Online Appendix to "Proportional payoffs in legislative bargaining with weighted voting: a characterization"

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Calculations for [5;3,2,2,1]

By contradiction, suppose $v = \left(\frac{3}{8}, \frac{2}{8}, \frac{2}{8}, \frac{1}{8}\right)$. What would be the optimal proposer behavior given v? The player with 3 votes needs to buy 2 votes, hence it always offers $v_{[2]}$ to one of the players with 2 votes (the player with 1 vote is of no use to this player, regardless of the value of $v_{[1]}$). The player with 1 vote needs to buy 4 votes, and buys them from the two players that control 2 votes each. A player with 2 votes needs to buy 3 votes, and is indifferent between buying them from the large player or from the other two players since $v_{[3]} = v_{[2]} + v_{[1]}$. Let p be the probability that a player with 2 votes proposes to the player with 3 votes (conditional on a player with 2 votes being selected as proposer). Expected payoffs for types [3] and [1] must satisfy the following equations:

$$\frac{3}{8} = \frac{3}{8}\left[1 - \frac{2}{8}\right] + \frac{4}{8}p\frac{3}{8}$$
$$\frac{1}{8} = \frac{1}{8}\left[1 - \frac{4}{8}\right] + \frac{4}{8}(1-p)\frac{1}{8}$$

From the second equation we find $p = 0$. This means that in order to sustain a payoff of $\frac{1}{8}$ for type [1], type [2] must always propose a coalition of type [221]. However, $p = 0$ does not solve the first equation: in order to sustain a payoff of $\frac{3}{8}$ for type [3], p must be $\frac{1}{2}$.

Interestingly, the equilibrium is still competitive in the sense that $v_{[3]} = v_{[2]} + v_{[1]}$. Below we construct an equilibrium strategy profile. Let player [3] propose to each of the two players of type [2] with probability $\frac{1}{2}$, and let each of the players of type [2] propose to player [3] with probability p. The equilibrium values of $v_{[3]}$, $v_{[2]}$, $v_{[1]}$ and p can be found from the following system:

$$\begin{aligned}
v_{[3]} &= \frac{3}{8}\left[1 - v_{[2]}\right] + \frac{4}{8} p v_{[3]} \\
v_{[2]} &= \frac{2}{8}\left[p(1 - v_{[3]}) + (1-p)(1 - v_{[2]} - v_{[1]})\right] + \frac{3}{8}\frac{1}{2} v_{[2]} + \frac{2}{8}(1-p) v_{[2]} + \frac{1}{8} v_{[2]} \\
v_{[1]} &= \frac{1}{8}[1 - 2v_{[2]}] + \frac{4}{8}(1-p) v_{[1]} \\
v_{[3]} &= v_{[2]} + v_{[1]}
\end{aligned}$$

The solution to this system is $v_{[3]} = \frac{5}{14}$, $v_{[2]} = \frac{4}{14}$, $v_{[1]} = \frac{1}{14}$ and $p = \frac{1}{2}$. This is an equilibrium since players are behaving optimally both as proposers and as responders. Because of the uniqueness result of Eraslan and McLennan (2013), all SSPE must have the same payoff vector.

Calculations for [13;7,6,4,3,3,1]

There are seven MWCs of five types: [76], [743], [733], [643], [6331]. If expected equilibrium payoffs were proportional, only types [76], [733], [643] and [6331] could be proposed in equilibrium. It can be checked that the necessary and sufficient condition for proportionality does not hold: for any probability distribution over those coalitions, type [6] and/or type [3] would appear in the final coalition disproportionately often.

It turns out that, even though there are five player types, SSPE payoffs divide the players in only three groups, which we denote as L, M and S. We now construct an equilibrium with $v_{[7]} = v_{[6]} := v_L$, $v_{[4]} = v_{[3]} := v_M$, $v_{[1]} := v_S$ and $v_L = 2v_M$. In this situation, player [7] is indifferent between proposing to the other large player and paying v_L, and proposing to two medium players, paying v_M to each (v_L in total). Type [6] is also indifferent between buying votes from the large player or from two medium players (except that, when buying votes from a medium player, one of the two medium players has to be of type [4] because otherwise the coalition would be losing). Coalition [6331] would be too expensive, since on top of $2v_M$ one needs to pay $v_S > 0$. Type [4] may propose [743] or [643]; in both cases it needs to pay $v_L + v_M$. Likewise, type [3] has three coalition types that are equally optimal: [743], [733] and [643]; coalition [6331] is too expensive. Player [1] has [6331] as its only MWC; given the prices, it could replace [6] with [7] or/and [3] with [4] at no extra cost, hence the surplus coalitions [7331] and [7431] would also be optimal for type [1] (we return to this point below).

We now construct a profile of SSPE strategies. The following table introduces a notation for the strategies. The rows in the table are player types and the columns are coalition types. Each entry in the table represents the probability that the player type in the corresponding row proposes the coalition type in the corresponding column. It is assumed that all players of the same type follow the same strategy and each coalition of the same type is proposed with equal probability.

	[76]	[743]	[733]	[643]	[6331]
[7]	α	β	$1-\alpha-\beta$	$-$	$-$
[6]	γ	$-$	$-$	$1-\gamma$	0
[4]	$-$	μ	$-$	$1-\mu$	$-$
[3]	$-$	π	ρ	$1-\pi-\rho$	0
[1]	$-$	$-$	$-$	$-$	1

Equilibrium strategies and payoffs solve the following system of equations[1]

$$\begin{aligned}
v_L &= \frac{7}{24}[1-v_L]+\left[\frac{6}{24}\gamma+\frac{4}{24}\mu+\frac{6}{24}(\pi+\rho)\right]v_L \\
v_L &= \frac{6}{24}[1-v_L]+\left[\frac{7}{24}\alpha+\frac{4}{24}(1-\mu)+\frac{6}{24}(1-\pi-\rho)+\frac{1}{24}\right]v_L \\
v_M &= \frac{4}{24}[1-v_L-v_M]+\left[\frac{7}{24}\beta+\frac{6}{24}(1-\gamma)+\frac{6}{24}(1-\rho)\right]v_M \\
v_M &= \frac{3}{24}[1-v_L-v_M]+\left[\frac{7}{24}(\frac{\beta}{2}+1-\alpha-\beta)+\frac{6}{24}\frac{1-\gamma}{2}+\frac{4}{24}\frac{1}{2}+\frac{3}{24}\rho+\frac{1}{24}\right]v_M \\
v_S &= \frac{1}{24}[1-v_L-2v_M] \\
v_L &= 2v_M
\end{aligned}$$

There are many solutions to this system, all with $v_L=\frac{46}{164}$, $v_M=\frac{23}{164}$ and $v_S=\frac{3}{164}$. The mixed strategies are not uniquely determined. A possible solution is $\alpha=\mu=\pi=0$, $\beta=\frac{5}{23}$, $\gamma=\frac{14}{23}$, $\rho=\frac{55}{138}$. These strategies constitute an SSPE since players are behaving optimally both as proposers and as responders: only optimal coalitions are proposed given the acceptance thresholds

[1]Note that we are simplifying the first five equations by using the sixth one (i.e., all coalitions proposed with positive probability in equilibrium must give the same payoff to the proposer). For example, player [6]'s proposer payoff is written as $1-v_L$ rather than $\gamma[1-v_L]+(1-\gamma)[1-2v_M]$.

(v_L, v_M and v_S), and the acceptance thresholds equal the continuation values given the strategies. Due to the uniqueness result of Eraslan and McLennan (2013), all SSPE must have the same v-values.

There are also equilibria in which surplus coalitions are proposed with positive probability. For example, if type [1] proposes [6331] with probability $\frac{1}{2}$ and [7431] with probability $\frac{1}{2}$, the system of equations can be amended accordingly and a new solution for the equilibrium strategies is $\alpha = \mu = \pi = 0$, $\beta = \frac{12}{161}$, $\gamma = \frac{14}{23}$, $\rho = \frac{29}{92}$ (the v-values are of course unaffected).

Predicted size of the deviations

The following tables compare equilibrium payoffs and weights for all games in the dataset with at most 7 players that fail to satisfy the condition (excluding games with a veto player, of which there are two in the database). For each game, the tables shows w_i (the MIWs), v_i (expected equilibrium payoffs), and two quantitative measures of how far v is from being proportional to w. One such measure is $\frac{v_i}{w_i/\sum_{j \in N} w_j}$, the ratio of payoffs to weights, where weights are normalized so that they add up to 1. This ratio measures how much of a player's weight is translated into expected equilibrium payoffs; if expected equilibrium payoffs were proportional to weights it would always be 1. Another measure is the relative payoffs v_i/v_n, i.e. the exchange rate between players according to equilibrium predictions. If expected equilibrium payoffs were proportional, this exchange rate would always be equal to w_i/w_n (in particular, if $w_n = 1$, this ratio would replicate the MIWs).

Expected payoffs for individual players can be substantially different from weight shares, and this is very often true for the smallest player type, who may get as little as 43% of its weight share. As a result, ratios between a player's payoff and the payoff of the smallest player are often very different from w_i/w_n. Nevertheless, if we focus on the ratio of expected payoffs to weights, we see that many players have a ratio close to 1.

Table A1. Homogeneous games with up to 6 players

Weights	**7**	**5**	**5**	**2**	**2**	**1**
Payoffs	0.323	0.226	0.226	0.097	0.097	0.032
Payoffs/weights	1.014	0.993	0.993	1.067	1.067	0.699
Relative payoffs	10.16	7.10	7.10	3.05	3.05	1
Weights	**5**	**4**	**4**	**1**	**1**	**1**
Payoffs	0.324	0.297	0.297	0.027	0.027	0.027
Payoffs/weights	1.038	1.190	1.190	0.430	0.430	0.430
Relative payoffs	12.06	11.06	11.06	1	1	1
Weights	**5**	**3**	**3**	**2**	**1**	
Payoffs	0.376	0.208	0.208	0.168	0.040	
Payoffs/weights	1.053	0.970	0.970	1.178	0.556	
Relative payoffs	9.47	5.24	5.24	4.24	1	
Weights	**5**	**2**	**2**	**2**	**1**	
Payoffs	0.412	0.176	0.176	0.176	0.059	
Payoffs/weights	0.988	1.059	1.059	1.059	0.706	
Relative payoffs	7	3	3	3	1	
Weights	**4**	**3**	**3**	**1**	**1**	
Payoffs	0.333	0.295	0.295	0.038	0.038	
Payoffs/weights	1.000	1.181	1.181	0.456	0.456	
Relative payoffs	8.77	7.77	7.77	1	1	
Weights	**3**	**2**	**2**	**1**		
Payoffs	0.357	0.286	0.286	0.071		
Payoffs/weights	0.952	1.143	1.143	0.571		
Relative payoffs	5	4	4	1		

Table A2. Homogeneous games with 7 players

Weights	**9**	**7**	**7**	**2**	**2**	**2**	**1**
Payoffs	0.302	0.233	0.233	0.069	0.069	0.069	0.026
Payoffs/weights	1.006	0.998	0.998	1.035	1.035	1.035	0.771
Relative payoffs	11.74	9.06	9.06	2.69	2.69	2.69	1
Weights	**9**	**6**	**6**	**3**	**2**	**1**	**1**
Payoffs	0.325	0.217	0.217	0.108	0.085	0.024	0.024
Payoffs/weights	1.013	1.013	1.013	1.013	1.188	0.661	0.661
Relative payoffs	13.79	9.19	9.19	4.60	3.60	1	1
Weights	**9**	**3**	**3**	**3**	**2**	**1**	**1**
Payoffs	0.416	0.139	0.139	0.139	0.109	0.023	0.023
Payoffs/weights	1.017	1.017	1.017	1.017	1.200	0.650	0.650
Relative payoffs	14.08	4.69	4.69	4.69	3.69	1	1

Table A3. Nonhomogeneous games with up to 6 players

Weights	**9**	**5**	**5**	**3**	**2**	**2**
Payoffs	0.364	0.182	0.182	0.091	0.091	0.091
Payoffs/weights	1.051	0.945	0.945	0.788	1.182	1.182
Relative payoffs	4	2	2	1	1	1
Weights	**8**	**6**	**5**	**3**	**3**	**1**
Payoffs	0.320	0.227	0.206	0.113	0.113	0.020
Payoffs/weights	1.039	0.983	1.073	0.983	0.983	0.528
Relative payoffs	15.75	11.16	10.16	5.58	5.58	1
Weights	**7**	**6**	**4**	**3**	**3**	**1**
Payoffs	0.280	0.280	0.140	0.140	0.140	0.018
Payoffs/weights	0.962	1.122	0.841	1.122	1.122	0.439
Relative payoffs	15.33	15.33	7.67	7.67	7.67	1
Weights	**5**	**4**	**3**	**2**	**2**	
Payoffs	0.290	0.280	0.150	0.140	0.140	
Payoffs/weights	0.928	1.119	0.801	1.119	1.119	
Relative payoffs	2.07	2	1.07	1	1	

Table A4. Nonhomogeneous games with 7 players

Weights	**13**	**11**	**9**	**6**	**5**	**4**	**2**
Payoffs	0.261	0.218	0.174	0.130	0.088	0.087	0.043
Payoffs/weights	1.003	0.989	0.968	1.082	0.877	1.082	1.082
Relative payoffs	6.03	5.03	4.03	3	2.03	2	1
Weights	**13**	**10**	**9**	**6**	**6**	**3**	**1**
Payoffs	0.264	0.198	0.198	0.132	0.132	0.066	0.010
Payoffs/weights	0.975	0.951	1.056	1.056	1.056	1.056	0.472
Relative payoffs	26.86	20.14	20.14	13.43	13.43	6.71	1
Weights	**12**	**10**	**7**	**5**	**4**	**3**	**1**
Payoffs	0.288	0.237	0.170	0.119	0.102	0.068	0.017
Payoffs/weights	1.009	0.996	1.017	0.997	1.070	0.947	0.704
Relative payoffs	17.20	14.16	10.12	7.08	6.08	4.04	1
Weights	**11**	**8**	**7**	**4**	**4**	**1**	**1**
Payoffs	0.320	0.222	0.209	0.111	0.111	0.014	0.014
Payoffs/weights	1.046	1.000	1.073	1.000	1.000	0.491	0.491
Relative payoffs	23.43	16.28	15.28	8.14	8.14	1	1
Weights	**10**	**9**	**7**	**3**	**3**	**3**	**1**
Payoffs	0.269	0.269	0.179	0.090	0.090	0.090	0.013
Payoffs/weights	0.969	1.077	0.923	1.077	1.077	1.077	0.462
Relative payoffs	21	21	14	7	7	7	1
Weights	**10**	**3**	**3**	**3**	**2**	**2**	**1**
Payoffs	0.426	0.120	0.120	0.120	0.093	0.093	0.027
Payoffs/weights	1.023	0.961	0.961	0.961	1.117	1.117	0.649
Relative payoffs	15.76	4.44	4.44	4.44	3.44	3.44	1
Weights	**9**	**8**	**5**	**4**	**4**	**1**	**1**
Payoffs	0.278	0.278	0.139	0.139	0.139	0.014	0.014
Payoffs/weights	0.988	1.111	0.889	1.111	1.111	0.444	0.444
Relative payoffs	20	20	10	10	10	1	1

An alternative way of checking the condition using linear programming

Consider the following linear programming problem

$$\begin{aligned} & \min \ e \qquad (1) \\ \text{s.t.} \sum_{i \in S} x_i + e & \geq 1 \text{ for all } S \in W \\ \sum_{i \in N} x_i & = 1 \\ x_i & \geq 0 \text{ for all } i \in N;\ e \geq 0 \end{aligned}$$

Its interpretation is the following. Take any $(x_1, ..., x_n)$ vector, and any winning coalition S. Coalition S can divide the dollar by itself, but it is getting only $\sum_{i \in S} x_i$ in this particular allocation. The difference $1 - \sum_{i \in S} x_i$ is known as the excess of the coalition, though perhaps deficit would be a better term. The linear program above finds allocations x that minimize the maximum excess.[2] This linear programming problem is well known in cooperative game theory and is related to the core (in particular, if the solution has $e = 0$, the core is nonempty; this is not the case in weighted majority games unless there are veto players).

The following result is adapted from Peleg and Rosenmüller's (1992) theorems 3.2 and 3.3, which concern the set W^m and homogeneous games.

Claim 1 *Let $[q; w_1, ..., w_n]$ be an arbitrary weighted majority game, normalized so that $\sum_{i \in N} w_i = 1$. Then W^* is weakly balanced if and only if $x = w$ and $e = 1 - \overline{q}$ solve linear programming problem (1).*

[2] Rewriting $\sum_{i \in S} x_i + e \geq 1$ as $e \geq 1 - \sum_{i \in S} x_i$, we see that the inequalities impose that excesses of the winning coalitions are at most e. This number e is then minimized.

This result allows us to check the weak balancedness of W^* by solving (1) and comparing the optimal value of e with $1 - \overline{q}$.

To see that claim 1 is correct, construct the dual program of (1) (see, for example, Vanderbei (2008), chapter 5), where λ_S is the dual variable associated to the constraint $\sum_{i \in S} x_i + e \geq 1$ and μ is the dual variable associated to $\sum_{i \in N} x_i = 1$ (rewritten as $\sum_{i \in N} x_i \leq 1$, or equivalently as $-\sum_{i \in N} x_i \geq -1$).

$$\max \sum_{S \in W} \lambda_S - \mu \tag{2}$$

$$\begin{aligned} \text{s. t.} \sum_{S \in W, S \ni i} \lambda_S - \mu &\leq 0 \text{ for all } i \in N \\ \sum_{S \in W} \lambda_S &\leq 1 \\ \lambda_S &\geq 0 \text{ for } S \in W,\ \mu \geq 0. \end{aligned}$$

The complementary slackness theorem (see theorem 5.3 in Vanderbei (2008)) tells us that a pair of feasible solutions for the primal (1) and for the dual (2) are optimal for their respective problems if and only if $\lambda_S(1 - \sum_{i \in S} x_i - e) = 0$ for all $S \in W$, $\mu(1 - \sum_{i \in N} x_i) = 0$, $x_i(\sum_{S \in W, S \ni i} \lambda_S - \mu) = 0$ for all $i \in N$, and $e(1 - \sum_{S \in W} \lambda_S) = 0$.

We now prove claim 1.

1. Sufficiency. Suppose W^* is weakly balanced. Then we can construct feasible solutions for the primal and for the dual such that the complementary slackness conditions are satisfied. For the primal, let $x = w$ and $e = 1 - \overline{q}$. This is clearly feasible for the primal since by definition $\overline{q} = \min_{S \in W} w_i$, hence $\sum_{i \in S} w_i + (1 - \overline{q}) \geq 1$ for all $S \in W$. As for the dual, we can construct λ_S in the same way we constructed $p(S)$ in the proof of the main proposition. Since W^* is weakly balanced, there are balancing weights $(\lambda'_S)_{S \in W^*}$ such that $\sum_{S \in W^*, S \ni i} \lambda'_S = 1$ for all $i \in N$. Now construct λ_S in the following way. Draw a player at random from i using w as probability vector, and,

given i, draw a coalition $S \in W^*, S \ni i$ at random using $(\lambda'_S)_{S\in W^*, S\ni i}$. For any $S \in W$, denote by λ_S the probability that S is drawn given this procedure. Clearly, $\sum_{S\in W} \lambda_S = 1$ (since the process always draws exactly one coalition), $\lambda_S > 0$ implies $S \in W^*$ (since only coalitions in W^* have been considered), and $\sum_{S\ni i} \lambda_S = \overline{q}$ (the probability that i appears in the final coalition is $\sum_{S\ni i} \sum_{j\in S} w_j \lambda'_S = \sum_{S\in W^*, S\ni i} \sum_{j\in S} w_j \lambda'_S = \sum_{S\in W^*, S\ni i} \overline{q}\lambda'_S = \overline{q}$). Take the $(\lambda_S)_{S\in W}$ constructed in this way and $\mu = \overline{q}$ as feasible solutions for the dual. They are clearly feasible, and moreover $\sum_{S\in W, S\ni i} \lambda_S - \mu \leq 0$ for all $i \in N$ and $\sum_{S\in W} \lambda_S \leq 1$ both hold with equality, which immediately implies two of the complementary slackness conditions, $x_i(\sum_{S\in W, S\ni i} \lambda_S - \mu) = 0$ for all $i \in N$, and $e(1 - \sum_{S\in W} \lambda_S) = 0$. The other two conditions are also immediate: by construction, $\lambda_S > 0$ implies $\sum_{i\in S} w_i = \overline{q}$. We have also assumed that weights are normalized, hence $1 = \sum_{i\in N} w_i$.

2. Necessity. Suppose $x = w$ and $e = 1 - \overline{q}$ solve the primal program, in which case the optimal value of the primal is $1 - \overline{q}$. By the strong duality theorem (see Vanderbei, 2008, theorem 5.2) the dual program also has a solution (and the optimal value of the objective function in the dual problem is also $1 - \overline{q}$). Since both the primal and the dual have a solution, the complementary slackness conditions must be satisfied for $x = w$, $e = 1 - \overline{q}$ and some suitable values of λ_S and μ. According to the complementary slackness conditions, if $\lambda_S > 0$, then $1 - \sum_{i\in S} w_i = 1 - \overline{q}$, that is, only coalitions in W^* have a positive value of λ_S. Also, $w_i > 0$ implies $\sum_{S\in W, S\ni i} \lambda_S = \mu$ for i, which, since only coalitions in W^* have a positive weight, can be written as $\sum_{S\in W^*, S\ni i} \lambda_S = \mu$.

If the weighted majority game is such that $\overline{q} = 1$, we are in the trivial case in which a winning coalition requires the presence of all players with positive weight. Then the optimal value of the primal is 0, and the optimal value of the dual is 0. This is a trivial case in which the set W^* is clearly weakly balanced since one can place a weight of 1 on the grand coalition and 0 on all others.

Let $\overline{q} < 1$. Then the optimal value of the primal is positive, and the optimal value of the dual must be positive as well. This in turn requires that $\mu > 0$ (if $\mu = 0$, feasibility of the dual program would imply $\lambda_S = 0$ for all $S \in W$, and the value of the objective function of the dual program would be 0). We can then construct weights $\lambda'_S = \frac{\lambda_S}{\mu}$. Are these weights balancing weights? If $w_i > 0$, complementary slackness requires that $\sum_{S \in W, S \ni i} \lambda_S = \mu$, or equivalently that $\sum_{S \in W^*, S \ni i} \lambda'_S = 1$. Once we have a collection of coalitions that is weakly balanced when only players with $w_i > 0$ are considered, we can construct a collection in which the result is also true for players with $w_i = 0$. Take a player j with $w_j > 0$, and add i to the coalition if and only if j is in it. Thus, coalitions including both i and j or neither are unchanged, coalitions including only i have i removed from them, and coalitions including only j have i added to them; the new coalitions inherit the weight of the old ones, and, since $w_i = 0$, i can be freely added or removed from coalitions in W^* to obtain coalitions still in W^*. The resulting weights λ''_S are such that $\sum_{S \in W^*, S \ni i} \lambda''_S = 1$ for all i, hence W^* is weakly balanced.

The nucleolus is always a solution to (1), hence, when W^* is weakly balanced, w has the same maximum excess as the nucleolus. This does not imply that w coincides with the nucleolus, or even that the nucleolus is a system of weights (see footnote 13 in the paper). Calculating the nucleolus is *not* a convenient way to solve (1): to calculate the nucleolus, one has to start by solving (1), which may have many solutions and, if this is the case, additional calculations have to be performed to determine which of the many solutions is the nucleolus. The upside of calculating the nucleolus is that researchers have developed algorithms and computer programs for this very purpose. Besides the more direct approach described in the supplementary files, all calculations in table 1 of the paper have been double-checked with the help of a computer program written by Jean Derks to compute the nucleolus.

Proof of the corollary

The proof of the corollary in the paper is very similar to the proof of the main proposition. For sufficiency, no changes need to be made since the proof does not rely on w being a set of voting weights. The proof of necessity rests on the following lemma.

Lemma 2 *Let x be such that $x_i > 0$ for all $i \in N$. If $v = x$ is a vector of equilibrium payoffs for the game with $\theta = x$, all players must belong to at least one of the cheapest winning coalitions in this equilibrium.*

We have denoted the set of cheapest winning coalitions according to x (the set of winning coalitions with minimum $\sum_{i \in S} x_i$) as $\mathcal{W}^*(x)$. Denote by $\overline{x} := \sum_{i \in S} x_i$ the total payoff of players in any such coalition (what we have denoted by $\overline{q}$ when x is a set of weights). The proof of the analogous result in the main text relies on $\overline{q} > \frac{1}{2}$, which is known to hold since w is a system of weights. The result holds more generally, but requires a longer proof.

Suppose an equilibrium exists with $v = x$. Consider the set C of players that belong to at least one coalition in $\mathcal{W}^*(x)$. Since players only propose a coalition if it is among the cheapest winning coalitions to which they belong, players in C only propose to other players in C, and they only propose coalitions of total payoff $\overline{x}$.

Take any coalition S that $i \in C$ proposes with positive probability in equilibrium. The total *expected* payoff of players in S, including i, is $\sum_{j \in S} x_j = \overline{x}$ (the total actual payoff if i is selected as a proposer and proposes S is of course

1). Player i may play a mixed strategy as a proposer, but he always proposes a coalition of total expected payoff $\overline{x}$; hence, $\sum_{S:i\in S} p_i(S) \sum_{j\in S} x_j = \overline{x}$ in equilibrium, since each S has a total expected payoff of $\overline{x}$ and $\sum_{S:i\in S} p_i(S) = 1$.

We can re-arrange the expression $\sum_{S:i\in S} p_i(S) \sum_{j\in S} x_j$ to highlight the probabilities p_{ij}, where p_{ij} is the probability that i includes j in the coalition (of course, $p_{ii} = 1$). We then get

$$\sum_{j\in C} p_{ij} x_j = \overline{x} \text{ for all } i \in C. \tag{3}$$

The next step is to look at i's expected payoff equation, where $i \in C$. We have $x_i = x_i(1 - \delta(\overline{x} - x_i)) + r_i \delta x_i$, where we are already using $\theta = x$, and r_i denotes the probability that i is included in the coalition as a responder. Dividing by x_i, which we have assumed to be positive, we find $1 = 1 - \delta(\overline{x} - x_i) + r_i\delta$, which we can re-arrange to find that all players in C must be in the final coalition with probability $\overline{x}$.

Now suppose $N \backslash C$ is nonempty. We now show that this leads to a contradiction, hence $N = C$.

If $N \backslash C$ is nonempty, at least one player in C must receive proposals from players in $N \backslash C$ since C is a winning coalition and, given that the game is proper, this makes $N \backslash C$ a losing coalition. Thus, if we only consider proposals from players in C to each other, we should find that at least one player in C is in the coalition with a probability less than $\overline{x}$.

Taking expression (3), we can multiply both sides by x_i to find $\sum_{j\in C} p_{ij} x_j x_i = \overline{x} x_i$, and then add all such expressions up over i to find

$$\sum_{i\in C} \sum_{j\in C} p_{ij} x_j x_i = \overline{x} \sum_{i\in C} x_i. \tag{4}$$

Now let us look at the left-hand side of (4). If we re-arrange the expression taking the point of view of the players j who receive proposals, we have

$\sum_{j\in C}\sum_{i\in C} p_{ij}x_j x_i = \sum_{j\in C} x_j \sum_{i\in C} p_{ij}x_i$. The expression $\sum_{i\in C} p_{ij}x_i$ is the probability that j is included in the final coalition when only proposers from C are considered; we know that this number is at most $\overline{x}$ for any j and it is strictly below $\overline{x}$ for some j. Thus, $\sum_{j\in C} x_j \sum_{i\in C} p_{ij}x_i < \sum_{j\in C} x_j\overline{x}$, but this contradicts (4).

Given that $N = C$ and that $x_i > 0$ for all i, since a player $i \in C$ with $x_i > 0$ must be in the coalition with probability $\overline{x}$, the set $\mathcal{W}^*(x)$ must be weakly balanced.