.
- We run simulations over synthetic and real networks and conduct a real user test to validate the correctness of our analyses and our proposed MELODY model.

The rest of this paper is organized as follows. In Section 2, we propose the BBM model and analyze the statistics of the messages. In Section 3, we propose the MELODY model and analyze the statistics of the opinions and the impact of message evolution on opinion dynamics. The simulation and real user test results are shown in Section 4 and Section 5, respectively. Finally, the conclusion is drawn in Section 6.

2 The BBM Model

In this section, we first introduce the BBM model and then analyze the statistical properties of the message, including its probability distribution and statistics.

2.1 Model Definition

Assume that there are a total of M message sources. We use $\mathcal{M} := \{1, \ldots, M\}$ to represent the index set of the message sources. Following the works by Mirtabatabaei *et al.* [43] and Mao *et al.* [38, 39], we model the messages at time t using a random vector $\boldsymbol{s}_t \in [0, 1]^M$. Let 0 and 1 represent two messages supporting opposite and extreme views on a topic, which we call the two *absorbing bounds*. In the BBM model, the message evolves following a stochastic process, and once it reaches one of the absorbing bounds, it remains unchanged. Let $s_{i,0}$ denote the initial message of the i-th message source. We assume that it follows a uniform distribution on $[\underline{\xi}, \overline{\xi}] \subset (0, 1)$, and all initial messages are independent. The two bounds $\underline{\xi}$ and $\overline{\xi}$ prevent $s_{i,0}$ from approaching values of 0 or 1 quickly, avoiding an immediate convergence to the absorbing bounds.

To model the continuous-time message evolution, we assume that each message $\{s_{i,t}\}_{t \geqslant 0}$ follows a Brownian motion. Let $(\Omega, \mathcal{F}, \{\mathscr{F}_t\}_{t \geqslant 0}, \mathbb{P})$ be a complete filtered probability space on which an M-dimensional standard Brownian motion $\{\boldsymbol{z}_t\}_{t \geqslant 0}$ is defined. The i-th component $\{z_{i,t}\}_{t \geqslant 0}$ represents the standard Brownian motion for the i-th message source, and we assume that all components are independent. Let $c > 0$ denote the changing rate of messages, which quantifies the level of randomness, and to simplify the analysis, we assume that c is identical for all messages in this work. Without considering the absorbing bounds, the message of the i-th message source at time t is

$$y_{i,t} = s_{i,0} + c z_{i,t}. \tag{1}$$

When we consider the absorbing bounds, i.e., the message of the i-th message source stops changing once it reaches 0 or 1, the message at time t can be expressed as

$$s_{i,t} = y_{i,u}, \tag{2}$$

where $u := t \wedge T_{i,0} \wedge T_{i,1}$. Here, we define $T_{i,0}$ as the first time $y_{i,t}$ hits 0, i.e., $T_{i,0} := \inf\{t : t > 0, y_{i,t} = 0\}$, and $T_{i,1}$ is defined similarly. The operator $\wedge$ returns the infimum of two variables. Equation (1) and (2) are called the BBM model.

In the BBM model, the message is a continuous-time stochastic process, which reflects the time continuity feature of the message. We use the Brownian motion to reflect the randomness feature. Additionally, the two absorbing bounds in the BBM model reflect the polarization feature. Therefore, the BBM model can quantitatively describe the complex characteristics of message evolution in mass media theory.

2.2 Statistical Analysis of the Messages

Given the BBM model, we analyze the probability distribution and statistics of the messages.

2.2.1 Probability distribution of the messages

Theorem 1 gives the probability distribution of $s_{i,t}$.

Theorem 1. *Given $s_{i,0}$, the conditional probabilities that $s_{i,t}$ reaches the absorbing bounds 0 and 1 at time t are*

$$\mathbb{P}(s_{i,t}=0|s_{i,0}) = \int_0^t \frac{\mathrm{d}\tau}{\tau} \sum_{j\in\mathbf{E}} (s_{i,0}-j) g(j,\tau|s_{i,0}), \text{ and} \tag{3}$$

$$\mathbb{P}(s_{i,t}=1|s_{i,0}) = \int_0^t \frac{\mathrm{d}\tau}{\tau} \sum_{j\in\mathbf{O}} (j-s_{i,0}) g(j,\tau|s_{i,0}), \tag{4}$$

respectively, where $\mathbf{E}$ *and* $\mathbf{O}$ *refer to the sets of the even and odd numbers, and* $g(x,t|x_0) := \frac{1}{\sqrt{2\pi c^2 t}} \exp\left\{-\frac{(x-x_0)^2}{2c^2 t}\right\}$ *is the transition probability from x_0 to x over time t of the Brownian motion with changing rate c. The conditional probability density function of $s_{i,t} \in (0,1)$ is*

$$f_{s_{i,t}}(x|s_{i,0}) = \sum_{n\in\mathbf{E}} g(n+x,t|s_{i,0}) - g(-n-x,t|s_{i,0}). \tag{5}$$

Proof. According to Karatzas and Shreve [29], we have Lemma 1.

Lemma 1. *Consider the Brownian motion $\{x_t\}_{t\geqslant 0}$ on $[0,1]$, whose changing rate is c. If the initial value $x_0 \in (0,1)$, then for $t>0$, $x \in (0,1)$, the probability density functions of the first hitting time T_0 and T_1 are*

$$f_{T_0}(t, T_0 < T_1|x_0) = \frac{1}{t} \sum_{n\in\mathbf{E}} (x_0-n) g(n,t|x_0), \text{ and} \tag{6}$$

$$f_{T_1}(t, T_1 < T_0|x_0) = \frac{1}{t} \sum_{n\in\mathbf{O}} (n-x_0) g(n,t|x_0), \tag{7}$$

respectively. For $t>0$, $x\in(0,1)$, the probability density function of x_t is

$$f_{x_t}(x|x_0) = \sum_{n\in\mathbf{E}} g(n+x,t|x_0) - g(-n-x,t|x_0). \tag{8}$$

From Lemma 1, we can easily show that (5) is satisfied. Note that

$$\begin{aligned}&\left\{\omega : s_{i,t}(\omega) = 0\middle| s_{i,0}\right\}\\ =&\left\{\omega : T_0(\omega) \leqslant t, T_0(\omega) < T_1(\omega)\middle| s_{i,0}\right\}\\ =&\bigcup_{0\leqslant\tau\leqslant t}\left\{\omega : T_0(\omega) = \tau, T_0(\omega) < T_1(\omega)\middle| s_{i,0}\right\}, \quad \text{and} \qquad (9)\\ &\left\{\omega : s_{i,t}(\omega) = 1\middle| s_{i,0}\right\}\\ =&\left\{\omega : T_1(\omega) \leqslant t, T_1(\omega) < T_0(\omega)\middle| s_{i,0}\right\}\\ =&\bigcup_{0\leqslant\tau\leqslant t}\left\{\omega : T_1(\omega) = \tau, T_1(\omega) < T_0(\omega)\middle| s_{i,0}\right\}. \qquad (10)\end{aligned}$$

Therefore, the conditional probabilities that $s_{i,t}$ reaches the absorbing bounds 0 and 1 at time t are

$$\begin{aligned}\mathbb{P}(s_{i,t} = 0|s_{i,0}) &= \int_0^t f_{T_0}(\tau, T_0 < T_1|s_{i,0})\mathrm{d}\tau\\ &= \int_0^t \frac{1}{\tau}\sum_{n\in\mathbf{E}}(s_{i,0} - n)g(n, \tau|s_{i,0})\mathrm{d}\tau, \quad \text{and}\\ \mathbb{P}(s_{i,t} = 1|s_{i,0}) &= \int_0^t f_{T_1}(\tau, T_1 < T_0|s_{i,0})\mathrm{d}\tau\\ &= \int_0^t \frac{1}{\tau}\sum_{n\in\mathbf{O}}(n - s_{i,0})g(n, \tau|s_{i,0})\mathrm{d}\tau, \qquad (11)\end{aligned}$$

respectively. Here, we have proved Theorem 1. □

From Theorem 1, we can show that when $s_{i,0} < \frac{1}{2}$, $\mathbb{P}(s_{i,t} = 0|s_{i,0}) > \mathbb{P}(s_{i,t} = 1|s_{i,0})$, and conversely, when $s_{i,0} > \frac{1}{2}$, $\mathbb{P}(s_{i,t} = 0|s_{i,0}) < \mathbb{P}(s_{i,t} = 1|s_{i,0})$. This indicates that if the initial message leans toward one side, then as the message evolves, the distribution will also tend toward that side. Furthermore, the probability of the message reaching the absorbing bounds increases monotonically with time, implying that as the message evolves, it tends to become more polarized. It can be proved that as t approaches infinity, the message almost surely reaches the absorbing bounds, and adheres to a binomial distribution, as stated in Theorem 2.

Theorem 2. *As t approaches infinity, the probability distribution of $s_{i,t}$ is a binomial distribution, and*

$$\begin{cases} \lim\limits_{t\to\infty} \mathbb{P}(s_{i,t}=1|s_{i,0}) \overset{a.s.}{=} s_{i,0}, \\ \lim\limits_{t\to\infty} \mathbb{P}(s_{i,t}=0|s_{i,0}) \overset{a.s.}{=} 1-s_{i,0}, \\ \lim\limits_{t\to\infty} f_{s_{i,t}}(x|s_{i,0}) \overset{a.s.}{=} 0, \quad \forall x\in(0,1). \end{cases} \tag{12}$$

Here, $\overset{a.s.}{=}$ *means the two variables are equal in probability almost surely. And* $\overset{a.s.}{\leqslant}$ *is defined similarly.*

Proof. As t approaches infinity, the transition probability function is

$$\lim_{t\to\infty} g(x,t|x_0) \overset{\text{a.s.}}{=} 0, \forall x\in(0,1). \tag{13}$$

From Theorem 1, the probability density function of $s_{i,t}$ is

$$\lim_{t\to\infty} f_{s_{i,t}}(x,t|s_{i,0}) \overset{\text{a.s.}}{=} 0, \forall x\in(0,1). \tag{14}$$

Because $\{y_{i,t}\}_{t\geqslant 0}$ is a Brownian motion, it is a continuous-time martingale. Because both T_0 and T_1 are stopping times with respect to $\{y_{i,t}\}_{t\geqslant 0}$, the first time $y_{i,t}$ hits either absorbing bound $T := T_0 \wedge T_1$ is also a stopping time. From Bhattacharya and Waymire [8], the conditional mean of T given $s_{i,0}$ is

$$\mathbb{E}(T|s_{i,0}) = \frac{s_{i,0}(1-s_{i,0})}{c^2} < \infty. \tag{15}$$

And it is easy to prove that

$$\mathbb{E}\left(\sup_{t\geqslant 0} |y_{i,T\wedge t}| \middle| s_{i,0}\right) \leqslant 1 < \infty. \tag{16}$$

According to (15), (16), and the Stopping Time Theorem [8], we can prove that

$$\mathbb{E}(y_{i,T}|s_{i,0}) = \mathbb{E}(y_{i,0}|s_{i,0}) \overset{\text{a.s.}}{=} s_{i,0}. \tag{17}$$

On the other hand, from the definition of mean,

$$\mathbb{E}(y_{i,T}|s_{i,0}) = \mathbb{P}(T_1 < T_0|s_{i,0}) \overset{\text{a.s.}}{=} s_{i,0}. \tag{18}$$

From (10), note that

$$\begin{aligned} &\lim_{t\to\infty} \left\{\omega : s_{i,t}(\omega)=1 \middle| s_{i,0}\right\} \\ =&\lim_{t\to\infty} \bigcup_{0\leqslant\tau\leqslant t} \left\{\omega : T_0(\omega)=\tau, T_1(\omega)<T_0(\omega) \middle| s_{i,0}\right\} \\ =&\bigcup_{\tau\geqslant 0} \left\{\omega : T_0(\omega)=\tau, T_1(\omega)<T_0(\omega) \middle| s_{i,0}\right\} \\ =&\left\{\omega : T_1(\omega)<T_0(\omega) \middle| s_{i,0}\right\}. \end{aligned} \tag{19}$$

Because

$$\left\{\omega : s_{i,\sigma}(\omega) = 1 \middle| s_{i,0}\right\} \subset \left\{\omega : s_{i,\tau}(\omega) = 1 \middle| s_{i,0}\right\}, \forall \sigma < \tau, \tag{20}$$

according to the continuity of probability, we have

$$\begin{aligned}\lim_{t\to\infty} \mathbb{P}(s_{i,t} = 1|s_{i,0}) &= \mathbb{P}\left(\lim_{t\to\infty} s_{i,t} = 1 \middle| s_{i,0}\right) \\ &= \mathbb{P}(T_1 < T_0|s_{i,0}) \stackrel{\text{a.s.}}{=} s_{i,0}.\end{aligned} \tag{21}$$

Similarly, we can prove that

$$\lim_{t\to\infty} \mathbb{P}(s_{i,t} = 0|s_{i,0}) \stackrel{\text{a.s.}}{=} 1 - s_{i,0}. \tag{22}$$

Here, we have proved Theorem 2. □

2.2.2 *Statistics of the messages*

In the BBM model, the message is a stochastic process. With the above probability distribution, we can further analyze its statistics. First, we analyze the mean of the messages.

Theorem 3. *The conditional mean of $s_{i,t}$ given $s_{i,0}$ is*

$$\mathbb{E}(s_{i,t}|s_{i,0}) \stackrel{a.s.}{=} s_{i,0}, \tag{23}$$

and the mean of $s_{i,t}$ is

$$\mathbb{E}s_{i,t} = \mu, \tag{24}$$

where $\mu := \frac{1}{2}(\underline{\xi} + \overline{\xi})$. In vector form, (24) is

$$\mathbb{E}\boldsymbol{s}_t = \mu \mathbf{1}_{M\times 1}, \tag{25}$$

where $\mathbf{1}_{M\times 1}$ is the all-one vector.

Proof. According to Bhattacharya and Waymire [8], we have Lemma 2.

Lemma 2. *If $\{x_t\}_{t\geqslant 0}$ is a martingale and T is a stopping time, then the stop process $\{x_{T\wedge t}\}_{t\geqslant 0}$ is also a martingale.*

From Lemma 2, since $\{y_{i,t}\}_{t\geqslant 0}$ is a martingale and T is a stopping time, the stopped process $\{s_{i,t}\}_{t\geqslant 0} = \{y_{i,T\wedge t}\}_{t\geqslant 0}$ is a martingale. Therefore, we have

$$\mathbb{E}(s_{i,t}|s_{i,0}) \stackrel{\text{a.s.}}{=} s_{i,0}, \quad \text{and} \tag{26}$$

$$\mathbb{E}s_{i,t} = \mathbb{E}[\mathbb{E}(s_{i,t}|s_{i,0})] = \mathbb{E}s_{i,0} = \mu. \tag{27}$$

Here, we have proved Theorem 3. □

From Theorem 3, the mean of the message remains the same over time and equals the mean of the initial message distribution.

Next, we analyze the variance of the messages. It is complicated to calculate the exact value of the variance of $s_{i,t}$ for each t, and we show its upper bound in Theorem 4.

Theorem 4. *The upper bound of the conditional variance of $s_{i,t}$ given $s_{i,0}$ is*

$$\mathbb{D}(s_{i,t}|s_{i,0}) \overset{a.s.}{\leqslant} (c^2 t) \wedge [s_{i,0}(1-s_{i,0})], \tag{28}$$

and the upper bound of the variance of $s_{i,t}$ is

$$\mathbb{D}s_{i,t} \leqslant (c^2 t + \delta^2) \wedge [\mu(1-\mu)], \tag{29}$$

where $\delta^2 := \frac{1}{12}(\bar{\xi} - \underline{\xi})^2$ is the variance of the uniform distribution in $[\underline{\xi}, \bar{\xi}]$. In vector form, (29) is

$$\mathbb{D}\boldsymbol{s}_t \leqslant (c^2 t + \delta^2) \wedge [\mu(1-\mu)]\mathbf{1}_{M\times 1}. \tag{30}$$

Proof. First, we have

$$\mathbb{D}(s_{i,t}|s_{i,0}) = \mathbb{D}(y_{i,T\wedge t}|s_{i,0}) = c^2\mathbb{D}(z_{i,T\wedge t}|s_{i,0}) \leqslant c^2 t. \tag{31}$$

From Theorem 2, as t approaches infinity, the message almost surely hits the absorbing bounds, so the variance is

$$\begin{aligned} \mathbb{D}(s_{i,t}|s_{i,0}) &\overset{\text{a.s.}}{\leqslant} \mathbb{P}(s_{i,t}=0|s_{i,0})s_{i,0}^2 + \mathbb{P}(s_{i,t}=1|s_{i,0})(1-s_{i,0})^2 \\ &= (1-s_{i,0})s_{i,0}^2 + s_{i,0}(1-s_{i,0})^2 \\ &= s_{i,0}(1-s_{i,0}). \end{aligned} \tag{32}$$

From (31) and (32), we can establish the upper bound of the variance:

$$\mathbb{D}(s_{i,t}|s_{i,0}) \overset{\text{a.s.}}{\leqslant} (c^2 t) \wedge [s_{i,0}(1-s_{i,0})]. \tag{33}$$

For the convenience of expression, we denote $\bar{\mathbb{D}}(s_{i,t}|s_{i,0}) := c^2 t$. According to the Law of Total Variance, when t is small, the variance is

$$\begin{aligned} \mathbb{D}s_{i,t} &= \mathbb{E}[\mathbb{D}(s_{i,t}|s_{i,0})] + \mathbb{D}[\mathbb{E}(s_{i,t}|s_{i,0})] \\ &= c^2 t + \mathbb{D}s_{i,0} = c^2 t + \delta^2. \end{aligned} \tag{34}$$

For the convenience of expression, we denote $\bar{\mathbb{D}}s_{i,t} := c^2 t + \delta^2$. As t approaches infinity, the variance is

$$\begin{aligned} \lim_{t\to\infty} \mathbb{D}s_{i,t} &= \mathbb{E}[s_{i,0}(1-s_{i,0})] + \delta^2 = \mathbb{E}s_{i,0} - \mathbb{E}(s_{i,0})^2 + \delta^2 \\ &= \mu - (\mu^2 + \delta^2) + \delta^2 = \mu(1-\mu). \end{aligned} \tag{35}$$

Here, we have proved Theorem 4. □

From Theorem 4, the upper bound of the variance is determined by two terms, each corresponding to the following two scenarios. When t is small, the message has not yet reached the absorbing bounds, and the stochastic process still exhibits Brownian motion features, with its conditional variance increasing linearly with time. As t approaches infinity, the message almost surely reaches the absorbing bound, and the distribution of the message takes on a binomial distribution, as described in Theorem 2. In this case, the variance reaches a steady state. Furthermore, it can be proved that the variance of the message is a monotonically increasing function with time.

In summary, from Theorem 1–4, in the BBM model, all messages initially follow a uniform distribution on a proper subinterval of $[0, 1]$, and eventually converge to a 0/1 binomial distribution. During the entire message evolution process, the mean remains constant while the variance gradually increases.

3 The MELODY Model

In this section, we incorporate the above BBM model into the traditional DeGroot model and propose the MELODY model. We quantitatively analyze the agents' opinions dynamics over networks and the impact of messages on agents' opinions.

3.1 The Traditional DeGroot Model

Assume that there are a total of N agents in a fully connected network. We use $\mathcal{N} := \{1, \dots, N\}$ to represent the index set of the agents. Let $\boldsymbol{W} \in \mathbf{R}^{N\times N}$ denote the adjacency matrix, which is a stochastic matrix with $\sum_{j=1}^{N} w_{ij} = 1, \forall i \in \mathcal{N}$ and $w_{ij} \geqslant 0, \forall (i, j) \in \mathcal{N} \times \mathcal{N}$. w_{ij} quantifies the j-th agent's influence on the i-th agent in opinion update.

Let $\boldsymbol{o}_t$ denote the vector of agents' opinion at time t. Given the initial opinion $\boldsymbol{o}_0$, the continuous-time form of the DeGroot model is described by the following differential equation [46]:

$$\dot{\boldsymbol{o}}_t = (\boldsymbol{W} - \boldsymbol{I})\boldsymbol{o}_t, \tag{36}$$

where $\boldsymbol{I} \in \mathbf{R}^{N\times N}$ is the identity matrix. In (36), without considering the influence of messages, the changing rate of agents' opinions $\dot{\boldsymbol{o}}_t$ is a linear combination of the agents' opinions $\boldsymbol{o}_t$ at time t, and the opinions are only determined by the adjacency matrix $\boldsymbol{W}$ and the initial opinion $\boldsymbol{o}_0$.

From Berger [5], the opinion at time t is

$$\boldsymbol{o}_t = \mathrm{e}^{(\boldsymbol{W}-\boldsymbol{I})t}\boldsymbol{o}_0, \tag{37}$$

where $\boldsymbol{o}_0$ is the vector of agents' initial opinions, and $\mathrm{e}^{(\boldsymbol{W}-\boldsymbol{I})t} := \sum_{k=0}^{\infty} \frac{t^k}{k!}(\boldsymbol{W}-\boldsymbol{I})^k$. As t approaches infinity, the *steady-state opinion* is

$$\lim_{t\to\infty} \boldsymbol{o}_t = \boldsymbol{\ell}^\top \boldsymbol{o}_0 \mathbf{1}_{N\times 1}, \tag{38}$$

where $\boldsymbol{\ell} \in \mathbf{R}^N$ is the eigenvector of $\boldsymbol{W}$ associated with 1 constrained to $\sum_{i=1}^{N} \ell_i = 1$.

3.2 The MELODY Model

3.2.1 Model definition

Following the works by Mao *et al.* [38], Yang *et al.* [59], and Mao *et al.* [39], we model the messages and agents' opinions as a dynamic system, where agents' opinions are the state variables, and messages are the input variables. The incorporation of the BBM model with the DeGroot model leads us to the following differential equation:

$$\dot{\boldsymbol{o}}_t = (\alpha \boldsymbol{W} - \boldsymbol{I})\boldsymbol{o}_t + (1-\alpha)\boldsymbol{U}\boldsymbol{s}_t, \tag{39}$$

where $\alpha \in (0,1)$ is the weight coefficient of the opinion, and $1-\alpha$ measures the extent to which the messages affect the opinion update. A smaller α indicates that messages exert a greater impact on the changing rate of agents' opinions. When $\alpha = 1$, the MELODY model degenerates into the traditional DeGroot model in (36). Following the works by Mao *et al.* [38, 39], the influence matrix $\boldsymbol{U} \in \mathbf{R}^{N\times M}$ is a stochastic matrix with $\sum_{j=1}^{M} u_{ij} = 1, \forall i \in \mathcal{N}$ and $u_{ij} \geqslant 0, \forall (i,j) \in \mathcal{N} \times \mathcal{M}$. u_{ij} quantifies the influence of the j-th source on the i-th agent in his/her opinion update. Equation (39) is the MELODY model.

In the MELODY model, the changing rate in opinions $\dot{\boldsymbol{o}}_t$ at time t is composed of two parts: the linear combination of the opinions from all agents $(\alpha\boldsymbol{W} - \boldsymbol{I})\boldsymbol{o}_t$ at time t and the linear combination of messages $(1-\alpha)\boldsymbol{U}\boldsymbol{s}_t$ at time t. We can solve (39) and calculate the opinion at time t. For the convenience of expression, we denote $\boldsymbol{A} := \alpha\boldsymbol{W} - \boldsymbol{I}$ and $\boldsymbol{B} := (1-\alpha)\boldsymbol{U}$ hereinafter.

Theorem 5. *The opinion $\boldsymbol{o}_t$ at time t is*

$$\boldsymbol{o}_t = \mathrm{e}^{\boldsymbol{A}t}\boldsymbol{o}_0 + \int_0^t \mathrm{e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{s}_\tau \mathrm{d}\tau. \tag{40}$$

From Theorem 5, agents' opinions are influenced not only by the adjacency matrix $\boldsymbol{W}$ and the initial opinions $\boldsymbol{o}_0$ but also by the messages $\{\boldsymbol{s}_t\}_{t\geqslant 0}$ and the influence matrix $\boldsymbol{U}$.

3.2.2 Statistics of the opinions

We analyze the mean and variance of agents' opinions in the MELODY model.

Theorem 6. *The mean of* $\boldsymbol{o}_t$ *is*

$$\mathbb{E}\boldsymbol{o}_t = \mathrm{e}^{\boldsymbol{A}t}\boldsymbol{o}_0 + \mu(\boldsymbol{I} - \mathrm{e}^{\boldsymbol{A}t})\mathbf{1}_{N\times 1}. \tag{41}$$

As t *approaches infinity,*

$$\lim_{t\to\infty} \mathbb{E}\boldsymbol{o}_t = \mu\mathbf{1}_{N\times 1}. \tag{42}$$

Proof. From Theorem 5, the mean of agents' opinions is

$$\begin{aligned}
\mathbb{E}\boldsymbol{o}_t &= \mathbb{E}\left[\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{o}_0 + \int_0^t \mathrm{e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{s}_\tau \mathrm{d}\tau\right] \\
&= \mathrm{e}^{\boldsymbol{A}t}\boldsymbol{o}_0 + \int_0^t \mathrm{e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\mathbb{E}\boldsymbol{s}_\tau \mathrm{d}\tau \\
&= \mathrm{e}^{\boldsymbol{A}t}\boldsymbol{o}_0 + \mu\int_0^t \mathrm{e}^{\boldsymbol{A}(t-\tau)}\mathrm{d}\tau\boldsymbol{B}\mathbf{1}_{N\times 1} \\
&= \mathrm{e}^{\boldsymbol{A}t}\boldsymbol{o}_0 + \mu(1-\alpha)(\mathrm{e}^{\boldsymbol{A}t} - \boldsymbol{I})\boldsymbol{A}^{-1}\mathbf{1}_{N\times 1}.
\end{aligned} \tag{43}$$

From Horn and Johnson [28], because $\mathbf{1}_{N\times 1}$ is an eigenvector of the row-stochastic matrix $\boldsymbol{W}$ with eigenvalue 1, it is also an eigenvector of $\boldsymbol{A}$ with eigenvalue $\alpha - 1$. Furthermore, from Horn and Johnson [28], because $\boldsymbol{W}$ is a row-stochastic matrix, the moduli of all eigenvalues are less than or equal to 1. So the moduli of all eigenvalues of matrix $\boldsymbol{A}$ are less than or equal to $\alpha - 1 < 0$. That is, $\boldsymbol{A}$ is nonsingular, and $\mathbf{1}_{N\times 1}$ is an eigenvector of $\boldsymbol{A}^{-1}$ with eigenvalue $(\alpha-1)^{-1}$, i.e.,

$$\boldsymbol{A}^{-1}\mathbf{1}_{N\times 1} = (\alpha-1)^{-1}\mathbf{1}_{N\times 1}. \tag{44}$$

Therefore, from (43) and (44), we have (41). Furthermore, because the moduli of all eigenvalues of matrix $\boldsymbol{A}$ are negative, as t approaches infinity, the matrix exponential function $\mathrm{e}^{\boldsymbol{A}t}$ tends to the zero matrix. Therefore, we have

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{E}\boldsymbol{o}_t &= \lim_{t\to\infty} \mathrm{e}^{\boldsymbol{A}t}\boldsymbol{o}_0 + \mu(1-\alpha)\boldsymbol{A}^{-1}(\mathrm{e}^{\boldsymbol{A}t} - \boldsymbol{I})\mathbf{1}_{N\times 1} \qquad (45) \\
&= \mu(\alpha-1)\boldsymbol{A}^{-1}\mathbf{1}_{N\times 1} = \mu\mathbf{1}_{N\times 1}.
\end{aligned}$$

□

From Theorem 6, $\mathbb{E}\boldsymbol{o}_t$ transitions from $\boldsymbol{o}_0$ to $\mu\mathbf{1}_{N\times 1}$ over time, and as t approaches infinity, $\mathbb{E}\boldsymbol{o}_t$ converges to the mean of the message distribution.

Next, we study the variance of $\boldsymbol{o}_t$. As it is difficult to analyze the variance of $\boldsymbol{o}_t$ for each t, we focus on the variance of the steady-state opinion $\boldsymbol{o}_t$ as t approaches infinity.

Theorem 7. *The variance of* $\boldsymbol{o}_t$ *is*

$$\mathbb{D}\boldsymbol{o}_t = \mathrm{diag}\int_0^t\int_0^t \mathrm{e}^{\boldsymbol{A}(t-\sigma)}\boldsymbol{B}\boldsymbol{\Sigma}_{\sigma,\tau}\boldsymbol{B}^\top\mathrm{e}^{\boldsymbol{A}^\top(t-\tau)}\mathrm{d}\sigma\mathrm{d}\tau, \tag{46}$$

where $\boldsymbol{\Sigma}_{\sigma,\tau} := \mathbb{E}(\boldsymbol{s}_\sigma\boldsymbol{s}_\tau^\top) - \mathbb{E}\boldsymbol{s}_\sigma\mathbb{E}\boldsymbol{s}_\tau^\top = \mathbb{E}(\boldsymbol{s}_\sigma\boldsymbol{s}_\tau^\top) - \mu^2\mathbf{1}_{M\times 1}\mathbf{1}_{M\times 1}^\top$. *As* t *approaches infinity,*

$$\lim_{t\to\infty}\mathbb{D}\boldsymbol{o}_t = \mu(1-\mu)\mathrm{diag}(\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{B}^\top\boldsymbol{A}^{-\top}), \tag{47}$$

where diag *transforms the diagonal of a matrix into a column vector.*

Proof. From Theorem 5, the variance of agents' opinions is

$$\begin{aligned}
\mathbb{D}\boldsymbol{o}_t &= \mathbb{D}\left[\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{o}_0 + \int_0^t \mathrm{e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{s}_\tau\mathrm{d}\tau\right] \\
&= \mathbb{D}\left[\int_0^t \mathrm{e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{s}_\tau\mathrm{d}\tau\right] \\
&= \mathrm{diag}\mathbb{E}\left[\int_0^t \mathrm{e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\boldsymbol{s}_\tau\mathrm{d}\tau\int_0^t \boldsymbol{s}_\tau^\top\boldsymbol{B}^\top\mathrm{e}^{\boldsymbol{A}^\top(t-\tau)}\mathrm{d}\tau\right] \\
&\quad - \mathrm{diag}\int_0^t \mathrm{e}^{\boldsymbol{A}(t-\tau)}\boldsymbol{B}\mathbb{E}\boldsymbol{s}_\tau\mathrm{d}\tau\int_0^t \mathbb{E}\boldsymbol{s}_\tau^\top\boldsymbol{B}^\top\mathrm{e}^{\boldsymbol{A}^\top(t-\tau)}\mathrm{d}\tau \\
&= \mathrm{diag}\int_0^t\int_0^t \mathrm{e}^{\boldsymbol{A}(t-\sigma)}\boldsymbol{B}\mathbb{E}(\boldsymbol{s}_\sigma\boldsymbol{s}_\tau^\top)\boldsymbol{B}^\top\mathrm{e}^{\boldsymbol{A}^\top(t-\tau)}\mathrm{d}\sigma\mathrm{d}\tau \\
&\quad - \mathrm{diag}\int_0^t\int_0^t \mathrm{e}^{\boldsymbol{A}(t-\sigma)}\boldsymbol{B}\mathbb{E}\boldsymbol{s}_\sigma\mathbb{E}\boldsymbol{s}_\tau^\top\boldsymbol{B}^\top\mathrm{e}^{\boldsymbol{A}^\top(t-\tau)}\mathrm{d}\sigma\mathrm{d}\tau \\
&= \mathrm{diag}\int_0^t\int_0^t \mathrm{e}^{\boldsymbol{A}(t-\sigma)}\boldsymbol{B}\boldsymbol{\Sigma}_{\sigma,\tau}\boldsymbol{B}^\top\mathrm{e}^{\boldsymbol{A}^\top(t-\tau)}\mathrm{d}\sigma\mathrm{d}\tau.
\end{aligned} \tag{48}$$

As t approaches infinity, by setting the right-hand side of (39) to zero vectors, we have

$$\lim_{t\to\infty}\boldsymbol{o}_t \overset{\text{a.s.}}{=} \lim_{t\to\infty}\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{s}_t. \tag{49}$$

From Theorem 1, as t approaches infinity, each element $s_{i,t}$ in $\boldsymbol{s}_t$ adheres to a binomial distribution, with a probability of $s_{i,0}$ for taking the value of 1, and a probability of $1-s_{i,0}$ for taking the value of 0. Therefore, the variance is

$$\lim_{t\to\infty}\mathbb{D}\boldsymbol{o}_t = \lim_{t\to\infty}\mathrm{diag}(\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{\Sigma}_{t,t}\boldsymbol{B}^\top\boldsymbol{A}^{-\top}). \tag{50}$$

We use Theorem 4 to calculate the covariance. The diagonal elements of $\lim_{t\to\infty}\boldsymbol{\Sigma}_{t,t}$ are equal to the variance as shown in (35), while the off-diagonal elements are zero because $\{z_{i,t}\}_{t\geqslant 0}$ are identical for $i\in\mathcal{M}$, i.e.,

$$\lim_{t\to\infty}\boldsymbol{\Sigma}_{t,t} = \mu(1-\mu)\boldsymbol{I}. \tag{51}$$

Here, we have proved Theorem 7. ☐

From (47), as t approaches infinity, the variance of agents' opinions is proportional to the variance of the messages $\mu(1-\mu)$, as shown in (35). The vector $\mathrm{diag}(\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{B}^\top\boldsymbol{A}^{-\top})$ is the weighted average coefficient of all messages. Specifically, when the influence matrix $\boldsymbol{U}=\frac{1}{M}\mathbf{1}_{N\times M}$, where $\mathbf{1}_{N\times M}:=\mathbf{1}_{N\times 1}\mathbf{1}_{M\times 1}^\top$ is the all-one matrix with dimension of $N\times M$, the influence of all messages on agent's opinion update is homogeneous. In this case, the variance of the steady-state opinions is shown in Corollary 1.

Corollary 1. *If $\boldsymbol{U}=\frac{1}{M}\mathbf{1}_{N\times M}$, as t approaches infinity, the variance of $\boldsymbol{o}_t$ is*

$$\lim_{t\to\infty}\mathbb{D}\boldsymbol{o}_t=\frac{\mu(1-\mu)}{M}\mathbf{1}_{N\times 1}. \tag{52}$$

Proof. Substituting $\boldsymbol{U}=\frac{1}{M}\mathbf{1}_{N\times M}$ into the right-hand side of (47), we have

$$\begin{aligned}\lim_{t\to\infty}\mathbb{D}\boldsymbol{o}_t&=\mu(1-\mu)\mathrm{diag}(\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{B}^\top\boldsymbol{A}^{-\top})\\&=\frac{\mu(1-\mu)(1-\alpha)^2}{M^2}\mathrm{diag}(\boldsymbol{A}^{-1}\mathbf{1}_{N\times 1}\mathbf{1}_{M\times 1}^\top\mathbf{1}_{M\times 1}\mathbf{1}_{N\times 1}^\top\boldsymbol{A}^{-\top})\\&=\frac{\mu(1-\mu)(1-\alpha)^2}{M}\mathrm{diag}[(\boldsymbol{A}^{-1}\mathbf{1}_{N\times 1})(\boldsymbol{A}^{-1}\mathbf{1}_{N\times 1})^\top],\end{aligned} \tag{53}$$

and from (44), we have

$$\lim_{t\to\infty}\mathbb{D}\boldsymbol{o}_t=\frac{\mu(1-\mu)}{M}\mathrm{diag}(\mathbf{1}_{N\times 1}\mathbf{1}_{N\times 1}^\top)=\frac{\mu(1-\mu)}{M}\mathbf{1}_{N\times 1}. \tag{54}$$

Here, we have proved (52). □

3.2.3 Impact of the messages on the opinions

Next, we compare the MELODY model with the DeGroot model and study the impact of the messages on agents' opinions.

First, due to the randomness of messages, agents' opinions in the MELODY model are a stochastic process. However, in the DeGroot model, the vector of the opinions is deterministic given the initial opinions and the adjacency matrix.

Second, as shown in (41), the mean of agents' opinions consists of two terms in the MELODY model. The first term quantifies the impact of the initial opinions and the adjacency matrix, which is similar to the DeGroot model, while the second term quantifies the impact of messages. As the time approaches infinity, the mean of agents' steady-state opinions converges to the mean of the message distribution. However, as shown in (38), in the DeGroot model, agents' steady-state opinions are determined by the initial opinions and the adjacency matrix.

Third, in the MELODY model, the variance of agents' opinions increases over time, which is attributed to the growing uncertainty in the messages. In contrast, the DeGroot model's opinions are deterministic, so the variance is always zero.

In summary, in the BBM model, external messages cause stochastic fluctuations in agents' opinions over time, leading to increased variance. Additionally, the mean of agents' steady-state opinions in the BBM model converges to the mean of the message distribution, reflecting the influence of external messages on opinion dynamics.

4 Simulation Results

In this section, first, we conduct simulations to verify the analyses of the BBM model in Section 2. Then, we conduct simulations over synthetic and real networks to verify the analyses of the MELODY model in Section 3.

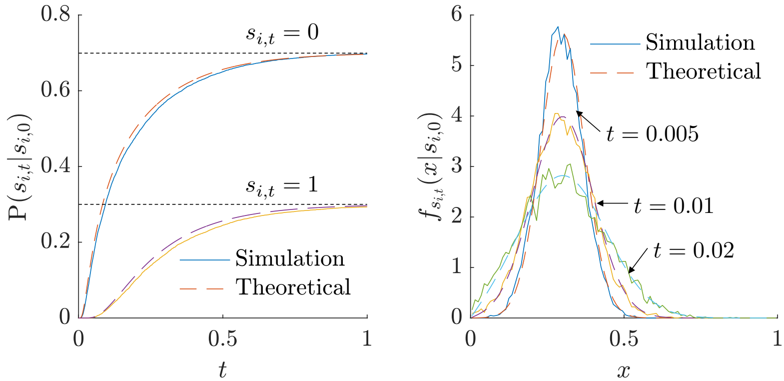
4.1 The BBM Model

We first simulate the message evolution process and verify our analyses of the statistical properties of the messages. We set the parameters as follows: $s_0 = 0.3$ and $c = 1$, and we observe the same trend for other values of the parameters. We generate a total of 10,000 message trajectories of the stochastic process according to (1) and (2). Because Brownian motion is a continuous function of time, we need to discretize it during simulation. The time interval for simulating Brownian motion is $\mathrm{d}t = 0.01$.

4.1.1 Probability distribution of the messages

The curves for $\mathbb{P}(s_t = 1|s_0 = 0.3)$ and $\mathbb{P}(s_t = 0|s_0 = 0.3)$ are shown in Figure 1a. Solid lines represent simulation values computed using the frequency of messages reaching the absorbing bounds at 0 and 1 at time t, respectively. Dashed lines represent theoretical values computed using (4). From Figure 1a, simulation results match our theoretical analysis results very well, validating the correctness of Theorem 1. Figure 1a also illustrates that the probability of a message reaching the absorbing bounds monotonically increases with time. The probability distribution functions $f_{s_{i,t}}(x|s_0 = 0.3)$ at $t = 0.005, 0.01, 0.02$ are plotted in Figure 1b. Again, the simulation and theoretical values exhibit a notable similarity, confirming the correctness of Theorem 1.

From Figure 1a, as t approaches infinity, $\lim_{t\to\infty} \mathbb{P}(s_t = 1|s_0 = 0.3) = 0.3$ and $\lim_{t\to\infty} \mathbb{P}(s_t = 0|s_0 = 0.3) = 0.7$, which is consistent with Theorem 2.



(a) Probability reaching 0 and 1

(b) Probability density function

Figure 1: Probability distribution of the messages in the BBM model.

4.1.2 Statistics of the messages

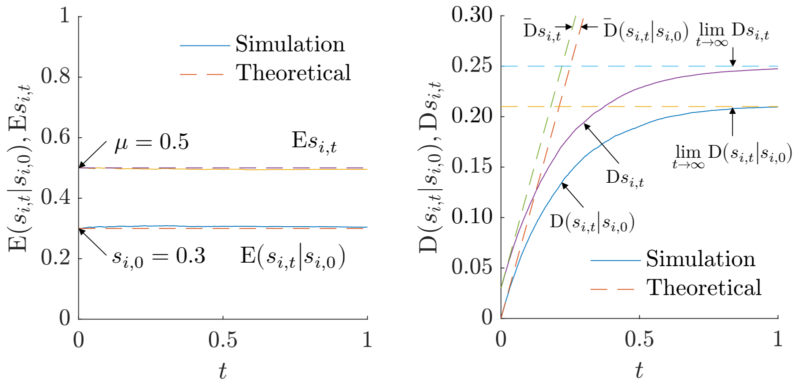
We simulate the mean and variance of the messages with $\underline{\xi} = 0.2$, $\overline{\xi} = 0.8$, $c = 1$. When we calculate the conditional mean and conditional variance, we fix $s_{i,0} = 0.3$. We also generate a total of 10,000 message trajectories. As shown in Figure 2, solid lines represent simulation values calculated by statistical methods, and dashed lines represent the theoretical values computed using (23), (24), (28), and (29), respectively. The simulation values closely align with theoretical values, validating Theorem 3 and Theorem 4.

4.2 The MELODY Model

Next, we simulate the opinion dynamics under the impact of messages over synthetic and real networks and verify our analyses of the statistical properties of the opinions. We first generate a small-scale synthetic network with 2 message sources and 3 agents and conduct simulations. The small numbers of message sources and agents facilitate better view of simulation results. To examine the generalization capability of our proposed MELODY model, we also conduct simulations over three real networks.

4.2.1 Simulation results over synthetic networks

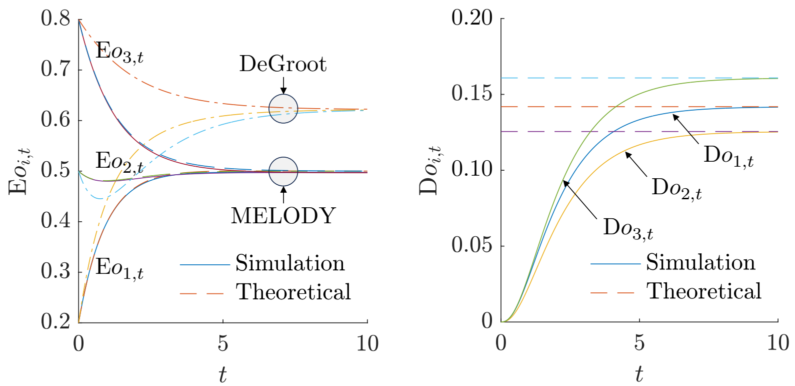
We set the parameters of the synthetic network as follows: $M = 2$, $N = 3$, $\underline{\xi} = 0.2$, $\overline{\xi} = 0.8$, $c = 1$, $\boldsymbol{W} = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.7 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$, $\boldsymbol{U} = \begin{bmatrix} 0.8 & 0.2 \\ 0.5 & 0.5 \\ 0.2 & 0.8 \end{bmatrix}$, $\alpha = 0.3$, and $\boldsymbol{o}_0 = \begin{bmatrix} 0.2 \\ 0.5 \\ 0.8 \end{bmatrix}$, and we observe the same trend for other values of the parameters.



(a) Mean

(b) Variance

Figure 2: Statistics of the messages in the BBM model.



(a) Mean

(b) Variance

Figure 3: Statistics of the opinions in the MELODY model in synthetic networks.

The messages are generated using the BBM model according to (1) and (2). The experiment repeats 10,000 times, and the mean and the variance of agents' opinions are plotted in Figure 3.

In Figure 3a, solid lines represent the simulated mean of agents' opinions, and dashed lines represent the theoretical mean of agents' opinions calculated using (41). Simulation results match our theoretical analysis results very well. As the time approaches infinity, all agents' opinions converge to the mean of the initial message distribution, validating Theorem 6. Given $\boldsymbol{W}$ and $\boldsymbol{o}_0$, the opinion curves based on the DeGroot model, with identical simulation parameters, are also included as dashed-dotted lines in Figure 3a. Unlike the MELODY model, the steady-state opinions in the DeGroot model are independent of the initial message distribution.

In Figure 3b, solid lines represent the simulated variance of agents' opinions. Dashed lines represent the theoretical variance of the steady-state opinion calculated using (47). Once again, a close correspondence between the simulated and theoretical variance of the steady-state opinions validates the correctness of Theorem 7.

4.2.2 Simulation results over real networks

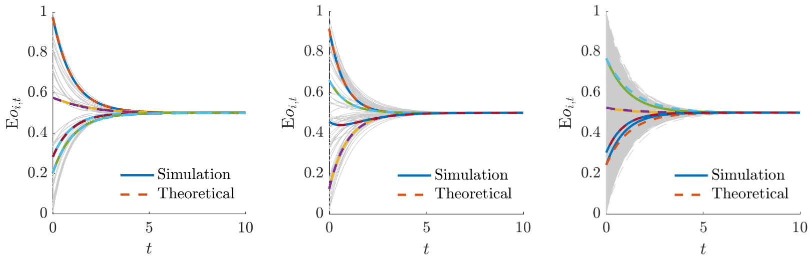
The real networks used in the simulations include the Karate Club graph [60], the Les Misérable graph [31], and the Facebook network [35]. These networks are undirected and connected. The numbers of nodes and edges of these networks are in Table 1. Note that in online social networks, the number of message sources is often smaller than that of agents. We set the numbers of message sources as $M = 10$, 20, and 100 for these three networks, respectively. The messages are generated using the BBM model according to (1) and (2).

Table 1: Numbers of nodes and edges of the real social networks.

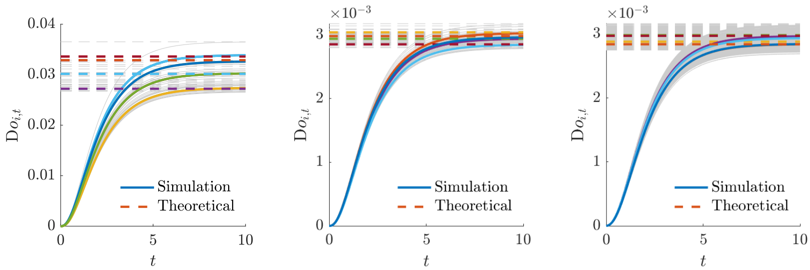
Network	Number of nodes N	Number of edges
Karate Club graph [60]	34	78
Les Misérable graph [31]	77	254
Facebook network [35]	4,039	88,234

We assume that agents are equally influenced by all their neighbors when updating their opinions. Therefore, in the adjacency matrix $\boldsymbol{W}$, we set $w_{ij} = 1/\deg(i)$ when the two nodes $i, j \in \mathcal{N}$ are connected, and we set $w_{ij} = 0$ otherwise. Here, $\deg(i) > 0$ is the degree of the node i. We assume that the influence of all messages on the agents' opinions is equal. Therefore, we set the influence matrix as a scaled all-one matrix, i.e., $\boldsymbol{U} = \frac{1}{M}\mathbf{1}_{N\times M}$. The agents' initial opinions $\boldsymbol{o}_0$ are randomly selected from $[0,1]^N$ and remain unchanged during each simulation run. We set other parameters as follows: $\underline{\xi} = 0.2$, $\overline{\xi} = 0.8$, $c = 1$, and $\alpha = 0.3$, and we observe the same trend for other values of the parameters. The experiment repeats 10,000 times, and we show the average results.

The mean and the variance of agents' opinions in these three real networks are plotted in Figure 4. Due to the large number of agents in the real networks, we plot the mean and the variance of all agents' opinions but highlight four of them using different colors as an example for a better view. We observe similar results for other agents. In Figure 4, solid lines represent the simulated mean and variance of agents' opinions, and dashed lines represent the theoretical



(a) Mean: Karate Club graph (b) Mean: Les Misérable graph (c) Mean: Facebook network



(d) Variance: Karate Club graph (e) Variance: Les Misérable graph (f) Variance: Facebook network

Figure 4: Statistics of the opinions in the MELODY model in real social networks.

mean and variance of the steady-state opinion calculated using (41) and (52), respectively. Simulation results match our theoretical results very well, validating the correctness of Theorem 6 and Theorem 7.

5 Real User Test

In this section, we conduct a real user test to verify the correctness of our proposed models. In the real user test, subjects receive messages and opinions expressed by other subjects, continually updating their opinions based on the messages and evolving opinions. Here, we analyze the impact of messages about stock prices on agents' opinions of financial market trends. Reddy and Clinton [51] has validated that stock prices adhere to Brownian motion, and the circuit breakers further validate the existence of two absorbing bounds [30], thereby validating the BBM model with real data. Consequently, we solely verify the accuracy of the MELODY model in this work.

5.1 Dataset Collection

We conduct 4 groups of real user tests, denoted as G_1, G_2, G_3, and G_4. In these groups, we recruit 24, 17, 19, and 15 subjects, respectively, i.e., $N_1 = 24$, $N_2 = 17$, $N_3 = 19$, and $N_4 = 15$. We use $\mathcal{G} := \{1, 2, 3, 4\}$ to represent the index set of the groups and $\mathcal{N}_i := \{1, \ldots, N_i\}$ to represent the index set of the subjects in group G_i. In group G_i, all N_i subjects complete 4 rounds of tests together, denoted as R_{i1}, R_{i2}, R_{i3}, R_{i4}. We use $\mathcal{R} := \{1, 2, 3, 4\}$ to represent the index set of the rounds. Due to feasibility constraints, we are unable to obtain continuous values of stock prices and opinions over throughout the entire experiment. Therefore, we sampled values at 10 discrete time points, corresponding to $\mathcal{T} := \{0, 1, 2, \ldots, 9\}$, within the continuous time interval. Each sampled time point is called a time step in the following.

In the real user test, the message is represented using 4 stock prices, i.e., $M = 4$, and the subject's opinion is quantified by the predicted average stock price, which represents the subject's views regarding the financial market trend. In the real-world communication process, the number of message sources is much smaller than that of agents, so the number of stock prices should be much smaller than the number of subjects in each group. However, too few message sources do not conform to real-world communication processes. In financial markets, relying on only one stock price may not adequately capture the complexity and variability of market dynamics [40]. Therefore, we select the number of the stock prices as 4. We also test the scenario where the number of stocks are 5 or 6, and our simulation results show that the number of stocks will not significantly impact the results.

The interface of the real user test is shown in Figure 5. In each round, at time step t, subjects are presented with 4 stock price curves before time step t and then submit their opinions. After all subjects submit their opinions, they are presented with a bar chart depicting the distribution of subjects' opinions at time step t. Then, they proceed to the next time step $t + 1$. To simplify the experimental operation, subjects select the interval where their opinions fall, including $[0, 0.2)$, $[0.2, 0.4)$, $[0.4, 0.6)$, $[0.6, 0.8)$, and $[0.8, 1]$, instead of inputting a real number. We use the midpoint of each interval as the opinion value, i.e., 0.1, 0.3, 0.5, 0.7, and 0.9, respectively.

At the end of the real user test in group G_i, subject l chooses the proportion α_{il} by which his/her opinion is influenced by the stock prices and all subjects' opinions. The options available are 10%/90%, 30%/70%, 50%/50%, 70%/30%, and 90%/10%, representing $\alpha_{il} = 0.9, 0.7, 0.5, 0.3$, and 0.1, respectively.

Note that in our real user test, since all subjects need to exchange opinions with each other, it is essential for them to do the test together in the same room and to share their opinions in real time. The experiment takes place in a medium-sized conference room, where each subject is provided with an internet-connected computer. Due to the physical space and computational

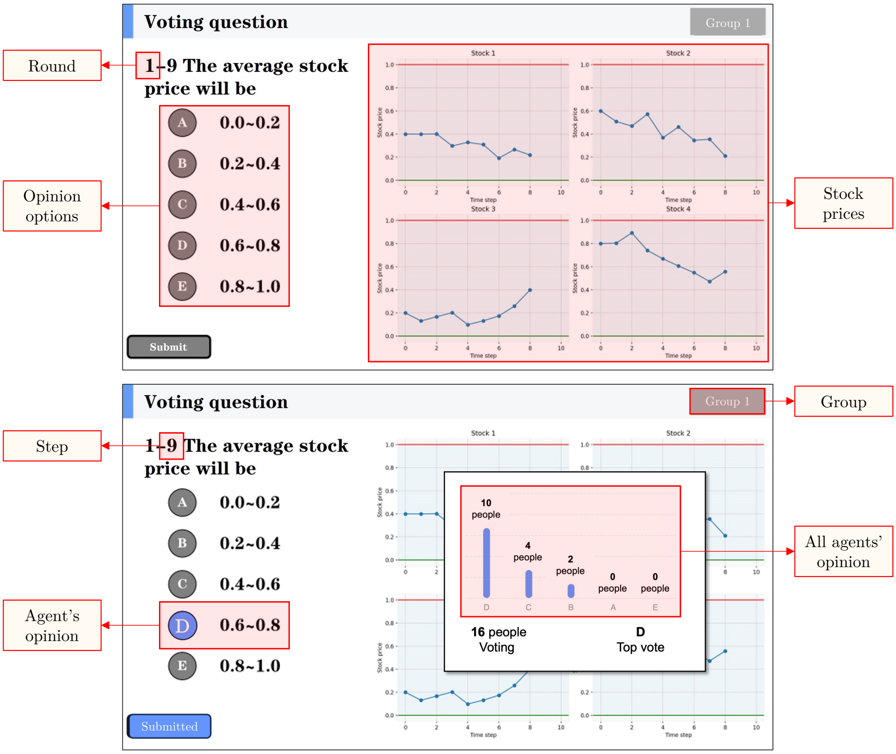


Figure 5: Interface of the real user test.

resources limitations, we are unable to recruit more than 25 subjects in each experiment. We plan to address this limitation in our future work.

5.2 Experimental Setup

In the following, we use $i \in \mathcal{G}$, $j \in \mathcal{R}$, $k \in \mathcal{M}$, $l \in \mathcal{N}_i$, and $t \in \mathcal{T}$ to represent the index of the groups, rounds, message sources, subjects, and time steps, respectively.

We denote the stock prices in round R_{ij} as $\{\boldsymbol{s}_{ij,t}\}_{t\in\mathcal{T}} := \{[s_{ijk,t}]^\top\}_{t\in\mathcal{T}}$, each of which is subjected to the Brownian motion with two absorbing bounds 0 and 1 and generated from the BBM model using (1) and (2). We set the initial price distribution as a uniform distribution on $[0.2, 0.8]$, and the changing rate $c = 0.1$ remained constant. After obtaining a Brownian motion trajectory representing the stock prices, we retained only the values at the time steps.

We denote the subjects' opinions in round R_{ij} as $\{\boldsymbol{o}_{ij,t}\}_{t\in\mathcal{T}} := \{[o_{ijl,t}]^\top\}_{t\in\mathcal{T}}$. In round R_{ij}, at time step $t = 1$, subjects are presented with the 4 stock initial price $\boldsymbol{s}_{ij,0}$, and then submit their initial opinions $\boldsymbol{o}_{ij,0}$. When $t = \tau > 1$, subjects are presented with 4 stock price curves before time step τ, i.e., $\{\boldsymbol{s}_{ij,t}\}_{t\leqslant\tau}$, and all subjects' opinions at time step $\tau - 1$, $\boldsymbol{o}_{ij,\tau-1}$, and then submit their opinions $\boldsymbol{o}_{ij,\tau}$.

Given that the 4 stock prices are randomly generated, the subjects' preferences for them are homogeneous. Therefore, we set the influence matrix $\boldsymbol{U}_i = \frac{1}{M}\mathbf{1}_{N_i\times M}$. Considering that subjects participate anonymously in the real user test, we set the adjacency matrix $\boldsymbol{W}_i = \frac{1}{N_i}\mathbf{1}_{N_i\times N_i}$. Since the parameter α for all agents in the MELODY model is homogeneous, we use its average value $\bar{\alpha} := \sum_{i\in\mathcal{G},l\in\mathcal{N}_i}\alpha_{il}/\sum_{i\in\mathcal{G}}N_i$.

5.3 Comparison with Existing Models

We compare the MELODY model with prior works including FJ-static [38], DW-static [12], HK-static [27], AHK-static [4], Sznajd-static [14], Voter-static [44], DW-dynamic [25], and HK-dynamic [43] models. To achieve a more comprehensive comparison, we additionally integrate dynamic messages into four existing opinion models including the FJ-static model, AHK-static model, Biased Opinion Formation model, and Stochastic Bounded Confidence model, and obtain the FJ-dynamic, AHK-dynamic, BOF-dynamic, and SBC-dynamic models, respectively. We then compare these models with the MELODY model. All parameters in these models are tuned to their optimal settings.

Among these works, the FJ-static, DW-static, HK-static, AHK-static, Sznajd-static, and Voter-static models are opinion models with static messages. Thus, when implementing these benchmark models, we keep the 4 stock prices as the mean of their initial prices. That is, in round R_{ij}, the time-invariant message is equal to $\frac{1}{M}\sum_{k\in\mathcal{M}}s_{ijk,0}$. In the Sznajd-static and Voter-static models, the values of the message and opinion are binary, i.e., 0 or 1. Therefore, we rounded the generated stock prices and subjects' opinions to the nearest integers at each time step.

The FJ-dynamic, DW-dynamic, HK-dynamic, AHK-dynamic, BOF-dynamic, and SBC-dynamic models are opinion models with dynamic messages. In the HK-dynamic model and the AHK-dynamic model, the stock prices follow a binomial distribution. Therefore, we rounded the generated stock prices to the nearest integers at each time step. In the DW-dynamic model, the stock prices follow a truncated normal distribution. As shown in Figure 1b, the probability distribution of the messages in the BBM model approximately conforms to the truncated normal distribution. Therefore, we use our generated stock prices directly to represent the message inputs in the DW-dynamic model. We incorporate the generated stock prices as the message inputs into the FJ-static model, and we call it the FJ-dynamic model. Additionally, inspired by Gündüç [25] and Mirtabatabaei *et al.* [43], we model the messages generated from the BBM model as agents' opinions, and we then combine them with other agents' opinions in the Biased Opinion Formation model and the Stochastic Bounded Confidence model, and we call them the BOF-dynamic model and the SBC-dynamic model, respectively.

For round R_{ij}, we calculate the predicted subjects' opinions using the above models, denoted by $\{\hat{\boldsymbol{o}}_{ij,t}\}_{t\in\mathcal{T}} := \{\hat{o}_{ijl,t}\}_{t\in\mathcal{T}}, \forall l \in \mathcal{N}_i$. We use the mean absolute error, abbreviated as MAE, between the predicted and the true opinion values as the metric to measure the prediction accuracy, which is defined as

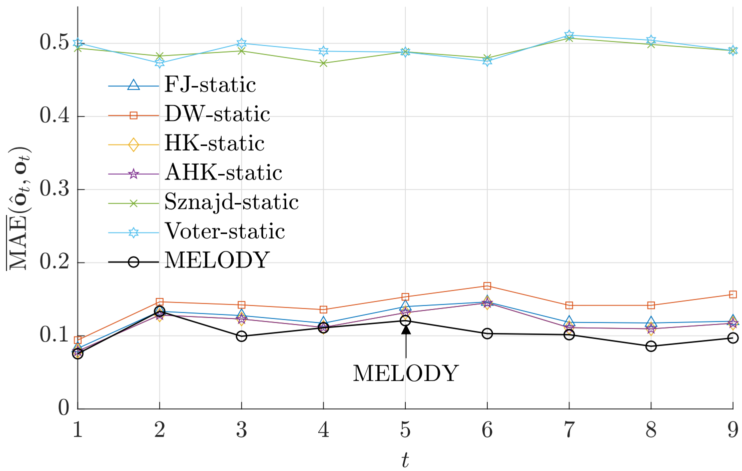
$$\mathrm{MAE}(\hat{\boldsymbol{o}}_{ij,t}, \boldsymbol{o}_{ij,t}) = \frac{1}{N_i}\sum_{l\in\mathcal{N}_i} |\hat{o}_{ijl,t} - o_{ijl,t}|. \tag{55}$$

We define the average MAE of the opinion dynamics model by averaging all rounds of tests, which is

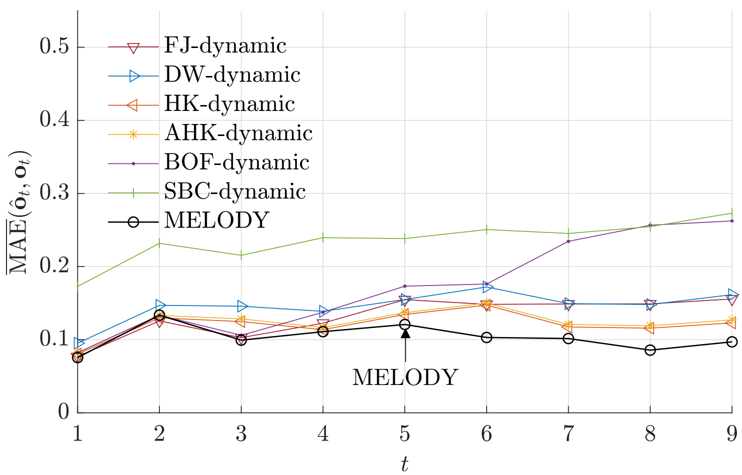
$$\overline{\mathrm{MAE}}(\hat{\boldsymbol{o}}_t, \boldsymbol{o}_t) = \frac{\sum_{i\in\mathcal{G}, j\in\mathcal{M}, l\in\mathcal{N}_i} |\hat{o}_{ijl,t} - o_{ijl,t}|}{\sum_{i\in\mathcal{G}} MN_i}. \tag{56}$$

We plot the average MAE curves between the predicted and the true opinion values of different models over time in Figure 6. It can be observed that the MELODY model exhibits the highest accuracy in predicting subjects' opinions. Since the Sznajd-static and Voter-static models utilize binary messages and opinions, their accuracy in predicting subjects' opinions in continuous opinion space is notably lower than all other models. For the SBC-dynamic model, at each time step, the opinion update of an agent is primarily influenced by one or two messages through numerical calculations, while the impact of other messages is small. Therefore, it also has limited accuracy. The prediction accuracy of DW-static and DW-dynamic models is relatively lower compared to other continuous opinion models. This is because the Deffuant-Weisbuch model assumes that at each time step, agents update their opinions only in pairs, which differs from the collective opinion updating mechanism in real social networks. Conversely, the MELODY, FJ-static, FJ-dynamic, HK-static, HK-dynamic, AHK-static, AHK-dynamic, and BOF-dynamic models assume that agents' opinion updates are influenced by either all agents or a group, leading to higher prediction accuracy.

In the above models, the ones that exhibit prediction performance closest to our proposed MELODY model are the HK-static and HK-dynamic models. However, as time increases, the difference in prediction accuracy for agents' opinions between these models and the MELODY model gradually widens. This is because, in the HK-static model, the stock prices remain unchanged, and in the HK-dynamic model, the stock prices follow a binomial distribution, which does not align with the message evolution in real social networks. Our proposed MELODY model assumes that the messages follow the BBM model, which characterizes the temporal continuity, randomness, and polarization features of the message evolution in real social networks. As time increases, the messages diverge significantly from their initial values, and both the HK-static



(a) Average MAE between the predicted and the true opinion values of the MELODY and static message models



(b) Average MAE between the predicted and the true opinion values of the MELODY and dynamic message models

Figure 6: Comparison of the average MAE between the predicted and the true opinion values of the MELODY and existing models.

and HK-dynamic models exhibit lower accuracy in predicting messages than the MELODY model, leading to lower accuracy in predicting opinions.

5.4 Impact of the Messages on the Opinion Dynamics

From Theorem 6 and Theorem 7, the mean and variance of the agents' opinions are influenced by the messages. In this part, we analyze the accuracy of the MELODY model's predictions for the mean and variance of subjects' opinions. We denote the sample mean and sample variance of subjects' opinions at time step t as $\mathbb{E}\boldsymbol{o}_{i,t}$ and $\mathbb{D}\boldsymbol{o}_{i,t}$, respectively. We also use the MAE of the mean and variance of subjects' opinions to measure the accuracy of the opinion dynamics model's predictions, which are

$$\mathrm{MAE}(\mathbb{E}) := \mathrm{MAE}(\mathbb{E}\hat{\boldsymbol{o}}_{i,t}, \mathbb{E}\boldsymbol{o}_{i,t}) = \frac{1}{N_i}\sum_{l\in\mathcal{N}_i} |\mathbb{E}\hat{o}_{il,t} - \mathbb{E}o_{il,t}|, \quad \text{and} \tag{57}$$

$$\mathrm{MAE}(\mathbb{D}) := \mathrm{MAE}(\mathbb{D}\hat{\boldsymbol{o}}_{i,t}, \mathbb{D}\boldsymbol{o}_{i,t}) = \frac{1}{N_i}\sum_{l\in\mathcal{N}_i} |\mathbb{D}\hat{o}_{il,t} - \mathbb{D}o_{il,t}|. \tag{58}$$

5.4.1 Steady-state opinions

We first consider the case when the time step $t = 9$. From the experimental data, it can be observed that at time step $t = 9$, subjects' opinions tend to stabilize. Therefore, we assume that the subjects' opinions approximately reach a steady state in round R_{i9}. We compare the prediction accuracy of the MELODY model with the DeGroot [17] and FJ-static [38]. In the MELODY model, we calculate the theoretical mean and variance of agents' steady-state opinions using (42) and (52), respectively.

We calculate the steady-state opinions of the DeGroot model using (38). In the DeGroot model, agents' opinions are not influenced by messages. By comparing the MELODY model with the DeGroot model, we can verify the influence of messages on opinions. As a baseline model considering the influence of messages on opinions, we choose the FJ-static model. From Parsegov *et al.* [48], we can obtain the mean of the steady-state opinions of the FJ-static model. As the Deffuant-Weisbuch and Hegselmann-Krause models lack the formula for calculating the mean of the steady-state opinions, we cannot compare them with the MELODY model. Because the DeGroot and FJ-static models are deterministic, the mean of the steady-state opinions is equal to the steady-state opinions, and the variance is zero.

The results are shown in Table 2. In group G_1, the mean of subjects' initial opinions differs from the mean of the stock initial prices, which highlights the influence of messages on the mean of the steady-state opinions. In this case, both the MELODY and FJ-static models exhibit much higher accuracy in predicting the mean of the steady-state opinions compared to the DeGroot model. In group G_2, G_3, and G_4, since the mean of subjects' initial opinions is equal to the mean of the initial prices, we cannot determine whether the mean of the steady-state opinions is influenced by the initial opinions or the

Table 2: Comparison of statistics of the steady-state opinions between the DeGroot, FJ-static, and MELODY model.

Model		**DeGroot**	**FJ-static**	**MELODY**		**Mean of $\boldsymbol{o}_{i,0}$**	**Mean of $\boldsymbol{s}_{i,0}$**
Metric		MAE($\mathbb{E}$)	MAE($\mathbb{E}$)	MAE($\mathbb{E}$)	MAE($\mathbb{D}$)		
Group	G_1	0.0937	0.0372	**0.0229**	0.0542	0.55	0.5
	G_2	**0.0441**	0.0586	0.0470	0.0295	0.5	0.5
	G_3	**0.0473**	0.0565	0.0552	0.0384	0.5	0.5
	G_4	**0.0633**	0.0881	0.0833	0.0413	0.5	0.5
Average		0.0621	0.0601	**0.0521**	0.0408	–	–

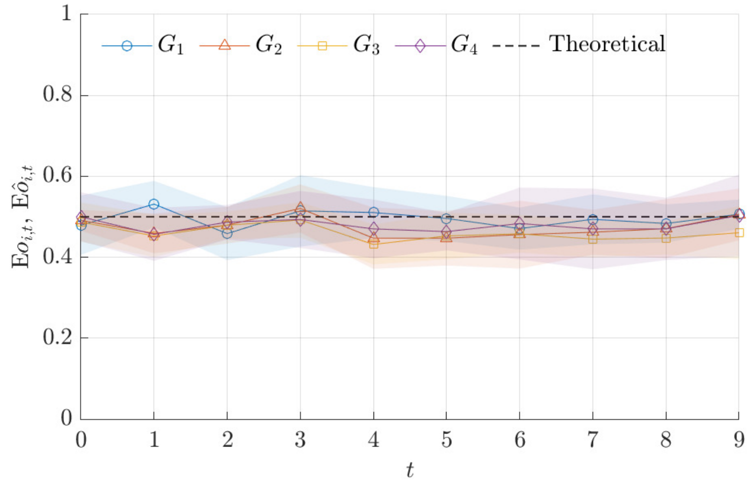
messages. In this case, the prediction accuracy of the three models is similar. Furthermore, in all groups, the MELODY model exhibits higher prediction accuracy for the mean of the steady-state opinion compared to the FJ-static model. It also exhibits a high prediction accuracy for the variance of the steady-state opinion. In summary, we have verified that the MELODY model can accurately quantify the influence of messages on steady-state opinions.

5.4.2 Transition to the steady-state opinions

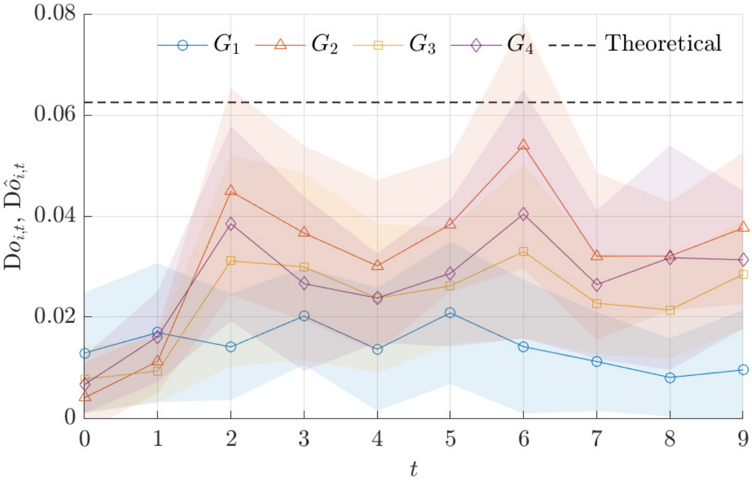
We plot the mean and variance of the opinions during the transition process $t \in \mathcal{T}$ in Figure 7. Since the mean and variance of subjects' opinions, $\mathbb{E}\boldsymbol{o}_{i,t}$ and $\mathbb{D}\boldsymbol{o}_{i,t}$ of group G_i, are vectors, we compute the mean of all elements and represent them with solid lines. Additionally, shaded areas represent one standard deviation of all elements above and below the mean. Dashed lines represent the theoretical mean and variance of the steady-state opinions calculated using (42) and (52), respectively. The mean of the opinions remains nearly constant over time, and its value is equal to the mean of the initial message distribution, which validates the correctness of Theorem 6. The variance of the opinions exhibits an increasing trend with time and gradually approaches the variance of the steady-state opinions, which validates the correctness of Theorem 7.

6 Conclusions

In this paper, we propose the MELODY model to describe the joint evolution of messages and opinions in social networks and analyze the impact of messages on opinion dynamics. We first propose the BBM model to capture the temporal continuity, randomness, and polarization features of the message evolution process. We then combine the BBM model with the classic DeGroot opinion dynamics model and analyze the statistical properties of the opinions with consideration of message evolution. Our analyses show that the mean of the opinions remains the same over time and equals the mean of the initial message distribution, and the variance of agents' opinions increases over time.



(a) Mean



(b) Variance

Figure 7: Mean and variance of the opinions in the transition process.

Simulation and real user test results validate the correctness of our analyses. This investigation is critical to a better understanding of how messages shape agents' opinions in social networks and how to guide agents' opinions.

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