Online Appendix

Contents

A. Proofs

A.1. Benchmark Models

Proof of **Lemma 1**: Let $(\mathcal{V}, \sigma)$ be an election. We prove the lemma by constructing another election which has binary results and yields the incumbent an ex-ante expected payoff that is at least as large as what $(\mathcal{V}, \sigma)$ does. Let

$$\begin{aligned} \mathcal{V}_1 &:= \{v \in \mathcal{V} : \mathbb{E}_\sigma[\theta|v] \geqslant 1 - c_O\} \\ \mathcal{V}_0 &:= \{v \in \mathcal{V} - \mathcal{V}_1 : \mathbb{E}_\sigma[\theta|v] \geqslant c_I\}. \end{aligned}$$

Note that for any $v \in \mathcal{V}_1$, the incumbent always prefers to claim the office and the opposition always acquiesces; for any $v \in \mathcal{V}_0$, the incumbent always prefers to stay in office but the opposition strictly prefers to start a confrontation, so that a confrontation occurs for sure; and for any $v \in \mathcal{V} - \mathcal{V}_1 - \mathcal{V}_0$, the opposition strictly prefers to start a confrontation and given that the opposition does so, the incumbent always prefers to step down. Thus, the ex-ante expected payoff of the incumbent under $(\mathcal{V}, \sigma)$ is

$$\begin{aligned} &\sum_{\theta=0,1} Pr(\theta)\left(\sigma(\mathcal{V}_1|\theta) + \sigma(\mathcal{V}_0|\theta)(\theta - c_I)\right) \\ &= \mathbb{E}[\theta]\sigma(\mathcal{V}_0 + \mathcal{V}_1|\theta = 1) + (1 - \mathbb{E}[\theta])\sigma(\mathcal{V}_1|\theta = 0) - \left(\mathbb{E}[\theta]\sigma(\mathcal{V}_0|\theta = 1) + (1 - \mathbb{E}[\theta])\sigma(\mathcal{V}_0|\theta = 0)\right) c_I \end{aligned}$$

Consider a new election $(\mathcal{V}, \sigma^*)$ such that

$$\sigma^*(v|\theta) = \begin{cases} \sigma(v|\theta), & v \in \mathcal{V}_1 \\ \theta\sigma(v|\theta=1), & v \in \mathcal{V}_0 \\ \theta\sigma(v|\theta=1) + (1-\theta)\sigma(v|\theta=0)\frac{\sigma(\mathcal{V}-\mathcal{V}_1|\theta=0)}{\sigma(\mathcal{V}-\mathcal{V}_1-\mathcal{V}_0|\theta=0)}, & v \in \mathcal{V}-\mathcal{V}_1-\mathcal{V}_0 \end{cases}.$$

Clearly, σ^* is an election and

$$\begin{aligned} \mathbb{E}_{\sigma^*}[\theta|v] &= \mathbb{E}_{\sigma}[\theta|v], \quad v \in \mathcal{V}_1 \\ \mathbb{E}_{\sigma^*}[\theta|v] &= 1, \quad v \in \mathcal{V}_0 \\ \mathbb{E}_{\sigma^*}[\theta|v] &\leqslant \mathbb{E}_{\sigma}[\theta|v], \quad v \in \mathcal{V}-\mathcal{V}_1-\mathcal{V}_0. \end{aligned}$$

Hence, $(\mathcal{V}, \sigma^*)$ yields the incumbent the ex-ante expected payoff of

$$\begin{aligned} &\mathbb{E}[\theta]\sigma(\mathcal{V}_0+\mathcal{V}_1|\theta=1) + (1-\mathbb{E}[\theta])\sigma(\mathcal{V}_1|\theta=0) \\ \geqslant\ &\mathbb{E}[\theta]\sigma(\mathcal{V}_0+\mathcal{V}_1|\theta=1) + (1-\mathbb{E}[\theta])\sigma(\mathcal{V}_1|\theta=0) - (\mathbb{E}[\theta]\sigma(\mathcal{V}_0|\theta=1) + (1-\mathbb{E}[\theta])\sigma(\mathcal{V}_0|\theta=0))\, c_I, \end{aligned}$$

which is at least as large as that under $(\mathcal{V}, \sigma)$.

Now define $\boldsymbol{m}$ such that

$$m_\theta := \sigma^*(\mathcal{V}_0+\mathcal{V}_1|\theta).$$

First, by definition,

$$\sigma^*(v|\theta=0) \leqslant \mu\sigma^*(v|\theta=1)$$

for each $v \in \mathcal{V}_0+\mathcal{V}_1$. Hence,

$$m_0 = \sigma^*(\mathcal{V}_0+\mathcal{V}_1|\theta=0) \leqslant \mu\sigma^*(\mathcal{V}_0+\mathcal{V}_1|\theta=1) = \mu m_1 \leqslant m_1.$$

Second,

$$\begin{aligned} \mathbb{E}_{\boldsymbol{m}}[\theta|v=1] &= \mathbb{E}_{\sigma^*}[\theta|\mathcal{V}_0+\mathcal{V}_1] \geqslant 1-c_O \\ \mathbb{E}_{\boldsymbol{m}}[\theta|v=0] &= \mathbb{E}_{\sigma^*}[\theta|\mathcal{V}-\mathcal{V}_1-\mathcal{V}_0] < c_I. \end{aligned}$$

Hence, $\boldsymbol{m}$ yields the incumbent the ex-ante expected payoff of

$$\mathbb{E}[\theta]m_1 + (1-\mathbb{E}[\theta])m_0 = \mathbb{E}[\theta]\sigma(\mathcal{V}_0+\mathcal{V}_1|\theta=1) + (1-\mathbb{E}[\theta])\sigma(\mathcal{V}_1|\theta=0),$$

which is at least as large as that under $(\mathcal{V}, \sigma)$. *(q.e.d.)*

Proof of **Proposition 1**: As shown in the proof of Lemma 1, the incumbent always finds it better to choose an $\boldsymbol{m}$ such that $\mathbb{E}_{\boldsymbol{m}}[\theta|v=1] \geqslant 1-c_O$ and $\mathbb{E}_{\boldsymbol{m}}[\theta|v=0] < c_I$

or, equivalently,

$$
\begin{aligned}
m_0 &\leqslant \mu m_1 \\
m_0 &< \frac{\mathbb{E}[\theta]}{1-\mathbb{E}[\theta]} \frac{1-c_I}{c_I} m_1 + 1 - \frac{\mathbb{E}[\theta]}{1-\mathbb{E}[\theta]} \frac{1-c_I}{c_I}
\end{aligned}
$$

For such an $\boldsymbol{m}$, the incumbent has the ex-ante expected payoff of

$$\mathbb{E}[\theta] m_1 + (1-\mathbb{E}[\theta]) m_0,$$

which is strictly increasing in both m_0 and m_1. Hence, it is optimal for the incumbent to choose $m_0 = \mu$ and $m_1 = 1$. *(q.e.d.)*

Proof of **Proposition 2**: Obviously, the incumbent would always stay in office after observing $v = 1$, so that $\theta = 1$. Hence, $a_1 = 1$. Suppose the incumbent stays in office with probability a_0 after observing $v = 0$, so that $\theta = 0$. Then, if the incumbent stays in office, the opposition has the posterior

$$\mathbb{E}_{(a_0,1)}[\theta | A = 1] = \frac{\mathbb{E}[\theta]}{\mathbb{E}[\theta] + (1-\mathbb{E}[\theta]) a_0}$$

and is not willing to start a confrontation if and only if $a_0 \leqslant \mu$. First, assume $a_0 < \mu$, then the opposition never starts a confrontation, which implies that it is optimal for the incumbent to always stay in office. This contradicts $a_0 < \mu$. Second, assume $a_0 > \mu$, then the opposition always starts a confrontation, which implies that it is optimal for the incumbent to step down after observing $v = 0$. This contradicts $a_0 > \mu$. Hence, $a_0 = \mu$. In turn, the strategy of the opposition has to make the incumbent indifferent after observing $v = 0$, so that $r = (1 + c_I)^{-1}$. *(q.e.d.)*

A.2. <u>Integrated Model: Post-Election Stage</u>

Proof of **Lemma 2**: First, because $m_0 \leqslant m_1$,

$$\mathbb{E}_{\boldsymbol{m}}[\theta | v = 1] \geqslant \mathbb{E}[\theta] > c_I,$$

so that $a_1 = 1$; and

$$\mathbb{E}_{\boldsymbol{m},\boldsymbol{a}}[\theta | A = 1, s = 1] = \mathbb{E}_{\boldsymbol{m}}[\theta | v = 0] \leqslant \mathbb{E}[\theta] < 1 - c_O,$$

so that $r_1 = 1$.

To prove that $m_1 = 1$, consider m_1 and m_1' such that $m_1 < m_1'$. Suppose $\boldsymbol{a} = (a_0, 1)$ and $\boldsymbol{r} = (r_0, 1)$ constitute an equilibrium following $\boldsymbol{m} = (m_1, m_0)$ while $\boldsymbol{a}' = (a_0', 1)$ and $\boldsymbol{r}' = (r_0', 1)$ constitute an equilibrium following $\boldsymbol{m}' = (m_1', m_0)$. Assume $r_0' > r_0$.

After choosing $\boldsymbol{m}$ and observing $v=0$, the incumbent has the expected payoff of

$$
\begin{aligned}
&1-(t+(1-t)r_0)+(t+(1-t)r_0)\left(\mathbb{E}_{\boldsymbol{m}}[\theta|v=0]-c_I\right)\\
&=1-(t+(1-t)r_0)\left(\frac{(1-\mathbb{E}[\theta])(1-m_0)}{\mathbb{E}[\theta](1-m_1)+(1-\mathbb{E}[\theta])(1-m_0)}+c_I\right)
\end{aligned}
$$

by staying in office. After choosing $\boldsymbol{m}'$ and observing $v=0$, the incumbent has

$$
\begin{aligned}
&1-(t+(1-t)r_0')\left(\frac{(1-\mathbb{E}[\theta])(1-m_0)}{\mathbb{E}[\theta](1-m_1')+(1-\mathbb{E}[\theta])(1-m_0)}+c_I\right)\\
&<1-(t+(1-t)r_0)\left(\frac{(1-\mathbb{E}[\theta])(1-m_0)}{\mathbb{E}[\theta](1-m_1)+(1-\mathbb{E}[\theta])(1-m_0)}+c_I\right)
\end{aligned}
$$

by staying in office. Hence, $a_0' \leqslant a_0$. But this implies that

$$
\begin{aligned}
\mathbb{E}_{\boldsymbol{m},\boldsymbol{a}}[\theta|A=1,s=0] &= \frac{\mathbb{E}[\theta]\left(m_1+(1-t)(1-m_1)a_0\right)}{\mathbb{E}[\theta]\left(m_1+(1-t)(1-m_1)a_0\right)+(1-\mathbb{E}[\theta])\left(m_0+(1-t)(1-m_0)a_0\right)}\\
&\leqslant \frac{\mathbb{E}[\theta]\left(m_1'+(1-t)(1-m_1')a_0'\right)}{\mathbb{E}[\theta]\left(m_1'+(1-t)(1-m_1')a_0'\right)+(1-\mathbb{E}[\theta])\left(m_0+(1-t)(1-m_0)a_0'\right)}\\
&= \mathbb{E}_{\boldsymbol{m}',\boldsymbol{a}'}[\theta|A=1,s=0],
\end{aligned}
$$

so that $r_0' \leqslant r_0$, which is a contradiction to $r_0' > r_0$. Hence, it must be true that $r_0' \leqslant r_0$. In turn, $\boldsymbol{m}$ yields the incumbent the ex-ante expected payoff of

$$
\begin{aligned}
&\left(\mathbb{E}[\theta]m_1+(1-\mathbb{E}[\theta])m_0\right)\left(1-r_0+r_0\left(\mathbb{E}_{\boldsymbol{m}}[\theta|v=1]-c_I\right)\right)\\
&+\left(\mathbb{E}[\theta](1-m_1)+(1-\mathbb{E}[\theta])(1-m_0)\right)\\
&\times \max\left\{1-(t+(1-t)r_0)+(t+(1-t)r_0)\left(\mathbb{E}_{\boldsymbol{m}}[\theta|v=0]-c_I\right),0\right\}.
\end{aligned}
$$

Because $r_0' \leqslant r_0$ and because

$$
\begin{aligned}
&1-r_0+r_0\left(\mathbb{E}_{\boldsymbol{m}}[\theta|v=1]-c_I\right)\\
&>\max\left\{1-(t+(1-t)r_0)+(t+(1-t)r_0)\left(\mathbb{E}_{\boldsymbol{m}}[\theta|v=0]-c_I\right),0\right\},
\end{aligned}
$$

the previous expression is smaller than the expected payoff that $\boldsymbol{m}'$ yields to the incumbent. Therefore, it is optimal for the incumbent to choose $m_1=1$.

At last, assume that the incumbent chooses $\boldsymbol{m}=(m,1)$ with $m>\mu$. Then,

$$
\mathbb{E}_{\boldsymbol{m},\boldsymbol{a}}[\theta|A=1,s=0] \leqslant \mathbb{E}_{\boldsymbol{m}}[\theta|v=1] < 1-c_O,
$$

so that it is optimal for the opposition to choose $r=1$. In turn, it is optimal for the incumbent to choose $a=0$. The ex-ante expected payoff of the incumbent following $\boldsymbol{m}=(m,1)$ is then $\mathbb{E}[\theta](1-c_I)$. Now assume that the incumbent chooses $\boldsymbol{m}=(0,1)$. Suppose $r=1$, then it is optimal for the incumbent to choose $a=0$. But this in turn implies that it is optimal for the opposition to choose $r = 0$, a contradiction to $r = 1$. Hence, $r < 1$. The ex-ante expected payoff of the incumbent is then

$$\mathbb{E}[\theta](1 - rc_I) + (1 - \mathbb{E}[\theta]) \max\{1 - (t + (1-t)r)(1 + c_I), 0\} > \mathbb{E}[\theta](1 - c_I).$$

Therefore, $\boldsymbol{m} = (0, 1)$ strictly dominates any $\boldsymbol{m} = (m, 1)$ such that $m > \mu$. *(q.e.d.)*

Proof of **Lemma 3**: Suppose the incumbent has chosen $\boldsymbol{m} = (m, 1)$ with $m \leqslant \mu$.

First, assume $t \geqslant (1 + c_I)^{-1}$. Then, after observing $v = 0$, the incumbent expects to gain

$$1 - (t + (1-t)r)(1 + c_I) \leqslant -(1-t)r(1 + c_I) \leqslant 0.$$

Hence, it is optimal for the incumbent to choose $a = 0$. In turn, because $m \leqslant \mu$,

$$\mathbb{E}_{\boldsymbol{m},(0,1)}[\theta | A = 1, s = 0] = \mathbb{E}_{\boldsymbol{m}}[\theta | v = 1] \geqslant 1 - c_O,$$

it is optimal for the opposition to choose $r = 0$.

Second, assume

$$t < \min\left(\frac{1-\mu}{1-m}, \frac{1}{1+c_I}\right).$$

Note that

$$\mathbb{E}_{\boldsymbol{m},\boldsymbol{a}}[\theta | A = 1, s = 0] = \frac{\mathbb{E}[\theta]}{\mathbb{E}[\theta] + (1 - \mathbb{E}[\theta])(m + (1-t)(1-m)a)} > 1 - c_O$$

if and only if $a < a^*(m)$ in the lemma. Assume $a < a^*(m)$, then $r = 0$. This, together with $t < (1 + c_I)^{-1}$, implies that it is optimal for the incumbent to choose $a = 1$, a contradiction to $a < a^*(m)$. Assume $a > a^*(m)$, then $r = 1$. This implies that it is optimal for the incumbent to choose $a = 0$, a contradiction to $a > a^*(m)$. Hence, it must be true that $a = a^*(m)$. Because $t < (1-m)^{-1}(1-\mu)$, $a^*(m) \in (0, 1)$, so that the strategy of the opposition has to make the incumbent indifferent after observing $v = 0$. This implies that $r = (1 + c_I)^{-1}$.

Third, assume

$$\frac{1-\mu}{1-m} \leqslant t < \frac{1}{1+c_I},$$

which necessitates

$$c_I < \frac{\mu - m}{1 - \mu}.$$

Because $t \geqslant (1-m)^{-1}(1-\mu)$,

$$\begin{aligned}\mathbb{E}_{\boldsymbol{m},\boldsymbol{a}}[\theta|A=1,s=0] &\geqslant \mathbb{E}_{\boldsymbol{m},\boldsymbol{a}}[\theta|A=1,s=0] \\ &= \frac{\mathbb{E}[\theta]}{\mathbb{E}[\theta]+(1-\mathbb{E}[\theta])\left(m+(1-t)(1-m)\right)} \\ &\geqslant \frac{\mathbb{E}[\theta]}{\mathbb{E}[\theta]+(1-\mathbb{E}[\theta])\mu} = 1-c_O\end{aligned}$$

for each $a \in [0,1]$. Hence, it is optimal for the opposition to choose $r=0$. Because $t < (1+c_I)^{-1}$ and $r=0$, it is optimal for the incumbent to choose $a=1$. (*q.e.d.*)

A.3. Integrated Model: Pre-Election Stage

Now we prove Proposition 3 by comparing the ex-ante expected payoff of the incumbent on the equilibrium path shown in Lemma 3 following each choice of $\boldsymbol{m}=(m,1)$ such that $m \leqslant \mu$, that is,

$$\begin{aligned}u(m) := (\mathbb{E}[\theta]+(1-\mathbb{E}[\theta])m)&\left(1-r^*(m)\left(\frac{(1-\mathbb{E}[\theta])m}{\mathbb{E}[\theta]+(1-\mathbb{E}[\theta])m}+c_I\right)\right) \\ &+(1-\mathbb{E}[\theta])(1-m)\max\left\{1-(t+(1-t)r^*(m))\,(1+c_I),0\right\}.\end{aligned}$$

Before proving Proposition 3, we first prove a lemma that takes care of simple cases with respect to t.

Lemma A.1: Suppose $t < 1-\mu$ or $t \geqslant (1+c_I)^{-1}$. It is optimal for the incumbent to chooses $m=\overline{m}$ ex ante.

Proof: First, by Lemma 3, $t \geqslant (1+c_I)^{-1}$ implies that $r^*(m)=0$ for each $m<\mu$. Hence,

$$u(m) = \mathbb{E}[\theta]+(1-\mathbb{E}[\theta])m,$$

which is strictly increasing in m. Hence, $m=\overline{m}$ is optimal.

Second, by Lemma 3, $t < 1-\mu \leqslant (1-m)^{-1}(1-\mu)$ implies that

$$r^*(m) = \frac{\frac{1}{1+c_I}-t}{1-t}$$

for each $m < \mu$. Hence,

$$u(m) = (\mathbb{E}[\theta] + (1-\mathbb{E}[\theta])m)\left(1 - \frac{\frac{1}{1+c_I} - t}{1-t}\left(\frac{(1-\mathbb{E}[\theta])m}{\mathbb{E}[\theta] + (1-\mathbb{E}[\theta])m} + c_I\right)\right)$$
$$= \mathbb{E}[\theta] + (1-\mathbb{E}[\theta])m - \frac{\frac{1}{1+c_I} - t}{1-t}\left(c_I\mathbb{E}[\theta] + (1+c_I)(1-\mathbb{E}[\theta])m\right)$$

and

$$u'(m) = (1-\mathbb{E}[\theta])\left(1 - \frac{1-(1+c_I)t}{1-t}\right) > 0,$$

so that $m = \overline{m}$ is optimal. *(q.e.d.)*

Proof of **Proposition 3**: By Lemma A.1, the proposition holds if either $t < 1-\mu$ or $t \geqslant (1+c_I)^{-1}$. Hence, we only have to prove it assuming that

$$1 - \mu \leqslant t < \frac{1}{1+c_I},$$

which necessitates

$$c_I < \frac{\mu}{1-\mu}.$$

Then, according to Lemma 3, $t < (1+c_I)^{-1}$ implies that

$$r^*(m) = \begin{cases} 0, & m \leqslant \mu - (1-\mu)\frac{1-t}{t} \\ \frac{\frac{1}{1+c_I}-t}{1-t}, & m > \mu - (1-\mu)\frac{1-t}{t} \end{cases}.$$

First, assume $\overline{m} \leqslant \mu - t^{-1}(1-t)(1-\mu)$, which is equivalent to

$$t \geqslant \frac{1-\mu}{1-\overline{m}}.$$

Then, $r^*(m) = 0$ for each $m \leqslant \overline{m}$, so that the incumbent has the ex-ante expected payoff of

$$u(m) = \mathbb{E}[\theta] + (1-\mathbb{E}[\theta])m + (1-\mathbb{E}[\theta])(1-m)\left(1-(1+c_I)t\right),$$

which is strictly increasing in m. Hence, it is optimal for the incumbent to choose

$$m = \overline{m} = \min\left(\mu - (1-\mu)\frac{1-t}{t}, \overline{m}\right).$$

Second, assume $\overline{m} > \mu - t^{-1}(1-t)(1-\mu)$, which is equivalent to

$$t < \frac{1-\mu}{1-\overline{m}}.$$

In this case, Lemma 3 implies that the incumbent has two qualitatively different choice: $m > \mu - t^{-1}(1-t)(1-\mu)$ or $m \leqslant \mu - t^{-1}(1-t)(1-\mu)$. We prove the proposition by three steps. First, derive the largest ex-ante expected payoff the incumbent can get by $m > \mu - t^{-1}(1-t)(1-\mu)$; second, derive the largest ex-ante expected payoff the incumbent gains by $m \leqslant \mu - t^{-1}(1-t)(1-\mu)$; and third, compare the two payoffs.

Suppose the incumbent chooses $m > \mu - t^{-1}(1-t)(1-\mu)$, then he has the ex-ante expected payoff

$$u(m) = \mathbb{E}[\theta] + (1-\mathbb{E}[\theta])m - (c_I\mathbb{E}[\theta] + (1+c_I)(1-\mathbb{E}[\theta])m)\,\frac{\frac{1}{1+c_I} - t}{1-t},$$

which is strictly increasing in m. Hence, the maximal payoff the incumbent expects by choosing $m > \mu - t^{-1}(1-t)(1-\mu)$ is $u(\overline{m})$.

Now suppose the incumbent chooses $m \leqslant \mu - t^{-1}(1-t)(1-\mu)$, then he has the ex-ante expected payoff

$$u(m) = \mathbb{E}[\theta] + (1-\mathbb{E}[\theta])m + (1-\mathbb{E}[\theta])(1-m)\left(1-(1+c_I)t\right),$$

which is strictly increasing in m. Hence, the maximal payoff the incumbent expects by choosing $m \leqslant \mu - t^{-1}(1-t)(1-\mu)$ is

$$\begin{aligned} u\left(\mu - (1-\mu)\frac{1-t}{t}\right) &= 1 - (1+c_I)(1-\mathbb{E}[\theta])(1-\mu) \\ &= \mathbb{E}[\theta] + (1-\mathbb{E}[\theta])\mu - c_I(1-\mathbb{E}[\theta])(1-\mu). \end{aligned}$$

Let

$$\begin{aligned} f(t, c_I, \overline{m}) &:= u\left(\mu - (1-\mu)\frac{1-t}{t}\right) - u(\overline{m}) \\ &= (c_I\mathbb{E}[\theta] + (1+c_I)(1-\mathbb{E}[\theta])\overline{m})\,\frac{\frac{1}{1+c_I} - t}{1-t} \\ &\quad + (1-\mathbb{E}[\theta])(\mu - \overline{m}) - c_I(1-\mathbb{E}[\theta])(1-\mu). \end{aligned}$$

Then, it is optimal for the incumbent to choose $m = \mu - t^{-1}(1-t)(1-\mu)$ instead of $m = \overline{m}$ if and only if $f(t, c_I, \overline{m}) > 0$.

Because

$$\frac{\partial^2}{\partial c_I^2} f(t, c_I, \overline{m}) = -\frac{1}{(1+c)^3}\frac{2\mathbb{E}[\theta]}{1-t} < 0,$$

which implies that $f(t, c_I, \overline{m})$ is strictly concave in c_I, and because

$$f\left(1-\mu, \frac{\mu}{1-\mu}, \overline{m}\right) = -(1-\mathbb{E}[\theta])\overline{m} < 0$$
$$f\left(1-\mu, \frac{\mu-\overline{m}}{1-\mu}, \overline{m}\right) = (c_I\mathbb{E}[\theta] + (1+c_I)(1-\mathbb{E}[\theta])\overline{m}) \frac{\frac{1}{1+c_I} - t}{1-t} > 0$$

there is a unique

$$\widehat{c}_I(\overline{m}) \in \left(\frac{\mu-\overline{m}}{1-\mu}, \frac{\mu}{1-\mu}\right)$$

that solves equation

$$f\left(1-\mu, \widehat{c}_I(\overline{m}), \overline{m}\right) = 0$$

Then, $f(1-\mu, c_I, \overline{m}) > 0$ if and only if $c < \widehat{c}_I(\overline{m})$.

Suppose $c \geqslant \widehat{c}_I(\overline{m})$, then

$$f(t, c_I, \overline{m}) \leqslant f(1-\mu, c_I, \overline{m}) \leqslant 0$$

for each $t \geqslant 1-\mu$. Hence, it is optimal for the incumbent to choose $m = \overline{m}$.

Now suppose $c < \widehat{c}_I(\overline{m})$, so that $f(1-\mu, c_I, \overline{m}) > 0$. Then, because

$$\frac{\partial}{\partial t} f(t, c_I, \overline{m}) = -\left(c_I\mathbb{E}[\theta] + (1+c_I)(1-\mathbb{E}[\theta])\overline{m}\right) \frac{c_I}{1+c_I} \frac{1}{(1-t)^2} < 0,$$

so that $f(t, c_I, \overline{m})$ is strictly decreasing in t, and because $f(1-\mu, c_I, \overline{m})$, there exists a unique

$$\widehat{t}(\overline{m}) \in \left(1-\mu, \frac{1}{1+c_I}\right)$$

that solves equation

$$f(t, c_I, \overline{m}) = 0$$

if $f\left((1+c_I)^{-1}, c_I, \overline{m}\right) < 0$. Extend the definition of $\widehat{t}(\overline{m})$ by letting $\widehat{t}(\overline{m}) = (1+c_I)^{-1}$ if $f\left((1+c_I)^{-1}, c_I, \overline{m}\right) \geqslant 0$. Note that by definition,

$$1-\mu < \widehat{t}(\overline{m}) \leqslant \frac{1}{1+c_I}.$$

If $1-\mu \leqslant t < \widehat{t}(\overline{m})$, then $f(1-\mu, c_I, \overline{m}) > 0$, so that it is optimal for the incumbent to choose $m = \mu - t^{-1}(1-t)(1-\mu)$. But if $\widehat{t}(\overline{m}) \leqslant t < (1+c_I)^{-1}$, then

$f(t, c_I, \overline{m}) \leqslant 0$, so that it is optimal for the incumbent to choose $\overline{m}$. *(q.e.d.)*

A.4. Accountability-Peace Trade-Off

Given m, a, r, the probability that the weak incumbent survives in office is

$$\begin{aligned} Pr(survive|\theta = 0) &= m(1-r) + (1-m)(1-t)(1-r)a \\ &= (m + (1-m)(1-t)a)(1-r) \end{aligned}$$

and the ex-ante probability of a confrontation is

$$\begin{aligned} Pr(conflict) &= (\mathbb{E}[\theta] + (1-\mathbb{E}[\theta])m)\, r + (1-\mathbb{E}[\theta])(1-m)\,(t + (1-t)r)\, a \\ &= (\mathbb{E}[\theta] + (1-\mathbb{E}[\theta])\,(m + (1-m)(1-t)a))\, r + (1-\mathbb{E}[\theta])(1-m)ta. \end{aligned}$$

Proof of **Proposition 4**: Assume $t \geqslant (1+c_I)^{-1}$, then

$$\begin{aligned} Pr(survive|\theta = 0) &= \overline{m} \\ Pr(conflict) &= 0, \end{aligned}$$

which readily proves the claim.

Now assume $t < (1+c_I)^{-1}$. There are three possible cases of $Pr(survive|\theta = 0)$ and $Pr(conflict)$. First,

$$\begin{aligned} Pr(survive|\theta = 0) &= \frac{c_I}{1+c_I}\frac{\mu}{1-t} \\ Pr(conflict) &= (\mathbb{E}[\theta] + (1-\mathbb{E}[\theta])\mu)\,\frac{\frac{1}{1+c_I} - t}{1-t} + (1-\mathbb{E}[\theta])(\mu - \overline{m})\frac{t}{1-t}, \end{aligned}$$

which is the case if $c_I \geqslant \widehat{c}_I(\overline{m})$ or if $c_I < \widehat{c}_I(\overline{m})$ and $t \notin \left[1-\mu, \widehat{t}(\overline{m})\right)$. In this case, $Pr(survive|\theta = 0)$ is independent to $\overline{m}$ and $Pr(conflict)$ is strictly decreasing in $\overline{m}$. Second,

$$\begin{aligned} Pr(survive|\theta = 0) &= \mu \\ Pr(conflict) &= (1-\mathbb{E}[\theta])(1-\mu), \end{aligned}$$

which is the case if $c_I \leqslant (1-\mu)^{-1}(\mu - \overline{m})$ and $1-\mu \leqslant t < (1-\overline{m})^{-1}(1-\mu)$ or if $(1-\mu)^{-1}(\mu - \overline{m}) < c_I < \widehat{c}_I(\overline{m})$ and $1-\mu \leqslant t < \widehat{t}(\overline{m})$. In this case, the two quantities are independent to $\overline{m}$. Third,

$$\begin{aligned} Pr(survive|\theta = 0) &= \overline{m} + (1-\overline{m})(1-t) \\ Pr(conflict) &= (1-\mathbb{E}[\theta])(1-\overline{m})t, \end{aligned}$$

which is the case if $c_I \leqslant (1-\mu)^{-1}(\mu - \overline{m})$ and $t \geqslant (1-\overline{m})^{-1}(1-\mu)$. In this case, $Pr(survive|\theta = 0)$ is increasing in $\overline{m}$ while $Pr(conflict)$ is decreasing in $\overline{m}$. Hence, the statement of the proposition holds for all these cases. (*q.e.d.*)

Proof of **Proposition 5**: The claim holds trivially if $t \geqslant (1+c_I)^{-1}$. Now assume $t < (1+c_I)^{-1}$.

Then, there are three possible cases of $Pr(survive|\theta = 0)$ and $Pr(conflict)$ as functions of t. First,

$$Pr(survive|\theta = 0) = \frac{c_I}{1+c_I}\frac{\mu}{1-t}$$
$$Pr(conflict) = (\mathbb{E}[\theta] + (1-\mathbb{E}[\theta])\mu)\,\frac{\frac{1}{1+c_I} - t}{1-t} + (1-\mathbb{E}[\theta])(\mu - \overline{m})\frac{t}{1-t},$$

which is the case if $c_I \geqslant \widehat{c}_I(\overline{m})$ or if $c_I < \widehat{c}_I(\overline{m})$ and $t \notin \left[1-\mu, \widehat{t}(\overline{m})\right)$. In this case, $Pr(survive|\theta = 0)$ is increasing in t. Moreover, the derivative of $Pr(conflict)$ with respect to t is

$$\frac{1-\mathbb{E}[\theta]}{(1-t)^2}\left(\mu - \left(\mu + \frac{\mathbb{E}[\theta]}{1-\mathbb{E}[\theta]}\right)\frac{1}{1+c_I} - \overline{m}\right),$$

which is non-positive if and only if

$$\overline{m} \geqslant \mu - \left(\mu + \frac{\mathbb{E}[\theta]}{1-\mathbb{E}[\theta]}\right)\frac{1}{1+c_I}.$$

Hence, $Pr(conflict)$ is decreasing in t as long as $\overline{m}$ is large enough. Second,

$$Pr(survive|\theta = 0) = \mu$$
$$Pr(conflict) = (1-\mathbb{E}[\theta])(1-\mu),$$

which is the case if $c_I \leqslant (1-\mu)^{-1}(\mu - \overline{m})$ and $1-\mu \leqslant t < (1-\overline{m})^{-1}(1-\mu)$ or if $(1-\mu)^{-1}(\mu-\overline{m}) < c_I < \widehat{c}_I(\overline{m})$ and $1-\mu \leqslant t < \widehat{t}(\overline{m})$. In this case, the two quantities are independent to t. Third,

$$Pr(survive|\theta = 0) = \overline{m} + (1-\overline{m})(1-t)$$
$$Pr(conflict) = (1-\mathbb{E}[\theta])(1-\overline{m})t,$$

which is the case if $c_I \leqslant (1-\mu)^{-1}(\mu - \overline{m})$ and $t \geqslant (1-\overline{m})^{-1}(1-\mu)$. In this case, $Pr(survive|\theta = 0)$ is decreasing in t while $Pr(conflict)$ is increasing in t. (*q.e.d.*)

B. Generalization

All aspects of the model remain the same except that

$$Pr(s = 1|A = 1, v) = t_v$$

with $t_0 > t_1$. In the main text, we discussed only the case with $t_1 = 0$, here we show that the same equilibrium properties hold with a general $t_1 < t_0$.

The incumbent has posterior beliefs

$$\mathbb{E}_{\boldsymbol{m}}[\theta|v = 1] = \frac{m_1\mathbb{E}[\theta]}{m_1\mathbb{E}[\theta] + m_0\,(1 - \mathbb{E}[\theta])},$$
$$\mathbb{E}_{\boldsymbol{m}}[\theta|v = 0] = \frac{(1 - m_1)\mathbb{E}[\theta]}{(1 - m_1)\mathbb{E}[\theta] + (1 - m_0)\,(1 - \mathbb{E}[\theta])}.$$

The opposition has posterior beliefs

$$\mathbb{E}_{\boldsymbol{m},\boldsymbol{a}}[\theta|s = 0, A = 1] = \left(1 + \frac{(1 - t_1)a_1m_0 + (1 - t_0)a_0(1 - m_0)}{(1 - t_1)a_1m_1 + (1 - t_0)a_0(1 - m_1)}\frac{1 - \mathbb{E}[\theta]}{\mathbb{E}[\theta]}\right)^{-1},$$
$$\mathbb{E}_{\boldsymbol{m},\boldsymbol{a}}[\theta|s = 1, A = 1] = \left(1 + \frac{t_1a_1m_0 + t_0a_0(1 - m_0)}{t_1a_1m_1 + t_0a_0(1 - m_1)}\frac{1 - \mathbb{E}[\theta]}{\mathbb{E}[\theta]}\right)^{-1}.$$

Obviously, $a_1 = 1$. Let $a_0 = a$ represent the ex-post strategy of the incumbent. Hence,

$$\mathbb{E}_{\boldsymbol{m},a}[\theta|s = 0, A = 1] = \left(1 + \frac{m_0 + \frac{1-t_0}{1-t_1}a(1 - m_0)}{m_1 + \frac{1-t_0}{1-t_1}a(1 - m_1)}\frac{1 - \mathbb{E}[\theta]}{\mathbb{E}[\theta]}\right)^{-1},$$
$$\mathbb{E}_{\boldsymbol{m},a}[\theta|s = 1, A = 1] = \left(1 + \frac{m_0 + \frac{t_0}{t_1}a(1 - m_0)}{m_1 + \frac{t_0}{t_1}a(1 - m_1)}\frac{1 - \mathbb{E}[\theta]}{\mathbb{E}[\theta]}\right)^{-1}.$$

Note that

$$\mathbb{E}_{\boldsymbol{m}}[\theta|v = 0] \leqslant \mathbb{E}_{\boldsymbol{m},a}[\theta|s = 1, A = 1] \leqslant \mathbb{E}_{\boldsymbol{m},a}[\theta|s = 0, A = 1] \leqslant \mathbb{E}_{\boldsymbol{m}}[\theta|v = 1]$$

and the inequalities hold strictly if $m_0 < m_1$ and $a > 0$.

Step 1

Lemma B.1: In equilibrium, $\max(r_0, r_1) > 0$ and $\min(r_0, r_1) < 1$.

Proof: First, assume that the incumbent chooses $\boldsymbol{m}$ in equilibrium that induces an equilibrium path on which $r_0 = r_1 = 0$. Then, it is optimal for the incumbent to choose $a = 1$. But this in turn implies that

$$\mathbb{E}_{\boldsymbol{m},a}[\theta|s=1, A=1] = \left(1 + \frac{m_0 + \frac{t_0}{t_1}(1-m_0)}{m_1 + \frac{t_0}{t_1}(1-m_1)} \frac{1-\mathbb{E}[\theta]}{\mathbb{E}[\theta]}\right)^{-1} < \mathbb{E}[\theta] < 1 - c_O,$$

so that $r_1 = 1$, a contradiction. Hence, it must be true that $\max(r_0, r_1) > 0$.

Second, assume that the incumbent chooses $\boldsymbol{m}$ in equilibrium that induces an equilibrium path on which $r_0 = r_1 = 1$. Then, the ex-ante expected payoff of the incumbent is no larger than $\mathbb{E}[\theta](1 - c_I)$. Now suppose the incumbent has $\boldsymbol{m}' = (0, 1)$ and let $\boldsymbol{r}' = (r_0', r_1')$ denote the equilibrium strategy $\boldsymbol{m}'$ induces. Assume $r_0' = r_1' = 1$, then it is optimal for the incumbent to choose $a = 0$. This in turn implies that

$$\mathbb{E}_{\boldsymbol{m},a}[\theta|s=1, A=1] = \mathbb{E}_{\boldsymbol{m},a}[\theta|s=0, A=1] = 1 > 1 - c_O,$$

so that $r_0' = r_1' = 0$, a contradiction. Hence, $\min(r_0, r_1) < 1$. It follows that by choosing $\boldsymbol{m}'$, the incumbent obtains the ex-ante expected payoff of

$$\mathbb{E}[\theta]\left(1 - (t_1 r_1 + (1-t_1) r_0)\, c_I\right) > \mathbb{E}[\theta](1 - c_I).$$

This contradicts that $\boldsymbol{m}$ is in equilibrium. Hence, $\min(r_0, r_1) < 1$. *(q.e.d.)*

Define

$$\mu := \frac{c_O}{1 - c_O} \frac{\mathbb{E}[\theta]}{1 - \mathbb{E}[\theta]}.$$

Lemma B.2: In equilibrium, $m_0 \leqslant \mu m_1$.

Proof: Assume otherwise. Then,

$$\mathbb{E}_{\boldsymbol{m},a}[\theta|s=1, A=1] \leqslant \mathbb{E}_{\boldsymbol{m},a}[\theta|s=0, A=1] \leqslant \mathbb{E}_{\boldsymbol{m}}[\theta|v=1] < 1 - c_O,$$

which implies that it is optimal for the opposition to choose $r_0 = r_1 = 1$. But this contradicts the previous lemma. *(q.e.d.)*

Define

$$\alpha(\boldsymbol{m}) := \frac{\mu m_1 - m_0}{1 - \mu + \mu m_1 - m_0}.$$

Lemma B.3: In equilibrium,

$$\frac{t_1}{t_0}\alpha(\boldsymbol{m}) \leqslant a \leqslant \frac{1-t_1}{1-t_0}\alpha(\boldsymbol{m}).$$

Proof: First, assume that $a < (t_1/t_0)\alpha(\boldsymbol{m})$, then

$$\mathbb{E}_{\boldsymbol{m},a}[\theta|s=0, A=1] \geqslant \mathbb{E}_{\boldsymbol{m},a}[\theta|s=1, A=1] > 1 - c_O,$$

which implies that $r_0 = r_1 = 0$. But this in turn implies that it is optimal for the incumbent to choose $a = 1$, a contradiction to $a < (t_1/t_0)\alpha(\boldsymbol{m}) < 1$. Hence, it must be true that $a \geqslant (t_1/t_0)\alpha(\boldsymbol{m})$.

Second, assume that $a > ((1-t_1)/(1-t_0))\,\alpha(\boldsymbol{m})$, then

$$\mathbb{E}_{\boldsymbol{m},a}[\theta|s=1, A=1] \leqslant \mathbb{E}_{\boldsymbol{m},a}[\theta|s=0, A=1] < 1 - c_O,$$

which implies that $r_0 = r_1 = 1$, a contradiction. *(q.e.d.)*

Step 2

Define

$$\gamma(\boldsymbol{m}) := \left(\frac{(1-m_0)\,(1-\mathbb{E}[\theta])}{(1-m_1)\mathbb{E}[\theta] + (1-m_0)\,(1-\mathbb{E}[\theta])} + c_I \right)^{-1}.$$

Lemma B.4: If $m_0 < \mu m_1$, on the equilibrium path,

$$r_1^*(\boldsymbol{m}) = \begin{cases} \frac{\gamma(\boldsymbol{m})}{t_0}, & t_0 \geqslant \gamma(\boldsymbol{m}) \\ 1, & t_0 < \gamma(\boldsymbol{m}) \end{cases}.$$

Proof: Note that $m_0 < \mu m_1$ implies that $a \geqslant (t_1/t_0)\alpha(\boldsymbol{m}) > 0$, which in turn implies that $r_1 \geqslant r_0$, so that $r_1 = \max(r_0, r_1) > 0$.

First, suppose $t_0 \geqslant \gamma(\boldsymbol{m})$. Assume that $r_1 > \gamma(\boldsymbol{m})/t_0$, then

$$(1-t_0)r_0 + t_0 r_1 > \gamma(\boldsymbol{m}),$$

so that it is optimal for the incumbent to choose $a = 0$, a contradiction. Assume that $r_1 < \gamma(\boldsymbol{m})/t_0 \leqslant 1$. Then, $0 < r_1 < 1$ and

$$\mathbb{E}_{\boldsymbol{m},a}[\theta|s=0, A=1] > \mathbb{E}_{\boldsymbol{m},a}[\theta|s=1, A=1] = 1 - c_O,$$

so that $r_0 = 0$. It follows that

$$(1-t_0)r_0 + t_0 r_1 < \gamma(\boldsymbol{m}),$$

so that it is optimal for the incumbent to choose $a = 1$. But $a = 1$ implies that

$$\mathbb{E}_{\boldsymbol{m},a}[\theta|s=1, A=1] < \mathbb{E}[\theta] < 1 - c_O,$$

which in turn implies that $r_1 = 1$, a contradiction to $r_1 < \gamma(\boldsymbol{m})/t_0$. Hence, it must be true that $r_1 = \gamma(\boldsymbol{m})/t_0$.

Second, suppose $t_0 < \gamma(\boldsymbol{m})$. Assume that $r_1 < 1$. Then, $0 < r_1 < 1$ and

$$\mathbb{E}_{\boldsymbol{m},a}[\theta|s=0, A=1] > \mathbb{E}_{\boldsymbol{m},a}[\theta|s=1, A=1] \geqslant 1 - c_O,$$

so that $r_0 = 0$. But then,

$$(1-t_0)r_0 + t_0 r_1 < \gamma(\boldsymbol{m}),$$

so that it is optimal for the incumbent to choose $a = 1$. But $a = 1$ implies that

$$\mathbb{E}_{\boldsymbol{m},a}[\theta|s=1, A=1] < \mathbb{E}[\theta] < 1 - c_O,$$

which in turn implies that $r_1 = 1$, a contradiction to $r_1 < 1$. Hence, it must be true that $r_1 = 1$. *(q.e.d.)*

Lemma B.5: If $m_0 < \mu m_1$, on the equilibrium path,

$$r_0^*(\boldsymbol{m}) = \begin{cases} 0, & t_0 \geqslant \min\left(\gamma(\boldsymbol{m}), 1-(1-t_1)\alpha(\boldsymbol{m})\right) \\ \frac{\gamma(\boldsymbol{m})-t_0}{1-t_0}, & t_0 < \min\left(\gamma(\boldsymbol{m}), 1-(1-t_1)\alpha(\boldsymbol{m})\right) \end{cases}.$$

Proof: First, suppose $t_0 \geqslant \gamma(\boldsymbol{m})$, then $r_1 = \gamma(\boldsymbol{m})/t_0$. Assume $r_0 > 0$, then

$$(1-t_0)r_0 + t_0 r_1 > \gamma(\boldsymbol{m}),$$

so that it is optimal for the incumbent to choose $a = 0$, a contradiction. Hence, it must be true that $r_0 = 0$.

Second, suppose $t_0 \geqslant 1-(1-t_1)\alpha(\boldsymbol{m})$, so that

$$\frac{1-t_1}{1-t_0}\alpha(\boldsymbol{m}) \geqslant 1.$$

Then, for any $0 \leqslant a \leqslant 1$,

$$\mathbb{E}_{\boldsymbol{m},a}[\theta|s=0, A=1] \geqslant 1 - c_O,$$

so that it is optimal for the opposition to choose $r_0 = 0$.

Third, suppose $t_0 < \gamma(\boldsymbol{m})$ and $t_0 < 1-(1-t_1)\alpha(\boldsymbol{m})$, then $r_1 = 1$. Assume that $r_0 > \left(\gamma(\boldsymbol{m}) - t_0\right)/(1-t_0)$, then

$$(1-t_0)r_0 + t_0 r_1 > \gamma(\boldsymbol{m}),$$

so that it is optimal for the incumbent to choose $a = 0$, a contradiction. Assume that

$r_0 < (\gamma(\boldsymbol{m}) - t_0)/(1 - t_0)$, then

$$(1 - t_0)r_0 + t_0 r_1 < \gamma(\boldsymbol{m}),$$

so that it is optimal for the incumbent to choose $a = 1$. But $a = 1$ implies that

$$\mathbb{E}_{\boldsymbol{m},a}[\theta | s = 1, A = 1] = \left(1 + \frac{m_0 + \frac{1-t_0}{1-t_1}(1 - m_0)}{m_1 + \frac{1-t_0}{1-t_1}(1 - m_1)} \frac{1 - \mathbb{E}[\theta]}{\mathbb{E}[\theta]}\right)^{-1} < 1 - c_O,$$

where the second inequality is due to $t_0 < 1 - (1 - t_1)\alpha(\boldsymbol{m})$, which in turn implies that $r_0 = 1$, a contradiction. Hence, $r_0 = (\gamma(\boldsymbol{m}) - t_0)/(1 - t_0)$. *(q.e.d.)*

Note that because the equilibrium correspondence is upper hemicontinuous, $\boldsymbol{r}^*(\boldsymbol{m}) = (r_0^*(\boldsymbol{m}), r_1^*(\boldsymbol{m}))$ remains to be in subgame equilibrium following $\boldsymbol{m}$ such that $m_0 \leqslant \mu m_1$. Then, the above three lemmas imply that for any $\boldsymbol{m}$ such that $m_0 \leqslant \mu m_1$, $r_0^*(\boldsymbol{m}) < r_1^*(\boldsymbol{m})$.

Step 3

Lemma B.6: In equilibrium, $m_1 = 1$.

Proof: The ex-ante expected payoff of the incumbent is

$$\begin{aligned} U(\boldsymbol{m}, \boldsymbol{r}) := &(m_1 \mathbb{E}[\theta] + m_0 (1 - \mathbb{E}[\theta])) \\ &\times \left(1 - ((1 - t_1)r_0 + t_1 r_1) \left(\frac{m_0 (1 - \mathbb{E}[\theta])}{m_1 \mathbb{E}[\theta] + m_0 (1 - \mathbb{E}[\theta])} + c_I\right)\right) \\ &+ ((1 - m_1)\mathbb{E}[\theta] + (1 - m_0)(1 - \mathbb{E}[\theta])) \\ &\times \max\left(0, 1 - ((1 - t_0)r_0 + t_0 r_1) \left(\frac{(1 - m_0)(1 - \mathbb{E}[\theta])}{(1 - m_1)\mathbb{E}[\theta] + (1 - m_0)(1 - \mathbb{E}[\theta])} + c_I\right)\right). \end{aligned}$$

Note that $U(\boldsymbol{m}, \boldsymbol{r})$ is strictly decreasing in r_0 and r_1 and

$$\frac{\partial}{\partial m_1} U(m_0, m_1, r_0, r_1) = (t_0 - t_1)(r_1 - r_0)c_I > 0$$

as long as $r_0 < r_1$. Consider $\boldsymbol{m}$ such that $m_0 \leqslant \mu m_1$ and $m_1 < 1$. Let $m_1' > m_1$, so that $m_0 < \mu m_1'$. According to the lemmas of the previous step, $r_s^*(m_0, m_1) \geqslant r_s^*(m_0, m_1')$ for each $s = 0, 1$. Then,

$$\begin{aligned} U(m_0, m_1, r_0^*(m_0, m_1), r_1^*(m_0, m_1)) &\leqslant U(m_0, m_1, r_0^*(m_0, m_1'), r_1^*(m_0, m_1')) \\ &< U(m_0, m_1', r_0^*(m_0, m_1'), r_1^*(m_0, m_1')). \end{aligned}$$

Thus, the incumbent always has a strictly larger ex-ante expected payoff on the equilibrium path by increasing m_1. Therefore, $m_1 = 1$ in equilibrium. *(q.e.d.)*

Given $m_1 = 1$, we let $m_0 = m$ denote the ex-ante strategy of the incumbent, then

$$\alpha(m,1) = \frac{\mu - m}{1-m}$$
$$\gamma(m,1) = \frac{1}{1+c_I}.$$

With a bit abuse of notation, we let

$$(r_0^*(m), r_1^*(m)) = \begin{cases} \left(\frac{\frac{1}{1+c_I} - t_0}{1-t_0}, 1\right), & t_0 < \min\left(\frac{1}{1+c_I}, 1-(1-t_1)\frac{\mu-m}{1-m}\right) \\ (0,1), & 1-(1-t_1)\frac{\mu-m}{1-m} \leqslant t_0 < \frac{1}{1+c_I} \\ \left(0, \frac{1}{t_0}\frac{1}{1+c_I}\right), & t_0 \geqslant \frac{1}{1+c_I} \end{cases}.$$

denote the subgame equilibrium strategy of the opposition given that the incumbent has chosen $\boldsymbol{m} = (m,1)$ ex ante.

Lemma B.7: If the incumbent chooses $\boldsymbol{m} = (m,1)$ ex ante, on the equilibrium path,

$$a^*(m) = \begin{cases} \frac{1-t_1}{1-t_0}\frac{\mu-m}{1-m}, & t_0 < \min\left(\frac{1}{1+c_I}, 1-(1-t_1)\frac{\mu-m}{1-m}\right) \\ 1, & 1-(1-t_1)\frac{\mu-m}{1-m} \leqslant t_0 < \frac{1}{1+c_I} \\ \frac{t_1}{t_0}\frac{\mu-m}{1-m}, & t_0 \geqslant \frac{1}{1+c_I} \end{cases}.$$

Proof: First, suppose

$$t_0 < \min\left(\frac{1}{1+c_I}, 1-(1-t_1)\frac{\mu-m}{1-m}\right).$$

Then, $0 < r_0^*(m) < 1$, so that the opposition is indifferent after observing $s = 0$, which implies that $\mathbb{E}_{m,a}[\theta | s = 0, A = 1] = 1 - c_O$ and solves

$$a = \frac{1-t_1}{1-t_0}\frac{\mu-m}{1-m}.$$

Second, suppose

$$1-(1-t_1)\frac{\mu-m}{1-m} \leqslant t_0 < \frac{1}{1+c_I},$$

then $r_0^*(m) = 0$ and $r_1^*(m) = 1$, so that

$$(1-t_0)r_0^*(m) + t_0 r_1^*(m) = t_0 < \frac{1}{1+c_I},$$

which implies that it is optimal for the incumbent to choose $a = 1$.

Third, suppose $t_0 \geqslant 1/(1+c_I)$. Then, $0 < r_1^*(m) \leqslant 1$ and the second inequality holds for all $t_0 > 1/(1+c_I)$, so that the opposition is indifferent after observing $s = 1$, which implies that $\mathbb{E}_{m,a}[\theta|s=1, A=1] = 1 - c_O$ and solves

$$a = \frac{t_1}{t_0}\frac{\mu - m}{1 - m}.$$

(q.e.d.)

Step 4

Let

$$V(m) := U\left(m, 1, r_0^*(m), r_1^*(m)\right)$$

be the ex-ante expected payoff of the incumbent on the equilibrium path when he chooses $\boldsymbol{m} = (m, 1)$.

Lemma B.8: If $t_0 < \min\left(1/(1+c_I), 1 - (1-t_1)\mu\right)$ or $t_0 \geqslant 1/(1+c_I)$, it is optimal for the incumbent to choose $m = \overline{m}$ ex ante.

Proof: First, suppose $t_0 < \min\left(1/(1+c_I), 1 - (1-t_1)\mu\right)$, then

$$\begin{aligned} r_0^*(m) &= \frac{\frac{1}{1+c_I} - t_0}{1 - t_0} \\ r_1^*(m) &= 1. \end{aligned}$$

It follows that

$$\begin{aligned} V(m) &= \left(\mathbb{E}[\theta] + m\left(1 - \mathbb{E}[\theta]\right)\right) \\ &\quad \times \left(1 - \left((1-t_1)\frac{\frac{1}{1+c_I} - t_0}{1 - t_0} + t_1\right)\left(\frac{m\left(1 - \mathbb{E}[\theta]\right)}{\mathbb{E}[\theta] + m\left(1 - \mathbb{E}[\theta]\right)} + c_I\right)\right) \\ &= \mathbb{E}[\theta] + m\left(1 - \mathbb{E}[\theta]\right) - \left((1-t_1)\frac{\frac{1}{1+c_I} - t_0}{1 - t_0} + t_1\right)\left(m\left(1 - \mathbb{E}[\theta]\right)(1 + c_I) + \mathbb{E}[\theta]c_I\right) \end{aligned}$$

and

$$V'(m) = \frac{(t_0 - t_1)c_I}{1 - t_0}\left(1 - \mathbb{E}[\theta]\right) > 0.$$

Hence, it is optimal for the incumbent to choose $m = \overline{m}$.

Second, suppose $t_0 \geqslant 1/(1+c_I)$, then

$$\begin{aligned} r_0^*(m) &= 0 \\ r_1^*(m) &= \frac{1}{t_0}\frac{1}{1+c_I}. \end{aligned}$$

It follows that

$$\begin{aligned} V(m) &= (\mathbb{E}[\theta] + m\,(1-\mathbb{E}[\theta])) \\ &\quad \times \left(1 - \frac{t_1}{t_0}\frac{1}{1+c_I}\left(\frac{m\,(1-\mathbb{E}[\theta])}{\mathbb{E}[\theta]+m\,(1-\mathbb{E}[\theta])} + c_I\right)\right) \\ &= \mathbb{E}[\theta] + m\,(1-\mathbb{E}[\theta]) - \frac{t_1}{t_0}\frac{1}{1+c_I}\,(m\,(1-\mathbb{E}[\theta])\,(1+c_I) + \mathbb{E}[\theta]c_I) \end{aligned}$$

and

$$V'(m) = \left(1 - \frac{t_1}{t_0}\right)(1-\mathbb{E}[\theta]) > 0.$$

Hence, it is optimal for the incumbent to choose $m = \overline{m}$. *(q.e.d.)*

Proposition B.1: The game has a unique equilibrium. In this equilibrium, there are two uniquely defined thresholds $\widetilde{c}_I(\overline{m}, t_1)$ and $\widetilde{t}_0(c_I, \overline{m}, t_1)$ such that

$$\begin{aligned} \widetilde{c}_I(\overline{m}, t_1) &\in \left(\frac{(1-t_1)\frac{\mu-\overline{m}}{1-\overline{m}}}{1-(1-t_1)\frac{\mu-\overline{m}}{1-\overline{m}}}, \frac{(1-t_1)\mu}{1-(1-t_1)\mu}\right) \\ \widetilde{t}_0(c_I, \overline{m}, t_1) &\in \left(1-(1-t_1)\mu, \frac{1}{1+c_I}\right] \end{aligned}$$

and the incumbent chooses $\boldsymbol{m}^* = (m^*, 1)$ ex ante, where

$$m^* = \begin{cases} \min\left(\overline{m}, \mu - (1-\mu)\frac{1-t_0}{t_0-t_1}\right), & c_I < \widetilde{c}_I(\overline{m}, t_1) \\ & 1-(1-t_1)\mu \leqslant t_0 < \widetilde{t}_0(c_I, \overline{m}, t_1) \\ \overline{m}, & \text{otherwise} \end{cases}.$$

Proof: By the previous lemma, the claim holds if $t_0 < \min(1/(1+c_I), 1-(1-t_1)\mu)$ or $t_0 \geqslant 1/(1+c_I)$. We only have to prove the case where

$$1-(1-t_1)\mu \leqslant t_0 < \frac{1}{1+c_I},$$

which necessitates

$$c_I < \frac{(1-t_1)\mu}{1-(1-t_1)\mu}.$$

Then,

$$r_0^*(m) = \begin{cases} 0, & m \leqslant \mu - (1-\mu)\frac{1-t_0}{t_0-t_1} \\ \frac{\frac{1}{1+c_I}-t_0}{1-t_0}, & m > \mu - (1-\mu)\frac{1-t_0}{t_0-t_1} \end{cases}$$

and $r_1^*(m) = 1$.

Assume $\overline{m} \leqslant \mu - (1-\mu)(1-t_0)/(t_0-t_1)$, which is equivalent to

$$t_0 \geqslant 1 - (1-t_1)\frac{\mu - \overline{m}}{1-\overline{m}}.$$

Then, $r_0^*(m) = 0$ for each $m \leqslant \overline{m}$, so that

$$\begin{aligned} V(m) &= (\mathbb{E}[\theta] + m\,(1-\mathbb{E}[\theta]))\left(1 - t_1\left(\frac{m\,(1-\mathbb{E}[\theta])}{\mathbb{E}[\theta] + m\,(1-\mathbb{E}[\theta])} + c_I\right)\right) \\ &\quad + (1-m)\,(1-\mathbb{E}[\theta])\,(1-t_0(1+c_I)) \\ &= 1 - t_1\,(m(1-\mathbb{E}[\theta])(1+c_I) + \mathbb{E}[\theta]c_I) - t_0(1-m)\,(1-\mathbb{E}[\theta])\,(1+c_I) \end{aligned}$$

and

$$V'(m) = (t_0 - t_1)(1+c_I)\,(1-\mathbb{E}[\theta]) > 0.$$

Hence, it is optimal for the incumbent to chooses

$$m = \overline{m} = \min\left(\overline{m}, \mu - (1-\mu)\frac{1-t_0}{t_0-t_1}\right).$$

Now assume $\overline{m} > \mu - (1-\mu)(1-t_0)/(t_0-t_1)$, which is equivalent to

$$t_0 < 1 - (1-t_1)\frac{\mu - \overline{m}}{1-\overline{m}}.$$

Then, the incumbent has two different choices: $m > \mu - (1-\mu)(1-t_0)/(t_0-t_1)$ or $m \leqslant \mu - (1-\mu)(1-t_0)/(t_0-t_1)$. We prove the proposition by three steps. First, we derive the maximal ex-ante expected payoff the incumbent obtains by choosing $m > \mu - (1-\mu)(1-t_0)/(t_0-t_1)$; second, we derive the maximal ex-ante expected payoff the incumbent gets by choosing $m \leqslant \mu - (1-\mu)(1-t_0)/(t_0-t_1)$; third, we compare the two maximal payoffs.

By choosing some $m > \mu - (1-\mu)(1-t_0)/(t_0-t_1)$, the incumbent expects that

$r_0^*(m) = (1/(1+c_I) - t_0)/(1-t_0)$ and his ex-ante expected payoff is

$$V(m) = \mathbb{E}[\theta] + m\,(1-\mathbb{E}[\theta]) - \left((1-t_1)\frac{\frac{1}{1+c_I} - t_0}{1-t_0} + t_1\right)(m\,(1-\mathbb{E}[\theta])\,(1+c_I) + \mathbb{E}[\theta]c_I)\,,$$

which has a derivative

$$V'(m) = \frac{(t_0-t_1)c_I}{1-t_0}\,(1-\mathbb{E}[\theta]) > 0.$$

Hence, the maximal payoff the incumbent can get from $m > \mu-(1-\mu)(1-t_0)/(t_0-t_1)$ is that of $m = \overline{m}$, which is $V(\overline{m})$.

By choosing some $m \leqslant \mu - (1-\mu)(1-t_0)/(t_0-t_1)$, the incumbent expects that $r_0^*(m) = 0$ and his ex-ante expected payoff is

$$V(m) = 1 - t_1\,(m(1-\mathbb{E}[\theta])(1+c_I) + \mathbb{E}[\theta]c_I) - t_0(1-m)\,(1-\mathbb{E}[\theta])\,(1+c_I),$$

which has a derivative

$$V'(m) = (t_0-t_1)(1+c_I)\,(1-\mathbb{E}[\theta]) > 0.$$

The maximal payoff the incumbent gets from $m \leqslant \mu - (1-\mu)(1-t_0)/(t_0-t_1)$ is

$$V\left(\mu - (1-\mu)\frac{1-t_0}{t_0-t_1}\right) = 1 - t_1c_I\mathbb{E}[\theta] - (1-\mu+t_1\mu)(1+c_I)\,(1-\mathbb{E}[\theta])\,.$$

Now define

$$\begin{aligned}
g(t_0,c_I,\overline{m},t_1) &:= V\left(\mu - (1-\mu)\frac{1-t_0}{t_0-t_1}\right) - V(\overline{m}) \\
&= \left((1-t_1)\frac{\frac{1}{1+c_I} - t_0}{1-t_0} + t_1\right)(\overline{m}\,(1-\mathbb{E}[\theta])\,(1+c_I) + \mathbb{E}[\theta]c_I) + (1-\overline{m})\,(1-\mathbb{E}[\theta]) \\
&\quad - t_1c_I\mathbb{E}[\theta] - (1-\mu+t_1\mu)(1+c_I)\,(1-\mathbb{E}[\theta]) \\
&= (\overline{m}\,(1-\mathbb{E}[\theta])\,(1+c_I) + \mathbb{E}[\theta]c_I)\,(1-t_1)\left(\frac{\frac{1}{1+c_I} - t_0}{1-t_0}\right) \\
&\quad + (1-\overline{m}-(\mu-\overline{m})(1-t_1))\,(1-\mathbb{E}[\theta])\left(\frac{(1-t_1)\frac{\mu-\overline{m}}{1-\overline{m}}}{1-(1-t_1)\frac{\mu-\overline{m}}{1-\overline{m}}} - c_I\right).
\end{aligned}$$

Then, it is optimal for the incumbent to choose $m = \mu-(1-\mu)(1-t_0)/(t_0-t_1)$ instead of $\overline{m}$ if and only if $g(t_0,c_I,\overline{m},t_1) > 0$.

Because

$$\frac{\partial}{\partial t_0} g(t_0, c_I, \overline{m}, t_1) = -\frac{(1-t_1)}{(1-t_0)^2} \frac{c_I}{1+c_I} \left(\overline{m}\left(1-\mathbb{E}[\theta]\right)(1+c_I) + \mathbb{E}[\theta] c_I\right) < 0,$$

$g(t_0, c_I, \overline{m}, t_1)$ is strictly decreasing in t_0. Hence,

$$g(t_0, c_I, \overline{m}, t_1) \leqslant \underline{g}(c_I, \overline{m}, t_1) := g\left(1-(1-t_1)\mu, c_I, \overline{m}, t_1\right)$$

for all

$$t_0 \geqslant 1-(1-t_1)\mu.$$

Note that

$$\begin{aligned} \underline{g}(c_I, \overline{m}, t_1) = {} & \left(\overline{m}\left(1-\mathbb{E}[\theta]\right)(1+c_I) + \mathbb{E}[\theta] c_I\right) \left(1 - t_1 - \frac{1}{\mu} \frac{c_I}{1+c_I}\right) \\ & + \left(1-\overline{m}-(\mu-\overline{m})(1-t_1)\right)\left(1-\mathbb{E}[\theta]\right) \left(\frac{(1-t_1)\frac{\mu-\overline{m}}{1-\overline{m}}}{1-(1-t_1)\frac{\mu-\overline{m}}{1-\overline{m}}} - c_I\right) \end{aligned}$$

and

$$\frac{\partial}{\partial c_I^2} \underline{g}(c_I, \overline{m}, t_1) = -\frac{1}{\mu} \frac{2}{(1+c_I)^3} < 0,$$

so that $\underline{g}(c_I, \overline{m}, t_1)$ is strictly concave in c_I. Then, because

$$\begin{aligned} \underline{g}\left(\frac{(1-t_1)\mu}{1-(1-t_1)\mu}, \overline{m}, t_1\right) &= \left(1-\overline{m}-(\mu-\overline{m})(1-t_1)\right)\left(1-\mathbb{E}[\theta]\right) \\ & \quad \times \left(\frac{(1-t_1)\frac{\mu-\overline{m}}{1-\overline{m}}}{1-(1-t_1)\frac{\mu-\overline{m}}{1-\overline{m}}} - \frac{(1-t_1)\mu}{1-(1-t_1)\mu}\right) < 0 \\ \underline{g}\left(\frac{(1-t_1)\frac{\mu-\overline{m}}{1-\overline{m}}}{1-(1-t_1)\frac{\mu-\overline{m}}{1-\overline{m}}}, \overline{m}, t_1\right) &= \left(\overline{m}\left(1-\mathbb{E}[\theta]\right)(1+c_I) + \mathbb{E}[\theta] c_I\right)(1-t_1) \\ & \quad \times \left(1 - \frac{1}{\mu} \frac{\mu-\overline{m}}{1-\overline{m}}\right) > 0, \end{aligned}$$

there exists a unique

$$\widetilde{c_I}(\overline{m}, t_1) \in \left(\frac{(1-t_1)\frac{\mu-\overline{m}}{1-\overline{m}}}{1-(1-t_1)\frac{\mu-\overline{m}}{1-\overline{m}}}, \frac{(1-t_1)\mu}{1-(1-t_1)\mu}\right)$$

that solves equation

$$\underline{g}(c_I, \overline{m}, t_1) = 0.$$

Moreover, $\underline{g}(c_I, \overline{m}, t_1) > 0$ if $c_I < \widetilde{c}_I(\overline{m}, t_1)$ and $\underline{g}(c_I, \overline{m}, t_1) < 0$ if $c_I > \widetilde{c}_I(\overline{m}, t_1)$.

First, suppose $c_I \geqslant \widetilde{c}_I(\overline{m}, t_1)$. Then,

$$g(t_0, c_I, \overline{m}, t_1) \leqslant \underline{g}(c_I, \overline{m}, t_1) \leqslant 0$$

for all $t_0 \geqslant 1 - (1 - t_1)\mu$. Hence, it is optimal for the incumbent to choose $m = \overline{m}$.

Second, suppose $c_I < \widetilde{c}_I(\overline{m}, t_1)$, so that $\underline{g}(c_I, \overline{m}, t_1) > 0$. Then, because $g(t_0, c_I, \overline{m}, t_1)$ is strictly decreasing in t and because

$$g\left(1 - (1 - t_1)\mu, c_I, \overline{m}, t_1\right) = \underline{g}(c_I, \overline{m}, t_1) < 0,$$

there exists a unique

$$\widetilde{t}_0(c_I, \overline{m}, t_1) \in \left(1 - (1 - t_1)\mu, \frac{1}{1 + c_I}\right)$$

that solves equation

$$g(t_0, c_I, \overline{m}, t_1) = 0$$

provided that

$$g\left(\frac{1}{1 + c_I}, c_I, \overline{m}, t_1\right) = \left(1 - \overline{m} - (\mu - \overline{m})(1 - t_1)\right)\left(1 - \mathbb{E}[\theta]\right)\left(\frac{(1 - t_1)\frac{\mu - \overline{m}}{1 - \overline{m}}}{1 - (1 - t_1)\frac{\mu - \overline{m}}{1 - \overline{m}}} - c_I\right) > 0$$

or equivalently

$$c_I < \frac{(1 - t_1)\frac{\mu - \overline{m}}{1 - \overline{m}}}{1 - (1 - t_1)\frac{\mu - \overline{m}}{1 - \overline{m}}}.$$

Now extend the definition of $\widetilde{t}_0(c_I, \overline{m}, t_1)$ by letting

$$\widetilde{t}_0(c_I, \overline{m}, t_1) = \frac{1}{1 + c_I}$$

if

$$c_I \in \left[\frac{(1 - t_1)\frac{\mu - \overline{m}}{1 - \overline{m}}}{1 - (1 - t_1)\frac{\mu - \overline{m}}{1 - \overline{m}}}, \widetilde{c}_I(\overline{m}, t_1)\right).$$

Note that by definition,

$$\widetilde{t}_0(c_I, \overline{m}, t_1) \in \left(1 - (1 - t_1)\mu, \frac{1}{1 + c_I}\right].$$

If $1 - (1 - t_1)\mu \leqslant t_0 < \widetilde{t}_0(c_I, \overline{m}, t_1)$, then $g(t_0, c_I, \overline{m}, t_1) > 0$, so that it is optimal for the incumbent to choose

$$m = \mu - (1 - \mu)\frac{1 - t_0}{t_0 - t_1} = \min\left(\overline{m}, \mu - (1 - \mu)\frac{1 - t_0}{t_0 - t_1}\right).$$

If $\widetilde{t}_0(c_I, \overline{m}, t_1) \leqslant t_0 < 1/(1 + c_I)$, then $g(t_0, c_I, \overline{m}, t_1) \leqslant 0$, os that it is optimal for the incumbent to choose $m = \overline{m}$. *(q.e.d.)*