Online Appendix

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August 17, 2020

Appendix:

Proof of Lemma 1. :

A civilian is about to betray the leader for sure regardless his type x and group size L, then no rebellion can succeed. Therefore each civilian has no incentive to deviate given other civilians' strategies. Given the civilians' strategies, it is obvious the best response for the leader is to choose $L = 0$. □

In the rest of this appendix, in the first step, I will prove the theoretical results under one-cutoff strategy Proposition 2 and Proposition 4. It is because these results are easier to establish and can also be used to prove the results of the two-cutoff strategy and Proposition 3.

Equation (6) can be rewritten as $u^R(x, L)|_{x=k} - u^B(x, L)|_{x=x^*} = 0$, then define $\hat{u}(x^*, M) \equiv (1 - \Phi(Ax^* - M(L, m_\theta)))((\alpha_R - \alpha_B)x^* + \beta_R - \beta_B + \gamma_R - \gamma_B) - (\gamma_R - \gamma_B)$, where $A = (1 - \lambda)/\sigma$, $M = \frac{\sigma_\varepsilon \Phi^{-1}(1 - \frac{\bar{R}}{L}) + (1-\lambda)m_\theta}{\sigma}$ when $L \leq \bar{B} + \bar{R}$ and $M = \frac{\sigma_\varepsilon \Phi^{-1}(\frac{\bar{B}}{L}) + (1-\lambda)m_\theta}{\sigma}$ when $L > \bar{B} + \bar{R}$. Since

$$(1 - \Phi(Ax^* - M(L, m_\theta)))((\alpha_R - \alpha_B)x^* + \beta_R - \beta_B + \gamma_R - \gamma_B) - (\gamma_R - \gamma_B) = 0, \quad \text{(A.1)}$$

is the equilibrium condition. The following lemma studies the shape of $\hat{u}(x^*)$.

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Lemma A.1. *For all parameter values:*

1. For a given M, $\hat{u}(x^*, M)$ is single peaked of x^*; $\lim_{x^* \to +\infty} \hat{u}(x^*, M) = -(\gamma_R - \gamma_B)$; and $\lim_{x^* \to -\infty} \hat{u}(x^*, M) = -\infty$.

2. For a given x^*, $\hat{u}(x^*, M)$ increases in M when $x^* \geq -(\beta_R - \beta_B + \gamma_R - \gamma_B)/(\alpha_R - \alpha_B)$, and decreases in M otherwise.

Proof of Lemma A.1. :

1. Let $f \equiv Ax^* - M$, and $T \equiv \frac{\beta_R - \beta_B + \gamma_R - \gamma_B}{\alpha_R - \alpha_B}$ $\hat{u}(x^*, M)$ is increasing in x^* if and only if $\frac{1-\Phi(f)}{\phi(f)}$ is greater than the finite positive constant $A(x^* + T)$. Since f is increasing in x^* and $\frac{1-\Phi(f)}{\phi(f)}$ is decreasing monotonically in x^* by the monotone hazard rate property of the normal density function. It implies that $\frac{1-\Phi(f)}{\phi(f)(x^*+T)}$ is decreasing monotonically in x^* for $x^* > -T$. Thus we need to show that $\frac{1-\Phi(f)}{\phi(f)(x^*+T)}$ passes through A. First we have $\lim_{x^* \to -T} \frac{1-\Phi(f)}{\phi(f)(x^*+T)} = +\infty$, it is because $\frac{1-\Phi(f)}{\phi(f)}$ is finite when $x^* = -T$.

$$
\begin{aligned}
& \lim_{x^* \to +\infty} \frac{1 - \Phi(f)}{\phi(f)(x^* + T)} \\
= & \lim_{x^* \to +\infty} \frac{-\phi(f) f_k}{\phi(f) - \phi(f) f f_k (x^* + T)} \\
= & \lim_{x^* \to +\infty} \frac{f_k}{f f_k (k + T) - 1} \\
= & \quad 0
\end{aligned}
$$

The first equality is due to l'Hopital's rule and the fact that $\phi'(x) = -x\phi(x)$, the second equality is algebra, and the last equality uses the fact that $f_k = A$ and f is increasing in x^*. Therefore, $\hat{u}$ is single peaked,

When x^* approaches $+\infty$, $1 - \Phi(f)$ goes to 0, therefore, $\lim_{x^* \to +\infty} \hat{u}(x^*.M) = -(\gamma_R - \gamma_B)$. Similarly x^* goes to $-\infty$, $1 - \Phi(f)$ approaches 1, Therefore $\lim_{x^* \to -\infty} \hat{u}(x^*.M) = -\infty$.

2. It is straightforward when take derivative with respect to M. □

Proof of Proposition 2. : Still define $T \equiv \frac{\beta_R - \beta_B + \gamma_R - \gamma_B}{\alpha_R - \alpha_B}$. When $x^* < -T$, $\hat{u}$ is always less than $-(\gamma_R - \gamma_B)$. Since $\hat{u}$ is single peaked and increases in M when $x^* \geq -T$, therefore there exists unique $M^{\min}$, such that the peak of $\hat{u}$ tangent to zero, i.e $\max_{x^*} (1-\Phi(Ax^*-M^{\min}))((\alpha_R-\alpha_B)x^*+\beta_R-\beta_B+\gamma_R-\gamma_B)-(\gamma_R-\gamma_B)=0$.

1. If $L \leq \bar{B}+\bar{R}$, $M=\frac{\sigma_\varepsilon\Phi^{-1}(1-\frac{\bar{R}}{L})+(1-\lambda)m_\theta}{\sigma}$ and increases with L. Therefore, when $m_\theta < \frac{\sigma}{1-\lambda}(M^{\min}-\frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(\frac{\bar{B}}{\bar{B}+\bar{R}}))$, no matter what L is chosen, M cannot exceed $M^{\min}$. If $L > \bar{B}+\bar{R}$, $M=\frac{\sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L})+(1-\lambda)m_\theta}{\sigma}$ and decreases with L. Consequently when $m_\theta < \frac{\sigma}{1-\lambda}(M^{\min}-\frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(\frac{\bar{B}}{\bar{B}+\bar{R}}))$, M cannot exceed $M^{\min}$ either. In summary, when $m_\theta < m'^{\min} \equiv \frac{\sigma}{1-\lambda}(M^{\min}-\frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(\frac{\bar{B}}{\bar{B}+\bar{R}}))$, no finite x^* can be solved from Equation (6).

2. For a given $m_\theta > m'^{\min}$, when $L \leq \bar{B}+\bar{R}$, $M=\frac{\sigma_\varepsilon\Phi^{-1}(1-\frac{\bar{R}}{L})+(1-\lambda)m_\theta}{\sigma}$ and increases with L. Then let $L^{l'} \equiv \bar{R}/(1-\Phi(\frac{M^{\min}\sigma-(1-\lambda)m_\theta}{\sigma_\varepsilon}))$. When $L > \bar{B}+\bar{R}$, $M=\frac{\sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L})+(1-\lambda)m_\theta}{\sigma}$ and decreases with L. Then let $L^{h'} \equiv \bar{B}/\Phi(\frac{M^{\min}\sigma-(1-\lambda)m_\theta}{\sigma_\varepsilon})$. It is easy to show that when $L^{l'} \leq L \leq L^{h'}$, we have $M > M^{\min}$. Next we need to show that the existence of the equilibrium cutoff x^* for such L.

Sufficiency: When $L \leq \bar{B}+\bar{R}$, $M=\frac{\sigma_\varepsilon\Phi^{-1}(1-\frac{\bar{R}}{L})+(1-\lambda)m_\theta}{\sigma}$, then when $M^{\min} \leq \frac{\sigma_\varepsilon\Phi^{-1}(1-\frac{\bar{R}}{L})+(1-\lambda)m_\theta}{\sigma}$, we have $\max_{x^*} \hat{u} \geq 0$. i.e. there exist finite x^* satisfying Equation (6). Similarly, when $L > \bar{B}+\bar{R}$, $M=\frac{\sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L})+(1-\lambda)m_\theta}{\sigma}$, then when we have $M^{\min} \leq \frac{\sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L})+(1-\lambda)m_\theta}{\sigma}$. Therefore there exist finite x^* satisfying Equation (6).

Necessity: When the finite x^* satisfying Equation (6) exists, then we must have $\max_{x^*} \hat{u} \geq 0$. When $L \leq \bar{B}+\bar{R}$, $M=\frac{\sigma_\varepsilon\Phi^{-1}(1-\frac{\bar{R}}{L})+(1-\lambda)m_\theta}{\sigma}$, then we must have $\frac{\sigma_\varepsilon\Phi^{-1}(1-\frac{\bar{R}}{L})+(1-\lambda)m_\theta}{\sigma} \geq M^{\min}$. Similarly, when $L > \bar{B}+\bar{R}$, we must have $\frac{\sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L})+(1-\lambda)m_\theta}{\sigma} \geq M^{\min}$.

Finally, because the payoffs satisfy the increasing return condition, we know for any type x, remain neutral is a weakly dominated strategy. Therefore Inequality (7) is held when $x = x^*$. □

To prove Proposition 4, we need the following lemma:

Lemma A.2. *When the finite cutoff can be solved from Equation (6), and let x^* denote the cutoff with smaller value when there are two solutions for Equation (6), then 1. $\frac{\partial x^*(L,m_\theta)}{\partial L} < 0$ when $L < \bar{B}+\bar{R}$, $\frac{\partial x^*(L,m_\theta)}{\partial L} > 0$ when $L \geq \bar{B}+\bar{R}$. 2. $\frac{\partial x^*(L,m_\theta)}{\partial m_\theta} < 0$.*

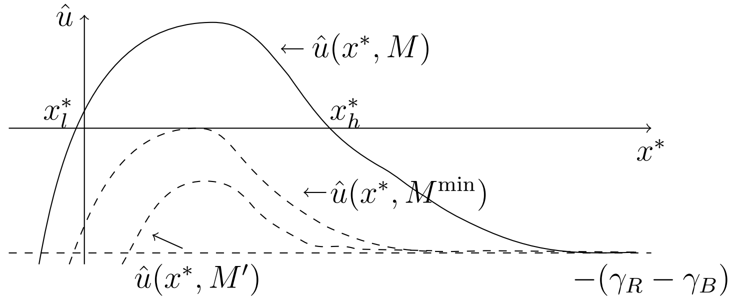

Figure 1: $\hat{u}$ represents the utility difference between joining the rebellion and turning in the leader. The cutoff which is consistent with the equilibrium is solved from $\hat{u}(x^*, M) = 0$. Solid curve-$\hat{u}(x^*, M)$ represents the case when there exist two cutoffs x^l and x^h. For a given x^*, $\hat{u}(x^*, M)$ is decreasing with M. The dashed curve $\hat{u}(x^*, M^{\min})$, $\hat{u}(x^*, M')$ represent cases when there are one cutoff and no cutoff respectively, with $M' < M^{\min} < M$.

Proof of Lemma A.2. : Since x^* is the smaller cutoff threshold, it means when other parameters are fixed, x^* is less than or equal to $x^{\max}$, where $x^{\max}$ is the maximal point of $\hat{u}$. For any $x^* \in (-\infty, x^{\max})$, $\partial \hat{u}/\partial x^* > 0$

Define

$$\bar{\theta} \equiv \max\left\{x^* - \sigma_\varepsilon \Phi^{-1}\left(1 - \frac{\bar{R}}{L}\right),\, x^* - \sigma_\varepsilon \Phi^{-1}\left(\frac{\bar{B}}{L}\right)\right\}.$$

1. By implicit function theory,

$$\frac{\partial x^*}{\partial L} = -\frac{\frac{\partial \hat{u}}{\partial L}}{\frac{\partial \hat{u}}{\partial x^*}} = \begin{cases} -\frac{\tilde{\phi}\frac{\sigma_\varepsilon}{\sigma}(\Phi_{\bar{R}}^{-1})'\frac{\bar{R}}{L^2}((\alpha_R-\alpha_B)x^*+\beta_R-\beta_B+\gamma_R-\gamma_B)}{-\tilde{\phi}A((\alpha_R-\alpha_B)x^*+\beta_R-\beta_B+\gamma_R-\gamma_B)+(1-\tilde{\Phi})(\alpha_R-\alpha_B)} < 0, & \text{if } L \le \bar{B} + \bar{R}, \\ \frac{\tilde{\phi}\frac{\sigma_\varepsilon}{\sigma}(\Phi_{\bar{B}}^{-1})'\frac{\bar{B}}{L^2}((\alpha_R-\alpha_B)x^*+\beta_R-\beta_B+\gamma_R-\gamma_B)}{-\tilde{\phi}A((\alpha_R-\alpha_B)x^*+\beta_R-\beta_B+\gamma_R-\gamma_B)+(1-\tilde{\Phi})(\alpha_R-\alpha_B)} > 0, & \text{if } L > \bar{B} + \bar{R}, \end{cases}$$

where $\Phi_{\bar{R}}^{-1} = \Phi^{-1}(1 - \frac{\bar{R}}{L})$, $\Phi_{\bar{B}}^{-1} = \Phi^{-1}(\frac{\bar{B}}{L})$, $\tilde{\phi} = \phi(\frac{\bar{\theta}-\lambda x-(1-\lambda)m_\theta}{\sigma})$, $\tilde{\Phi} = \Phi(\frac{\bar{\theta}-\lambda x-(1-\lambda)m_\theta}{\sigma})$.

2.

$$\frac{\partial x^*}{\partial m_\theta} = -\frac{\frac{\partial \hat{u}}{\partial m_\theta}}{\frac{\partial \hat{u}}{\partial x^*}} = -\frac{\tilde{\phi}\frac{1-\lambda}{\sigma}((\alpha_R - \alpha_B)x^* + \beta_R - \beta_B + \gamma_R - \gamma_B)}{-\tilde{\phi}A((\alpha_R - \alpha_B)x^* + \beta_R - \beta_B + \gamma_R - \gamma_B) + (1 - \tilde{\Phi})(\alpha_R - \alpha_B)} < 0.$$

□

Proof of Proposition 4. Let's proof the second part first.

2.

$$\frac{\partial u_0}{\partial L} = \begin{cases} -\frac{(V+\gamma_R)}{\sigma_\theta}(\frac{\partial x^*}{\partial L} - \sigma_\varepsilon(\Phi_{\bar{R}}^{-1})'\frac{\bar{R}}{L^2})\phi(\frac{\hat{\theta}-m_\theta}{\sigma_\theta}) > 0 & \text{if } L \leq \bar{B}+\bar{R} \\ -\frac{(V+\gamma_R)}{\sigma_\theta}(\frac{\partial x^*}{\partial L} + \sigma_\varepsilon(\Phi_{\bar{B}}^{-1})'\frac{\bar{B}}{L^2})\phi(\frac{\hat{\theta}-m_\theta}{\sigma_\theta}) < 0 & \text{if } L > \bar{B}+\bar{R} \end{cases}$$

By, Lemma A.2, u_0 is increasing with L when $L \leq \bar{B}+\bar{R}$, and decreasing with L when $L > \bar{B}+\bar{R}$. Therefore $L = \bar{B}+\bar{R}$ can let u_0 achieve the maximal value for any given m_θ.

1. Furthermore u_0 increases with m_θ because $\frac{\partial u_0(m_\theta,\bar{B}+\bar{R})}{\partial m_\theta} = -\phi(\frac{\partial x^*(m_\theta,\bar{B}+\bar{R})}{\partial m_\theta} - 1)\frac{1}{\sigma_\theta} > 0$. When $m_\theta \to -\gamma$, we have $u_0(m_\theta, \bar{B}+\bar{R}) \to -\infty$; and when $m_\theta \to +\infty$, $u_0(m_\theta, \bar{B}+\bar{R}) \to V > 0$. Therefore there exists unique m_θ^*, such that $u_0(m^{\min}, \bar{B}+\bar{R}) = 0$. Moreover $m_\theta^* > m^{\min}$, because $u(m^{\min}, \bar{B}+\bar{R}) = -\gamma_R$. □

Next we begin to proof the results for the two-cutoff strategy.

Proof of Lemma 2. Since the equilibrium condition only involve one equation: Equation (4). It is straightforward to prove this Lemma by following the proof of Proposition 2. Let the minimal m_θ needed to solve the cutoff as $m_1^{\min}$ □

Proof of Lemma 3. Same proof as Proposition 2. Let the minimal m_θ needed to solve the cutoff as $m_2^{\min}$ □

Define the following notations:

$$\begin{aligned} u_r(x, M) &\equiv (1-\Phi(Ax-M))(\alpha_R x + \beta_R + \gamma_R) - \gamma_R, \\ u_n(x, M) &\equiv (1-\Phi(Ax-M))(\alpha_N x + \beta_N + \gamma_N) - \gamma_N, \\ u_b(x, M) &\equiv (1-\Phi(Ax-M))(\alpha_B x + \beta_B + \gamma_B) - \gamma_B, \\ \Delta u_{rn}(x, M) &\equiv u_r - u_n = (1-\Phi(Ax-M))((\alpha_R-\alpha_N)x + \beta_R - \beta_N + \gamma_R - \gamma_N) - (\gamma_R - \gamma_N), \\ \Delta u_{nb}(x, M) &\equiv u_n - u_b = (1-\Phi(Ax-M))((\alpha_N-\alpha_B)x + \beta_N - \beta_B + \gamma_N - \gamma_B) - (\gamma_N - \gamma_B). \end{aligned}$$

Proof of Proposition 1. :

First, we prove some properties for the notations defined above. Similar as Lemma A.1, u_{rn} is also single peaked, and increases with M when $x > -(\beta_R - \beta_N + \gamma_R - \gamma_N)/(\alpha_R - \alpha_N)$. Then there exists a $M_1^{\min}$ such that $\Delta u_{rn}(x, M_1^{\min}) = 0$ has one finite root. For any $M > M_1^{\min}$, $\Delta u_{rn}(x, M) = 0$ has two roots. For any $M < M_1^{\min}$, $\Delta u_{rn}(x, M) = 0$ has no finite root.

We use x^l and x^h to denote the small and large root respectively, for $\Delta u_{rn}(x) = 0$, when they exist. Since we only care about small root x^l, I only focus on x^l, and x^h has the similar properties. By the Implicate Function Theory, it is easy to show x^l is a decreasing function of M.

Next, we will show $\frac{\partial u_n(x^l,M)}{\partial M} < 0$. It is because $\frac{\partial u_n(x^l,M)}{\partial M} = \phi(\alpha_N x^l + \beta_N + \gamma_N)[(-A + \frac{1-\Phi}{\phi}\frac{\alpha_N}{\alpha_N x^l+\beta_N+\gamma_N})\frac{\partial x^l}{\partial M} - 1]$. We have $\frac{\partial x^l}{\partial M} < 0$. $(\alpha_N x^l + \beta_N + \gamma_N) > 0$ because $x^l > -\frac{\beta_N+\gamma_N}{\alpha_N}$ (actually $x^l > 0$). Similar as the proof in Lemma A.1, we have $-A + \frac{1-\Phi}{\phi}\frac{\alpha_N}{\alpha_N x^l+\beta_N+\gamma_N} > 0$, it is because $u_n(x, M)$ is an increasing function of x before its peak. Therefore $\frac{\partial u_n(x^l,M)}{\partial M} < 0$, and $\frac{\partial u_r(x^l,M)}{\partial M} < 0$ because x^l is the solution of $u_r - u_n = 0$.

Similarly there exists a $M_2^{\min}$ such that $\Delta u_{nb}(x, M_2^{\min}) = 0$ has finite roots. For any $M > M_2^{\min}$, $\Delta u_{nb}(x^*, M) = 0$ has two roots. For any $M < M_2^{\min}$, $\Delta u_{nb}(x, M) = 0$ has no finite root. We use x'^l to denote the small root for $\Delta u_{nb}(x) = 0$, if it exists. It is easy to show that x'^l is a decreasing function of M and $u_r(x'^l(M), M)$ is decreasing with M. Due to the decreasing return condition, we have $x'^l(M) < x^l(M)$

If there exists a pair (x_N^*, x_R^*) which is consistent with the equilibrium for given (m_θ, L), it must satisfies:

$$(1-\Phi(Ax_R^* - M(m_\theta, L)))((\alpha_R - \alpha_N)x_R^* + \beta_R - \beta_N + \gamma_R - \gamma_N) - (\gamma_R - \gamma_N) = 0, \tag{A.2}$$

$$(1-\Phi(\frac{1}{\sigma}x_R^* - \frac{\lambda}{\sigma}x_N^* - M(m_\theta, L)))((\alpha_N - \alpha_B)x_N^* + \beta_N - \beta_B + \gamma_N - \gamma_B) - (\gamma_N - \gamma_B) = 0, \tag{A.3}$$

$$x_R^* - \sigma_\varepsilon\Phi^{-1}(1 - \frac{\bar{R}}{L}) \geq x_N^* - \sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L}), \tag{A.4}$$

$$x_R^* \geq x_N^*, \tag{A.5}$$

where $M = \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1 - \frac{\bar{R}}{L}) + \frac{1-\lambda}{\sigma}m_\theta$, or

$$(1-\Phi(\frac{1}{\sigma}x_N^* - \frac{\lambda}{\sigma}x_R^* - M(m_\theta, L)))((\alpha_R - \alpha_N)x_R^* + \beta_R - \beta_N + \gamma_R - \gamma_N) - (\gamma_R - \gamma_N) = 0, \tag{A.6}$$

$$(1-\Phi(Ax_N^* - M(m_\theta, L)))((\alpha_N - \alpha_B)x_N^* + \beta_N - \beta_B + \gamma_N - \gamma_B) - (\gamma_N - \gamma_B) = 0, \tag{A.7}$$

$$x_R^* - \sigma_\varepsilon \Phi^{-1}(1 - \frac{\bar{R}}{L}) < x_N^* - \sigma_\varepsilon \Phi^{-1}(\frac{\bar{B}}{L}), \tag{A.8}$$

$$x_R^* \geq x_N^*, \tag{A.9}$$

where $M = \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(\frac{\bar{B}}{L}) + \frac{1-\lambda}{\sigma}m_\theta$.

Next we need to prove the following lemma

Lemma A.3. *For any given m_θ and L, two cutoff thresholds $(x_N^*(m_\theta, L), x_R^*(m_\theta, L))$ exist, which is consistent with the equilibrium, if and only if $M_1^{\min} \leq \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1 - \frac{\bar{R}}{L}) + \frac{1-\lambda}{\sigma}m_\theta$ and $M_2^{\min} \leq \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(\frac{\bar{B}}{L}) + \frac{1-\lambda}{\sigma}m_\theta$.*

Proof of Lemma A.3.

Sufficiency: For any (m_θ, L) with $\frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1 - \frac{\bar{R}}{L}) + \frac{1-\lambda}{\sigma}m_\theta > M_1^{\min}$ and $\frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(\frac{\bar{B}}{L}) + \frac{1-\lambda}{\sigma}m_\theta > M_2^{\min}$, let $x_R^*(m_\theta, L)$ and $x_N^*(m_\theta, L)$ be the solutions for equation (A.2) and (A.7) respectively.

For this fixed m_θ, if $x_R^* - \sigma_\varepsilon\Phi^{-1}(1 - \frac{\bar{R}}{L}) > x_N^* - \sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L})$, then when we choose a larger L, the solution $x_N^*(m_\theta, L)$ will increase and $x_R^*(m_\theta, L)$ will decrease, therefore the left hand side of this inequality will decrease and the right hand side of this inequality will increase. We continuously increase L until $L = L'$ such that $x_R^* - \sigma_\varepsilon\Phi^{-1}(1 - \frac{\bar{R}}{L'}) = x_N^* - \sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L'})$. That L' exists because $\Phi^{-1}(1 - \frac{\bar{R}}{L})$ approaches $+\infty$ and $\Phi^{-1}(\frac{\bar{B}}{L})$ approaches $-\infty$, and both $x_R^*(m_\theta, L)$ and $x_N^*(m_\theta, L)$ exists with finite value when L is sufficiently large.

Similarly, when $x_R^* - \sigma_\varepsilon\Phi^{-1}(1 - \frac{\bar{R}}{L}) < x_N^* - \sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L})$, we can decrease L to get the equality, and let L'' be the solution to hold the equality.

We must have $L'(m_\theta) = L''(m_\theta)$, it is because any $x_R^*(m_\theta, L)$ satisfying (A.2) is monotonously decreasing with L, and $x_N^*(m_\theta, L)$ satisfying (A.7) is monotonously increasing with L, therefore $(x_R^*(m_\theta, L), x_N^*(m_\theta, L))$ satisfying (A.2), (A.7) and $x_R^* - \sigma_\varepsilon\Phi^{-1}(1 - \frac{\bar{R}}{L}) = x_N^* - \sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L})$ is unique for a given m_θ.

For a given m_θ and L, first, we focus on the case when $M_1^{min} < \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1-\frac{\bar{R}}{L}) + \frac{1-\lambda}{\sigma}m_\theta$ and $L < L'(m_\theta)$.

Claim 1. *For given (m_θ, L) When $M_1^{\min} < \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1-\frac{\bar{R}}{L})+\frac{1-\lambda}{\sigma}m_\theta$ and the pair $(x_R^*(m_\theta, L), x_N^*(m_\theta, L))$ satisfies (A.2)-(A.4), it must satisfy (A.5).*

Proof of Claim 1: When $M_1^{\min} < \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1-\frac{\bar{R}}{L}) + \frac{1-\lambda}{\sigma}m_\theta$. Let x_R^* be the solution solved from (A.2). For a given x_R^* solved from (A.2), let x_N^* be the solution solved from (A.3).

When (x_R^*, x_N^*) satisfies $x_R^* - \sigma_\varepsilon\Phi^{-1}(1-\frac{\bar{R}}{L}) \geq x_N^* - \sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L})$, we have

$$\begin{aligned} & (1-\Phi(\frac{x_R^*-\lambda x_R^*}{\sigma} - \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1-\frac{\bar{R}}{L}) - \frac{1-\lambda}{\sigma}m_\theta))(\alpha_N x_R^* + \beta_N + \gamma_N) - \gamma_N \\ = \; & u_n(x_R^*) \\ > \; & u_b(x_R^*) \\ = \; & (1-\Phi(\frac{x_R^*-\lambda x_R^*}{\sigma} - \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1-\frac{\bar{R}}{L}) - \frac{1-\lambda}{\sigma}m_\theta)))(\alpha_B x_R^* + \beta_B + \gamma_B) - \gamma_B. \end{aligned} \quad (A.10)$$

The first equality comes from that x_R^* is solved from (A.2) and the definition of u_n. The second inequality comes from that $M = \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1-\frac{\bar{R}}{L}) + \frac{1-\lambda}{\sigma}m_\theta$ and x_R^* is the indifference point between participation and remain neutral, at this point, the utility from betraying the leader is less than that from participation. The last equality is the definition of u_b. Since

$$(1-\Phi(\frac{x_R^*-\lambda x_N^*}{\sigma} - \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1-\frac{\bar{R}}{L}) - \frac{1-\lambda}{\sigma}m_\theta))((\alpha_N-\alpha_B)x_N^* + \beta_N - \beta_B + \gamma_N - \gamma_B).$$

is an increasing function of x_N^*, and we have

$$(1-\Phi(\frac{x_R^*-\lambda x_R^*}{\sigma} - \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1-\frac{\bar{R}}{L}) - \frac{1-\lambda}{\sigma}m_\theta))((\alpha_N-\alpha_B)x_R^* + \beta_N - \beta_B + \gamma_N - \gamma_B) - (\gamma_N - \gamma_B) > 0.$$

Therefore we have x_N^* solved from (A.3) is less than x_R^* solved from (A.2). $\square$,

Next, define $m_\theta^{\min}$ satisfies $M_1^{\min} = \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1-\frac{\bar{R}}{L}) + \frac{1-\lambda}{\sigma}m_\theta^{\min} = M_2^{\min} = \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(\frac{\bar{B}}{L}) + \frac{1-\lambda}{\sigma}m_\theta^{\min}$. For any (m_θ, L), when $m_\theta^{\min} < m_\theta$, we have the following claim:

Claim 2. *For any (m_θ, L), when $M_1^{\min} < \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1-\frac{\bar{R}}{L})+\frac{1-\lambda}{\sigma}m_\theta$, $m_\theta^{\min} < m_\theta$, and $x_R^*(m_\theta, L)$, $x_N^*(m_\theta, L)$ satisfying (A.2) and (A.3), then we have $x_R^* - \sigma_\varepsilon\Phi^{-1}(1-\frac{\bar{R}}{L}) > x_N^* - \sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L})$.*

Proof of Claim 2: For the given m_θ with $m_\theta^{\min} < m_\theta$, we know there exists a $L'(m_\theta)$ such that $(x_N^*(m_\theta, L'(m_\theta)), x_R^*(m_\theta, L'(m_\theta)))$ satisfy (A.2) and (A.7) with $x_R^*(m_\theta, L'(m_\theta)) - \sigma_\varepsilon\Phi^{-1}(1 - \frac{\bar{R}}{L'(m_\theta)}) = x_N^*(m_\theta, L'(m_\theta)) - \sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L'(m_\theta)})$.

$x_R^* - \sigma_\varepsilon\Phi^{-1}(1 - \frac{\bar{R}}{L}) = x_N^* - \sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L})$ is not possible, because $L < L'(m_\theta)$ and $L'(m_\theta)$ is the unique L satisfying satisfy (A.2) and (A.7) with $x_R^*(m_\theta, L'(m_\theta)) - \sigma_\varepsilon\Phi^{-1}(1 - \frac{\bar{R}}{L'(m_\theta)}) = x_N^*(m_\theta, L'(m_\theta)) - \sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L'(m_\theta)})$.

Now assume $x_R^* - \sigma_\varepsilon\Phi^{-1}(1 - \frac{\bar{R}}{L}) < x_N^* - \sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L})$. By the continuity of the solution, we can choose a L^c with $L' - L^c < \varepsilon^c$ such that the solution $x_R^*(m_\theta, L^c)$ and $x_N^*(m_\theta, L^c)$ solved from (A.2) and (A.3) has the following property that $x_R^*(m_\theta, L^c) - x_R^*(m_\theta, L'(m_\theta)) > x_N^*(m_\theta, L^c) - x_N^*(m_\theta, L'(m_\theta))$. It means that we choose L^c smaller than but close enough to $L'(m_\theta)$ such that the increase of x_R^* is larger than the increase of x_N^*. Then we have $x_R^*(m_\theta, L^c) - \sigma_\varepsilon\Phi^{-1}(1 - \frac{\bar{R}}{L^c}) > x_N^*(m_\theta, L^c) - \sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L^c})$.

Next, we need to show that this L^c can be found. We have

$$\frac{\partial x_N^*(m_\theta, L)}{\partial L} = \frac{\frac{1-\hat{\Phi}}{\hat{\phi}}\frac{1}{x_R^* + \frac{\beta_R - \beta_N + \gamma_R - \gamma_N}{\alpha_R - \alpha_N}} + \frac{(1-\lambda)}{\sigma}}{\frac{1-\tilde{\Phi}}{\tilde{\phi}}\frac{1}{x_N^* + \frac{\beta_N - \beta_B + \gamma_N - \gamma_B}{\alpha_N - \alpha_B}} + \frac{(1-\lambda)}{\sigma}}\frac{\partial x_R^*(m_\theta, L)}{\partial L}$$

where $\hat{\Phi} = \Phi(\frac{1-\lambda}{\sigma}x_R^* - \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1 - \frac{\bar{R}}{L^a}) - \frac{1-\lambda}{\sigma}m_\theta)$, $\hat{\phi}$ is $\hat{\Phi}$'s density function and $\tilde{\Phi} = \Phi(\frac{x_N^* - \lambda x_R^*}{\sigma} - \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1 - \frac{\bar{R}}{L}) - \frac{1-\lambda}{\sigma}m_\theta)$ and $\tilde{\Phi}$'s density function.

At $m_\theta, L'(m_\theta)$, $\hat{\Phi} = \tilde{\Phi}$ and $\hat{\phi} = \tilde{\phi}$. Because $x_R^*(m_\theta, L'(m_\theta)) > x_N^*(m_\theta, L'(m_\theta))$ and decreasing return condition, we have $x_R^*(m_\theta, L'(m_\theta)) + \frac{\beta_R - \beta_N + \gamma_R - \gamma_N}{\alpha_R - \alpha_N} > x_N^*(m_\theta, L'(m_\theta)) + \frac{\beta_N - \beta_B + \gamma_N - \gamma_B}{\alpha_N - \alpha_B} > 0$. Therefore $|\frac{\partial x_N^*(m_\theta, L)}{\partial L}| < |\frac{\partial x_R^*(m_\theta, L)}{\partial L}|$ at $(m_\theta, L'(m_\theta))$, furthermore, it means there exists an ε^c such that any L satisfies $0 < L'(m_\theta) - L < \varepsilon^c$ can be our L^c.

After that, since $x_R^*(m_\theta, L^c) - \sigma_\varepsilon\Phi^{-1}(1 - \frac{\bar{R}}{L^c}) > x_N^*(m_\theta, L^c) - \sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L^c})$ and $x_R^*(m_\theta, L) - \sigma_\varepsilon\Phi^{-1}(1 - \frac{\bar{R}}{L}) < x_N^*(m_\theta, L^c) - \sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L})$, by the continuity, we must have a L^d such that $x_N^*(m_\theta, L^d)$ and $x_R^*(m_\theta, L^d)$ satisfy (A.2) and (A.3) with $x_R^*(m_\theta, L^d) - \sigma_\varepsilon\Phi^{-1}(1 - \frac{\bar{R}}{L^d}) = x_N^*(m_\theta, L^d) - \sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L^d})$. It is a contradiction with the uniqueness of $L'(m_\theta)$.

Therefore we have $x_R^*(m_\theta, L) - \sigma_\varepsilon\Phi^{-1}(1 - \frac{\bar{R}}{L}) > x_N^*(m_\theta, L) - \sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L})$. □

So far, we prove that when $M_1^{\min} < \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1 - \frac{\bar{R}}{L}) + \frac{1-\lambda}{\sigma}m_\theta$ and $L \leq L'(m_\theta)$, there exist $(x_N^*(m_\theta, L), x_R^*(m_\theta, L))$, which is consistent with the cutoff equilibrium.

The case that when $M_2^{min} < \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(\frac{\bar{B}}{L}) + \frac{1-\lambda}{\sigma}m_\theta$ and $L > L'(m_\theta)$ is similar with the discussion above.

Necessity: Suppose two-cutoff thresholds $(x_N^*(m_\theta, L), x_R^*(m_\theta, L))$ exist,

For any m_θ and L with $M_1^{\min} < \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1-\frac{\bar{R}}{L})+\frac{1-\lambda}{\sigma}m_\theta$ but $M_2^{\min} > \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1-\frac{\bar{R}}{L})+\frac{1-\lambda}{\sigma}m_\theta$, then x_R^* and x_N^* can only be solved from (A.2) and (A.3), however, these solutions cannot satisfy (A.4) by Claim 2. Similarly, for any m_θ and L with $M_2^{\min} < \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(\frac{\bar{B}}{L}) + \frac{1-\lambda}{\sigma}m_\theta$ but $M_1^{\min} > \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(\frac{\bar{B}}{L}) + \frac{1-\lambda}{\sigma}m_\theta$, then x_R^* and x_N^* can only be solved from (A.6) and (A.7), however, these solutions cannot satisfy (A.8). Finally, when $M_1^{\min} > \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1 - \frac{\bar{R}}{L}) + \frac{1-\lambda}{\sigma}m_\theta$ and $M_2^{\min} > \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(\frac{\bar{B}}{L}) + \frac{1-\lambda}{\sigma}m_\theta$, then no finite x_R^* and x_N^* can be solved.

Now we have finished the proof of Lemma A.3 □

From the above proof, let $m^{\min} \equiv m_\theta^{\min}$, then we prove the first part of the proposition.

For any given m_θ, let L^l be the solution of L such that $M_1^{\min} = \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1 - \frac{\bar{R}}{L}) + \frac{1-\lambda}{\sigma}m_\theta$, and let L^h be the solution of L such that $M_2^{\min} = \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(\frac{\bar{B}}{L}) + \frac{1-\lambda}{\sigma}m_\theta$

Then we prove the second part of the proposition. □

Define some notations:

$$\Phi_1 = \Phi(Ax_R^* - \tau\Phi^{-1}(1 - \frac{\bar{R}}{L}) - D),$$

and ϕ_1 is Φ_1's density function. $A = (1-\lambda)/\sigma$, $D = (1-\lambda)m_\theta/\sigma$ and $\tau = \sigma_\varepsilon/\sigma$.

$$\Phi_2 = \Phi(Ax_N^* - \tau\Phi^{-1}(\frac{\bar{B}}{L}) - D),$$

and ϕ_2 is Φ_2's density function.

$$\Phi_3 = \Phi(Ax^* - \tau\Phi^{-1}(1 - \frac{\bar{R}}{L}) - D),$$

and ϕ_3 is Φ_3's density function.

On the equilibrium path, x_R^* is solved from

$$(1 - \Phi(Ax_R^* - \tau\Phi^{-1}(1 - \frac{\bar{R}}{L}) - D))((\alpha_R - \alpha_N)x_R^* + \beta_R - \beta_N + \gamma_R - \gamma_N) = \gamma_R - \gamma_N$$

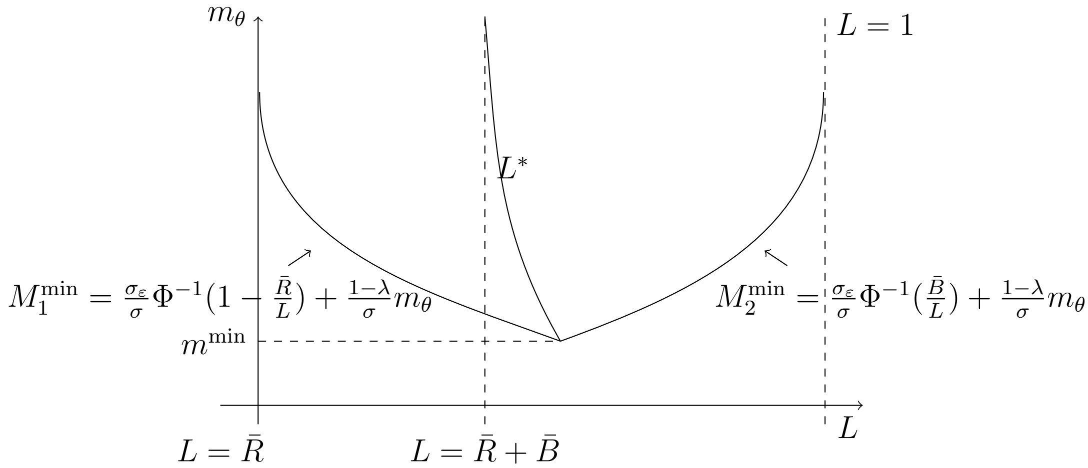

Figure 2: 1. Any point (L, m_θ) between $M_1^{\min}$ and $M_2^{\min}$ represents the finite cutoff pair (x_R^*, x_N^*) existing, which is consistent with the equilibrium. 3. Curve $M_1^{\min}$ and $M_2^{\min}$ represent the influence thresholds separating the two-cutoff strategy and no finite cutoff case when the participation condition is the dominant condition and when the maintaining secrecy condition is the dominant condition respectively. 4. $m_\theta^{\min}$ represents the lowest level of m_θ by which the leader can motivate positive participants. 6. L^* represents the balance points for the participation and maintaining secrecy conditions.

By Implicit Function Theorem,

$$\begin{aligned}\frac{\partial x_R^*}{\partial m_\theta} &= -\frac{1}{\frac{1-\Phi_1}{\phi_1}\frac{1}{x_R^*+\frac{\beta_R-\beta_N+\gamma_R-\gamma_N}{\alpha_R-\alpha_N}} - A}(\tau\Phi_c'^{-1}\frac{\bar{R}}{L^2}\frac{\partial L}{\partial m_\theta} + \frac{1-\lambda}{\sigma}) \\ &\equiv -\frac{1}{B_1}(\tau\Phi_c'^{-1}\frac{\bar{R}}{L^2}\frac{\partial L}{\partial m_\theta} + \frac{1-\lambda}{\sigma})\end{aligned}$$

where $B_1 \equiv \frac{1-\Phi_1}{\phi_1}\frac{1}{x_R^*+\frac{\beta_R-\beta_N+\gamma_R-\gamma_N}{\alpha_R-\alpha_N}} - A$

x_N^* is solved from

$$(1-\Phi(Ax_N^* - \tau\Phi^{-1}(\frac{\bar{B}}{L}) - D))((\alpha_N-\alpha_B)x_N^* + \beta_N - \beta_B + \gamma_N - \gamma_B) = \gamma_N - \gamma_B$$

By Implicit Function Theorem,

$$\begin{aligned}\frac{\partial x_N^*}{\partial m_\theta} &= -\frac{1}{\frac{1-\Phi_2}{\phi_2}\frac{1}{x_N^*+\frac{\beta_N-\beta_B+\gamma_N-\gamma_B}{\alpha_N-\alpha_B}} - A}(-\tau\Phi_{\bar{B}}'^{-1}\frac{\bar{B}}{L^2}\frac{\partial L}{\partial m_\theta} + \frac{1-\lambda}{\sigma}) \\ &\equiv -\frac{1}{B_2}(-\tau\Phi_{\bar{B}}'^{-1}\frac{\bar{B}}{L^2}\frac{\partial L}{\partial m_\theta} + \frac{1-\lambda}{\sigma})\end{aligned}$$

where $B_2 \equiv \frac{1-\Phi_2}{\phi_2}\frac{1}{x_N^*+\frac{\beta_N-\beta_B+\gamma_N-\gamma_B}{\alpha_N-\alpha_B}} - A$

x^* is solved from

$$(1-\Phi(Ax^*-\tau\Phi^{-1}(1-\frac{\bar{R}}{L})-D))((\alpha_R-\alpha_B)x^*+\beta_R-\beta_B+\gamma_R-\gamma_B)=\gamma_R-\gamma_B$$

By Implicit Function Theorem,

$$\begin{aligned}\frac{\partial x^*}{\partial m_\theta} &= -\frac{1}{\frac{1-\Phi_3}{\phi_3}\frac{1}{x^*+\frac{\beta_R-\beta_B+\gamma_R-\gamma_B}{\alpha_R-\alpha_B}}-A}(\tau\Phi_c'^{-1}\frac{\bar{R}}{L^2}\frac{\partial L}{\partial m_\theta}+\frac{1-\lambda}{\sigma}) \\ &\equiv -\frac{1}{B_3}(\frac{\sigma_\varepsilon}{\sigma}\Phi_c'^{-1}\frac{\bar{R}}{L^2}\frac{\partial L}{\partial m_\theta}+\frac{1-\lambda}{\sigma})\end{aligned}$$

where $B_3 \equiv \frac{1-\Phi_3}{\phi_3}\frac{1}{x^*+\frac{\beta_R-\beta_B+\gamma_R-\gamma_B}{\alpha_R-\alpha_B}} - A$.

Proof of Proposition 3. We begin with the second part of this proposition. If leader starts a rebellion, he will choose l to minimize θ^*. From the proof of Proposition **??**, we know for any given m_θ, there exist $L'(m_\theta)$ such that $x_R - \sigma_\varepsilon\Phi^{-1}(1-\frac{\bar{R}}{L}) = x_N - \sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L})$, where x_R is solved from (A.2) and x_N is solved from (A.6). Then if $L < L'$, we have $x_R^* - \sigma_\varepsilon\Phi^{-1}(1-\frac{\bar{R}}{L}) > x_N^* - \sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L})$. Similar as the proof of Lemma A.2, we have x_R^* is a decreasing function of L and $x_R^* - \sigma_\varepsilon\Phi^{-1}(1-\frac{\bar{R}}{L})$ is a decreasing function of L too. If $L > L'$, we have $x_R^* - \sigma_\varepsilon\Phi^{-1}(1-\frac{\bar{R}}{L}) < x_N^* - \sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L})$, and x_N^* is an increasing function of L and $x_N^* - \sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L})$ is an increasing function of L too.

Therefore, the optimal group size under two-cutoff case is L such that $x_R^*(m_\theta, L) - \sigma_\varepsilon(\Phi^{-1}(1-\frac{\bar{R}}{L})) = x_N^*(m_\theta, L(m_\theta)) - \sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L})$, through which $\bar{\theta}(L, m_\theta)$ can reach the minimal point and the leader's utility achieves the maximal point.

Next we will show that L^* is a decreasing function of m_θ. If any given m_θ, $\frac{\partial L^*}{\partial m_\theta} > 0$. Let x_N^* and x_R^* are solved from Equation (A.6) and (A.7). Then we have

$$\frac{\partial x_N^*(m_\theta, L)}{\partial m_\theta} = \frac{\frac{1-\hat{\Phi}}{\hat{\phi}}\frac{1}{x_R^*+\frac{\beta_R-\beta_N+\gamma_R-\gamma_N}{\alpha_R-\alpha_N}}+\frac{\lambda}{\sigma}}{\frac{1-\tilde{\Phi}}{\tilde{\phi}}\frac{1}{x_N^*+\frac{\beta_N-\beta_B+\gamma_N-\gamma_B}{\alpha_N-\alpha_B}}+\frac{\lambda}{\sigma}}\frac{\partial x_R^*(m_\theta, L)}{\partial m_\theta}$$

where $\hat{\Phi} = \Phi(\frac{x_N^*-\lambda x_R^*}{\sigma} - \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(\frac{\bar{B}}{L^a}) - \frac{1-\lambda}{\sigma}m_\theta)$, $\hat{\phi}$ is $\hat{\Phi}$'s density function and $\tilde{\Phi} = \Phi(\frac{(1-\lambda)x_N^*}{\sigma} - \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(\frac{\bar{B}}{L}) - \frac{1-\lambda}{\sigma}m_\theta)$ and $\tilde{\phi}$ is $\tilde{\Phi}$'s density function.

At any $L^*(m_\theta)$ and m_θ, $\hat{\Phi} = \tilde{\Phi}$ and $\hat{\phi} = \tilde{\phi}$, so we have $|\frac{x_N^*(m_\theta,L)}{\partial m_\theta}| < |\frac{x_R^*(m_\theta,L)}{\partial m_\theta}|$, and we also know both of these derivatives are less than zero. By the continuity, for another m'_θ that is less than but every close to m_θ, for the same $L^*(\theta)$, we have $x_R^* - \sigma_\varepsilon \Phi^{-1}(1 - \frac{\bar{R}}{L^*}) > x_N^* - \sigma_\varepsilon \Phi^{-1}(\frac{\bar{B}}{L^*})$.

However, since $\frac{\partial L^*}{\partial m_\theta} > 0$, for any $(L^*(m_\theta), m')$ with $m'_\theta < m_\theta$, we muse solve x_R^* and x_N^* from Equation (A.6)-(A.9). It means $x_R^* - \sigma_\varepsilon \Phi^{-1}(1 - \frac{\bar{R}}{L^*}) < x_N^* - \sigma_\varepsilon \Phi^{-1}(\frac{\bar{B}}{L^*})$. It is a contradiction.

If $\frac{\partial L^*}{\partial m_\theta} = 0$, take derivative on $x_R^* - \sigma_\varepsilon \Phi^{-1}(1 - \frac{\bar{R}}{L}) = x_N^* - \sigma_\varepsilon \Phi^{-1}(\frac{\bar{B}}{L})$ with respect to m_θ, then we have $\frac{x_N^*(m_\theta,L)}{\partial m_\theta} = \frac{x_R^*(m_\theta,L)}{\partial m_\theta}$. It is a contradiction with $|\frac{x_N^*(m_\theta,L)}{\partial m_\theta}| < |\frac{x_R^*(m_\theta,L)}{\partial m_\theta}|$.

So we conclude that $\frac{\partial L^*}{\partial m_\theta} < 0$,

On the equilibrium path, x_R^* is solved from Equation (A.2) and x_N^* is solved from Equation (A.6). When m_θ goes to positive infinity, both x_R^* and x_N^* approach zero, so $L^*(\theta)$ goes to $\bar{B} + \bar{R}$.

Since the leader's utility function is an increasing function of m_θ on the equilibrium path, and when $m_\theta < m^{\min}$, his payoff is negative. When m_θ goes to infinity, his payoff is positive. So there exist a m_θ^* such that leader will initiate a rebellion iff $m_\theta \geq m_\theta^*$. □

Proof of Corollary 1. When $\gamma_R = \gamma_N$, $\frac{\alpha_R - \alpha_N}{\gamma_R - \gamma_N} = \infty > \frac{\alpha_N - \alpha_B}{\gamma_N - \gamma_B}$, which satisfies the increasing return condition. The rest of the proof just follow the proof for one-cutoff strategy. □

The leader would select $L^*(m_\theta)$ to stage a rebellion and the threshold for the civilians would be $x_N^*(L^*(m_\theta), m_\theta)$ and $x_R^*(L^*(m_\theta), m_\theta)$ on the equilibrium path. The thresholds satisfy the following equations:

$$(1 - \Phi(Ax_R^*(L^*) - \tau\Phi^{-1}(1 - \frac{\bar{R}}{L^*}) - B))((\alpha_R - \alpha_N)x_R^*(L^*) + \beta_R - \beta_N + \gamma_R - \gamma_N) = \gamma_R - \gamma_N, \tag{A.11}$$

$$(1 - \Phi(Ax_N^*(L^*) - \tau\Phi^{-1}(\frac{\bar{B}}{L^*}) - B))((\alpha_N - \alpha_B)x_N^*(L^*) + \beta_N - \beta_B + \gamma_N - \gamma_B) = \gamma_N - \gamma_B, \tag{A.12}$$

$$x_R^*(L^*) - \sigma_\varepsilon \Phi^{-1}(1 - \frac{\bar{R}}{L^*}) = x_N^*(L^*) - \sigma_\varepsilon \Phi^{-1}(\frac{\bar{B}}{L^*}). \tag{A.13}$$

Under collective punishment, the leader chooses $L^* = \bar{B} + \bar{R}$ and we have $x_R^*(L^*) = x_N^*(L^*)$. cutoff threshold for the civilians satisfies the following equation.

$$(1-\Phi(Ax^*(L^*)-\tau\Phi^{-1}(1-\frac{\bar{R}}{L^*})-B))[(\alpha_R-\alpha_B)x^*(L^*)+\beta_R-\beta_B+\gamma_R-\gamma_B] = \gamma_R-\gamma_B), \quad \text{(A.14)}$$

we have $x_R^*(\bar{B}+\bar{R}, m^{med}) = x^C(\bar{B}+\bar{R}, m^{med}) = x^*(\bar{B}+\bar{R}, m^{med})$.

Proof of Proposition 5. First, since when m_θ goes to infinity, both x_R^* and x_B^* approach x^*. So we pick a very large m_θ and denote it as m^{med} such that $x_R^* = x_B^* = x^*$ (They are not exactly the same, but very very close).

Since $\alpha_R - \alpha_N$ is finite, we assume it is less a constant g, and α_N is greater than a constant $\hat{v}$. On the equilibrium path, since $\alpha_R > \alpha_N$, we use a constant d to denote the distance between α_R and α_N, i.e. $\alpha_R - \alpha_N = d$ and bounded by 0 and g, for any given ε_0, there exist a v_1 such that when $\alpha_N > v_1$, then $-\frac{\sigma_\varepsilon}{\sigma}\Phi(\frac{\bar{B}}{L})'^{-1}\frac{\bar{B}}{(\bar{B}+\bar{R})^2}\frac{\partial L}{\partial m_\theta} > \varepsilon_0 > 0$. It is because for given α_R and α_N, $\frac{\partial L}{\partial m_\theta} < 0$ at $m_\theta < m^{med}$. And when α_N increases x_N^* decreases, and $x_R^* - x_N^*$ increases, therefore $\frac{\partial L}{\partial m_\theta}$ is an increasing function of α_N on the equilibrium path when $m_\theta < m^{med}$.

When $\alpha_N > v_1$, we have

$$\begin{aligned} & \frac{1}{B_2}(-\frac{\sigma_\varepsilon}{\sigma}\Phi'^{-1}_{\bar{B}}\frac{\bar{B}}{L^2}\frac{\partial L}{\partial m_\theta} + \frac{1-\lambda}{\sigma}) - \frac{1}{B_3}\frac{1-\lambda}{\sigma}; \\ > \; & \frac{1}{B_2}(\varepsilon_0 + \frac{1-\lambda}{\sigma}) - \frac{1}{B_3}\frac{1-\lambda}{\sigma}; \\ = \; & \frac{1}{B_2}\varepsilon_0 + (\frac{1}{B_2} - \frac{1}{B_3})\frac{1-\lambda}{\sigma}; \end{aligned} \quad \text{(A.15)}$$

There exists a v_2 such that when $v > v_2$ we have $0 < (\frac{1}{B_3} - \frac{1}{B_2}) < \varepsilon$ and $\frac{1}{B_2}\varepsilon_0 + (\frac{1}{B_2} - \frac{1}{B_3})\frac{1-\lambda}{\sigma} > \frac{1}{B_2}\varepsilon_0 - \varepsilon\frac{1-\lambda}{\sigma} > 0$

Therefore when $\alpha_N > \bar{v} \equiv \max\{v_1, v_2\}$, we have that $\frac{\partial x_N^*}{\partial m_\theta} < \frac{\partial x^*}{\partial m_\theta} < 0$ at m^{med}. By the continuity, we have $x_N^* > x^*$ at least in a small interval $(m^{med} - \varepsilon', m^{med}]$.

Start from any $m_\theta < m^{med}$ with $x_N^*(L^*, m_\theta) > x^*(\bar{B}+\bar{R})$, we have $x_N^*(L^*, m_\theta) - \sigma\Phi^{-1}(\frac{\bar{B}}{L^*}) > x^*(\bar{B}+\bar{R}) - \sigma\Phi^{-1}(\frac{\bar{B}}{\bar{B}+\bar{R}})$ and $Ax_N^*(L^*) - \tau\Phi^{-1}(\frac{\bar{B}}{L^*}) - D > Ax^* - \tau\Phi^{-1}(\frac{\bar{B}}{\bar{B}+\bar{R}}) - D$. Therefore $\frac{1-\Phi_2}{\phi_2} < \frac{1-\Phi_3}{\phi_3}$ and $B_2 < B_3$ (under the condition $\alpha_N > v_2$ and $\alpha_R - \alpha_N = d > 0$).

As a results $\frac{\partial x_N^*}{\partial m_\theta} < \frac{\partial x^*}{\partial m_\theta}$. It means once $m_\theta < m^{med}$, $x_N^* > x^*$ always. Furthermore $\bar{\theta}^{TP} = x_N^*(L^*) - \sigma\Phi^{-1}(\frac{\bar{B}}{L^*}) > \theta^{*CP} = x^*(\bar{B}+\bar{R}) - \sigma\Phi^{-1}(\frac{\bar{B}}{\bar{B}+\bar{R}})$. Therefore, TP has higher survival probability.

□

Proof of Proposition 6. Since $\frac{\gamma_N-\gamma_B}{\alpha_N-\alpha_B} = h$ is finite, we denote its upper bound as h, and $\alpha_N - \alpha_B$ is less than a constant v'. The proof is similar as the proof of Proposition 5, we just give a sketch as follow. Let $\gamma_N - \gamma_B$ and $\alpha_N - \alpha_B$ smaller enough to guarantee $\frac{1}{B_1} - \frac{1}{B_3}$ smaller enough, then since $\frac{\sigma_\varepsilon}{\sigma}\Phi'^{-1}_{\bar{R}}\frac{\bar{R}}{L^2}\frac{\partial L}{\partial m_\theta} + \frac{1-\lambda}{\sigma} < \frac{1-\lambda}{\sigma}$. Then we have $\frac{\partial x_R^*}{\partial m_\theta} < \frac{\partial x^*}{\partial m_\theta}$ at m^{med}. Since $\frac{\gamma_N-\gamma_B}{\alpha_N-\alpha_B} = h$ is finite, we denote its upper bound as h, which can guarantee $\frac{\partial L}{\partial m_\theta}$ has a uniform low bound which is greater than 0, when $\gamma_N - \gamma_B$ and $\alpha_N - \alpha_B$ are smaller enough. By the continuity, we have $x_R^* < x^*$ at least in a small interval $(m^{med} - \varepsilon'', m^{med}]$. Start from any $m_\theta < m^{med}$ with $x_R^*(m_\theta, L^*(m_\theta)) < x^*(m_\theta, \bar{B}+\bar{R})$, we have $x_R^*(m_\theta, L^*(m_\theta)) - \sigma\Phi^{-1}(1-\frac{\bar{R}}{L(m_\theta)}) < x^*(m_\theta, \bar{B}+\bar{R}) - \sigma\Phi^{-1}(1-\frac{\bar{R}}{\bar{B}+\bar{R}})$ and $Ax_R^*(m_\theta, L^*(m_\theta)) - \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1-\frac{\bar{R}}{L^*(m_\theta)}) - \frac{1-\lambda}{\sigma}m_\theta < Ax^* - \frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1-\frac{\bar{R}}{\bar{B}+\bar{R}}) - \frac{1-\lambda}{\sigma}m_\theta$. Therefore $\frac{1-\Phi_1}{\phi_1} > \frac{1-\Phi_3}{\phi_3}$ and $B_1 > B_3$. As a results $\frac{\partial x_R^*}{\partial m_\theta} < \frac{\partial x^*}{\partial m_\theta}$. It means once $m_\theta < m^{med}$, $x_R^* < x^*$ always. Furthermore $\hat{\theta}^{TP} = x_R^*(m_\theta, L^*(m_\theta)) - \sigma\Phi^{-1}(1-\frac{\bar{R}}{L^*(m_\theta)}) < \bar{\theta}^{CP} = x^*(m_\theta, \bar{B}+\bar{R}) - \sigma\Phi^{-1}(1-\frac{\bar{R}}{\bar{B}+\bar{R}})$. Consequently, the leader has a higher starting point in CP i.e. $m_{CP}^{\min} > m_{TP}^{\min}$. Therefore, CP has higher survival probability in this case.

□

Proof of Corollary 2. For a given m_θ, using Equation (1) and (2), when $\omega = 0$, x_R^* and x_N^* can be solved from

$$\frac{\alpha_R - \alpha_N}{\gamma_R - \gamma_N} \cdot x_R^* = \frac{1-P}{P} + \xi. \tag{A.16}$$

$$\frac{\alpha_N - \alpha_B}{\gamma_N - \gamma_B} \cdot x_N^* = \frac{1-P}{P} + \xi. \tag{A.17}$$

We $x_R^* > x_N^*$. By adding ω into the first equation, we have

$$\frac{\alpha_R - \alpha_N}{\gamma_R - \gamma_N} \cdot x_R^* + \frac{\omega}{\gamma_R - \gamma_N} = \frac{1-P}{P} + \xi. \tag{A.18}$$

. Then we can always find ω large enough to to let the x_R^* solved from this equation less than x_N^* from Equation (A.17). It means there is no two-cutoff strategy, only the one-cutoff strategy exists under targeted punishment, then two punishment rules have the same survival probability. □

Proof of Corollary 3. :

Assume $\bar{B}^1$ is the government's optimal choice under collective punishment. Since the parameters α and β satisfy the conditions of Proposition 5, we have

$$R \cdot P_{\text{sur}}^{TP}(\theta, \bar{B}^1) - C(\bar{B}^1, N) \geq R \cdot P_{\text{sur}}^{CP}(\theta, \bar{B}^1) - C(\bar{B}^1, N).$$

By the continuity, there exists a $\bar{B}^2 \leq \bar{B}^1$ such that $C(\bar{B}^2, N) \leq C(\bar{B}^1, N)$ and

$$R \cdot P_{\text{sur}}^{TP}(\theta, \bar{B}^2) - C(\bar{B}^2, N) \geq R \cdot P_{\text{sur}}^{CP}(\theta, \bar{B}^1) - C(\bar{B}^1, N).$$

Therefore, under targeted punishment, the government can achieve no worse than the optimal expected payoff under collective punishment by choosing a weakly smaller $\bar{B}$. □

Proof of Corollary 4. : Similar as the proof of Corollary 3. □

Proof of Lemma 4. In this case, the type x civilian's expected payoffs for remaining neutral under TP is

$$(1 - \Phi(\frac{\bar{\theta}^{TP}(m_\theta, L) - \lambda x - (1-\lambda)m_\theta}{\sigma}))(\alpha_N x + \beta_N + \gamma_N - \bar{\gamma}) + \gamma_N.$$

The equilibrium strategy for the supporters can be calculated using the same way as before. Let $x_{N_3}^*(m_\theta, L)$ and $x_{R_3}^*(m_\theta, L)$ be the two thresholds which are consistent with the cutoff equilibrium for the two-cutoff strategy. For a given m_θ on the equilibrium path, $x_{N_3}^*(m_\theta, L)$, $x_{R_3}^*(m_\theta, L)$ and the optimal $L_3(m_\theta)$ chosen by the leader must satisfy the following equations:

$$(1-\Phi(Ax^*_{R_3}-\frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(1-\frac{\bar{R}}{L})-\frac{1-\lambda}{\sigma}m_\theta))((\alpha_R-\alpha_N)x^*_{R_3}+\beta_R-\beta_N+\gamma_R-\gamma_N-\bar{\gamma})-(\gamma_R-\gamma_N)=0 \tag{A.19}$$

$$(1-\Phi(Ax^*_{N_3}-\frac{\sigma_\varepsilon}{\sigma}\Phi^{-1}(\frac{\bar{B}}{L})-\frac{1-\lambda}{\sigma}m_\theta))((\alpha_N-\alpha_B)x^*_{N_3}+\beta_N-\beta_B+\gamma_N-\gamma_B-\bar{\gamma})-(\gamma_N-\gamma_B)=0 \tag{A.20}$$

$$x^*_{R_3}(m_\theta,L)-\sigma_\varepsilon\Phi^{-1}(1-\frac{\bar{R}}{L})=x^*_{N_3}(m_\theta,L)-\sigma_\varepsilon\Phi^{-1}(\frac{\bar{B}}{L}). \tag{A.21}$$

It is easy to see when $\bar{\gamma}$ is very large, then no solution can be found from this system of equations, so there is no two-cutoff strategy. But the one-cutoff strategy always exists because $\bar{\gamma}$ is not involved into it. □