Supplemental Appendix to “Signaling Policy Intentions in Fundraising Contests”

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Contents

1 Properties of Equilibrium

Lemma 1. *Let (σ, μ) be an equilibrium. For each type $t \in T$, $Eu_t = U_t$ σ_t-almost everywhere.*

Proof. Take either type t, and suppose there exists a set $S \subseteq \mathbb{R}_+$ such that $\sigma_t(S) > 0$ and $U_t \neq Eu_t(s)$ for all $s \in S$. If $U_t < Eu_t(s')$ for some $s' \in S$, then it would be profitable for a candidate of type t to deviate to a strategy that places probability one on s', contradicting the assumption of equilibrium. But if $U_t > Eu_t(s)$ for all $s \in S$, then a candidate of type t could strictly benefit by deviating to a strategy that places probability zero on S, which again violates the assumption of equilibrium. □

Lemma 2. *Let (σ, μ) be an equilibrium. For each type $t \in T$ and each $s \in \operatorname{supp} \sigma_t$, if σ_t places positive probability on s or λ is continuous at s, then $Eu_t(s) = U_t$.*

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Proof. Take either type t. The first part of the claim, concerning mass points of σ_t, is immediate from Lemma 1. To prove the second part, take any $s \in \operatorname{supp} \sigma_t$ such that λ is continuous at s, and suppose $Eu_t(s) < U_t$. It is apparent from Equation 6, the definition of Eu_t, that continuity of λ at s implies continuity of Eu_t at s. Therefore, there exists $\epsilon > 0$ such that $Eu_t(s') < U_t$ for all s' in an ϵ-neighborhood of s, contradicting the indifference condition of equilibrium.[1] □

Lemma 3. *Let (σ, μ) be an equilibrium. For each type $t \in T$ and each $s \in \mathbb{R}_+$ such that $Eu_t(s) = U_t$,*

$$\begin{aligned} \lambda(s) &> \lambda(s') && \textit{for all } s' < s, \\ Eu_m(s) &> Eu_m(s') && \textit{for all } s' < s. \end{aligned} \tag{1}$$

Proof. Take either type t, and take any s such that $Eu_t(s) = U_t$. In equilibrium, then, we have $Eu_t(s) \geq Eu_t(s')$ for all s'. For $s' < s$, this implies $\lambda(s) > \lambda(s')$, proving the first claim. The second claim is then immediate from the fact that $\lambda(s) > \lambda(s')$ only if $Eu_m(s) > Eu_m(s')$. □

Lemma 4. *In any equilibrium (σ, μ), $U_A \geq U_D$.*

Proof. The assumption $c_A < c_D$ implies $Eu_A \geq Eu_D$. Therefore, the optimality condition of equilibrium gives

$$U_A = \max_{s \in \mathbb{R}_+} Eu_A(s) \geq \max_{s \in \mathbb{R}_+} Eu_D(s) = U_D. \qquad \square$$

Lemma 5. *In any equilibrium (σ, μ), $\max \operatorname{supp} \sigma_D \leq \hat{s} \leq \min \operatorname{supp} \sigma_A$, where*

$$\hat{s} = \frac{U_A - U_D}{c_D - c_A}. \tag{2}$$

Proof. Let (σ, μ) be an equilibrium. For all $s > \hat{s}$,

$$(c_D - c_A)s > U_A - U_D \geq \lambda(s) - c_A s - U_D.$$

A rearrangement of terms yields

$$U_D > \lambda(s) - c_D s = Eu_D(s),$$

so a Disadvantaged candidate's mixed strategy may not place positive probability on $(\hat{s}, \infty)$. An analogous argument establishes that $\min \operatorname{supp} \sigma_A \geq \hat{s}$. □

2 Equilibria Surviving D1

Lemma 6. *An equilibrium (σ, μ) survives the D1 refinement if and only if*

$$\begin{aligned} s < \hat{s} &\quad \Rightarrow \quad \mu(s) = 0, \\ s > \hat{s} &\quad \Rightarrow \quad \mu(s) = 1, \end{aligned}$$

for all $s \leq \max_{t \in T}\{(1 - U_t)/c_t\}$.

[1] The same logic applies if λ is right-continuous at s, as long as $\sigma_t((s, s + \epsilon)) > 0$ for all $\epsilon > 0$.

Proof. The claim is true for all $s \neq \hat{s}$ on the equilibrium path by Bayes' rule and Lemma 5, so now consider off-the-path values of s. As in Banks (1990) and Callander and Wilkie (2007), beliefs under D1 depend on the probability of victory that would be necessary to give a candidate an incentive to deviate to an off-the-path spending choice. For each $s \geq 0$ and each type t, let $q_t(s) = U_t + c_t s$ denote the minimal probability of victory that would give a candidate of type t a weak incentive to deviate to spending s. Notice that

$$\hat{s} - s = \frac{U_A - U_D - s(c_D - c_A)}{c_D - c_A} = \frac{q_A(s) - q_D(s)}{c_D - c_A}.$$

Therefore, if $\hat{s} < s \leq (1-U_A)/c_A$, then $q_A(s) < q_D(s)$ and $q_A(s) \leq 1$, so D1 requires that a deviation to s be ascribed to an Advantaged candidate. Similarly, if $s < \hat{s}$ and $s \leq (1-U_D)/c_D$, then $q_D(s) < q_A(s)$ and $q_D(s) \leq 1$, so D1 requires that a deviation to s be ascribed to a Disadvantaged candidate. Finally, if $s > \max_{t \in T}\{(1-U_t)/c_t\}$, then $q_A(s) > 1$ and $q_D(s) > 1$, so D1 places no restriction on beliefs. □

Lemma 7. *Let (σ, μ) be an equilibrium that survives D1.*

(a) Neither type of candidate's strategy contains any mass points besides $\hat{s}$.

(b) If centrists are Advantaged, then neither type's strategy contains any mass points.

(c) If non-centrists are Advantaged and the Advantaged type's strategy places positive probability on $\hat{s}$, then $\hat{s} = 0$ and the Disadvantaged type spends 0 for certain.

Proof. I begin by proving a necessary intermediate claim, namely that no $s \geq (1-U_t)/c_t$ can be a mass point for a candidate of type t. Suppose not, so $\sigma_t(\{s'\}) > 0$ with $s' \geq (1-U_t)/c_t$. We then have, by Lemma 2,

$$U_t = \lambda(s') - c_t s' \leq \lambda(s') - (1-U_t).$$

Rearranging terms gives $\lambda(s') \geq 1$. This is a contradiction, as a candidate who spends s' has a positive probability of tying and thus cannot win the election for certain.

To prove claim (a), suppose some type t places probability $\pi > 0$ on $s' \neq \hat{s}$. Because $s' < (1-U_t)/c_t$ and $s' \neq \hat{s}$, we have from Lemma 6 that the electorate's beliefs are constant in an ϵ-neighborhood of s'. A candidate who spent $s \in (s', s'+\epsilon)$ would thus defeat any candidate who spent s'. Therefore, by spending infinitesimally more than s' and thereby defeating rather than tying those who spend s', a candidate would raise her chance of victory by at least $\pi p_t/2 > 0$. This contradicts the assumption of equilibrium.

To prove claim (b), suppose that centrists are Advantaged and that some type t places positive probability on $\hat{s}$. By Lemma 6, there exists $s' > \hat{s}$ such that $\mu(s) = 1$ for all $s \in (\hat{s}, s')$. This implies that Eu_m is strictly increasing on $[\hat{s}, s')$, as centrists are Advantaged, so any candidate who spent $s \in (\hat{s}, s')$ would defeat a candidate who spent $\hat{s}$. As in the proof of the last claim, a sufficiently small deviation would therefore be profitable, violating the assumption of equilibrium.

To prove claim (c), suppose that non-centrists are Advantaged and that their mixed strategy places positive probability on $\hat{s} > 0$. By Bayes' rule, then, the electorate's beliefs are $\mu(\hat{s}) > 0$. However, under D1, we have $\mu(s) = 0$ for all $s \in [0, \hat{s})$, per Lemma 6. Because non-centrists are Advantaged, this means there exists $s' < \hat{s}$ such that $Eu_m(s) > Eu_m(\hat{s})$ for all $s \in (s', \hat{s})$. We have

$U_A = Eu_A(\hat{s})$, as Advantaged candidates spend $\hat{s}$ with positive probability, so this contradicts Lemma 3. □

Lemma 8. *Let (σ, μ) be an equilibrium that survives D1. For each type $t \in T$, $\operatorname{supp} \sigma_t \setminus \{\hat{s}\}$ is convex.*

Proof. Take either type $t \in T$, and suppose $\operatorname{supp} \sigma_t \setminus \{\hat{s}\}$ is not convex, so there exist $s', s'' \in \operatorname{supp} \sigma_t \setminus \{\hat{s}\}$ such that $s' < s''$ and $\sigma_t((s', s'')) = 0$. Because $s' \neq \hat{s}$ and $s'' \neq \hat{s}$, neither of these is a mass point of σ_t, per Lemma 7(a). Therefore, there exists $\delta > 0$ such that $[s' - \delta, s'] \cup [s'', s'' + \delta] \subseteq \operatorname{supp} \sigma_t \setminus \{\hat{s}\}$. Let $S = [s' - \delta, s'' + \delta]$. By Lemma 6, the electorate's beliefs μ are constant on S, which in turn implies the median voter's expected payoff Eu_m is continuous and strictly increasing on S. Consequently, the set of $s \notin S$ such that $Eu_m(s) \in Eu_m(S)$ has $\tilde{\sigma}$-measure zero, per Lemma 3. Two implications follow from this claim. First, because there is not positive mass on any s such that $Eu_m(s) \in Eu_m(S)$, the probability of victory λ is continuous on S. Second, because the set of s such that $Eu_m(s) \in Eu_m((s', s''))$ has $\tilde{\sigma}$-measure zero, $\lambda(s') = \lambda(s'')$. We therefore have $Eu_t(s') > Eu_t(s'')$. But, by Lemma 2, continuity of λ implies $Eu_t(s') = Eu_t(s'') = U_t$, a contradiction. □

Lemma 9. *In any equilibrium (σ, μ) that survives D1, $Eu_m(\max \operatorname{supp} \sigma_D) = Eu_m(\min \operatorname{supp} \sigma_A)$.*

Proof. Let $\bar{s}_D = \max \operatorname{supp} \sigma_D$, and let $\underline{s}_A = \min \operatorname{supp} \sigma_A$. The claim is trivial if $\bar{s}_D = \underline{s}_A$, so suppose $\bar{s}_D < \underline{s}_A$ and $Eu_m(\bar{s}_D) \neq Eu_m(\underline{s}_A)$. The first step of the proof is to establish that $U_A = Eu_A(\underline{s}_A)$. We know that $\underline{s}_A$ is not a mass point of σ_A, as the only amount on which an Advantaged candidate's strategy may place positive mass is 0, by Lemma 7. Therefore, there exists $s' > \underline{s}_A$ such that $[\underline{s}_A, s'] \subseteq \operatorname{supp} \sigma_A$. By Bayes' rule, the electorate's beliefs are $\mu(s) = 1$ for all $s \in [\underline{s}_A, s']$, so Eu_m is continuous and strictly increasing on this interval. Because $\bar{s}_D$ is the only possible mass point of either type's strategy and $Eu_m(\bar{s}_D) \neq Eu_m(\underline{s}_A)$, this in turn implies λ is right-continuous at $\underline{s}_A$. Then, by Lemma 2 (see note 1), $U_A = Eu_A(\underline{s}_A)$.

We are now prepared to show that $Eu_m(\bar{s}_D) = Eu_m(\underline{s}_A)$. Suppose $Eu_m(\bar{s}_D) > Eu_m(\underline{s}_A)$. By definition, then, $\lambda(\bar{s}_D) \geq \lambda(\underline{s}_A)$. Because $\bar{s}_D < \underline{s}_A$ and $U_A = Eu_A(\underline{s}_A)$, this contradicts the optimality requirement of equilibrium, per Lemma 3. On the other hand, suppose $Eu_m(\bar{s}_D) < Eu_m(\underline{s}_A)$. If $\bar{s}_D$ is not a mass point of a Disadvantaged candidate's mixed strategy, then we have $\lambda(\bar{s}_D) = \lambda(\underline{s}_A)$, again contradicting the optimality requirement of equilibrium. Otherwise, we have $\hat{s} = \bar{s}_D$, so the electorate's beliefs μ are constant on $(\bar{s}_D, \underline{s}_A]$ under D1, per Lemma 6. The median voter's expected utility function Eu_m is thereby continuous and strictly increasing on $(\bar{s}_D, \underline{s}_A]$, so there exists $s'' \in (\bar{s}_D, \underline{s}_A)$ such that $Eu_m(s'') > Eu_m(\bar{s}_D)$. Because the set of s such that $Eu_m(s'') \leq Eu_m(s) \leq Eu_m(\underline{s}_A)$ has $\tilde{\sigma}$-measure zero, we have $\lambda(\underline{s}_A) = \lambda(s'')$, once again contradicting the optimality requirement of equilibrium. Consequently, we must have $Eu_m(\bar{s}_D) = Eu_m(\underline{s}_A)$. □

3 Equilibrium with Advantaged Centrists

Lemma 10. *If centrists are Advantaged, then in any equilibrium (σ, μ) that survives D1, the CDF of the Disadvantaged type's mixed strategy is given by Equation 3, and the CDF of the Advantaged* *type's mixed strategy is given by Equation 4.*

Proof. Assume $\alpha \leq 0$, and let (σ, μ) be an equilibrium that survives D1. Because centrists are Advantaged, we have from Lemma 7(b) that neither type's strategy contains any mass points. Consequently, the support of each type t's mixed strategy is a closed interval, $\operatorname{supp} \sigma_t = [\underline{s}_t, \bar{s}_t]$, per Lemma 8. Moreover, we have $0 \leq \underline{s}_D < \bar{s}_D \leq \hat{s} \leq \underline{s}_A < \bar{s}_A$, per Lemma 5. Let $S = [0, \bar{s}_A]$. Because centrists are Advantaged, the median voter's expected utility Eu_m is strictly increasing on S. Then, as neither type's mixed strategy contains any mass points, the probability of victory function λ is continuous and non-decreasing on S. Therefore, each type's expected utility function Eu_t is continuous on $[0, \bar{s}_A]$, and we have $U_t = Eu_t(s)$ for all $s \in \operatorname{supp} \sigma_t$, per Lemma 2.

I begin by characterizing the Disadvantaged type's mixed strategy. First, as $\lambda(\underline{s}_D) = 0$ and $U_D = Eu_D(\underline{s}_D)$, we must have $\underline{s}_D = 0$, or else it would be profitable for Disadvantaged candidates to deviate to spending 0. It follows immediately that $U_D = 0$. Next, to derive the expression for Disadvantaged candidates' mixed strategy, take any $s \in \operatorname{supp} \sigma_D$. Because Eu_m is strictly increasing on S and neither type's strategy contains any mass points, we have $\lambda(s) = p_D F_D(s)$. Moreover, by continuity of Eu_D at s and the indifference condition of equilibrium, we have

$$Eu_D(s) = p_D F_D(s) - c_D s = 0 = Eu_D(0).$$

Rearranging terms gives $F_D(s) = c_D s / p_D$. Then, from $F_D(\bar{s}_D) = 1$, we yield $\bar{s}_D = p_D / c_D$. Therefore, the Disadvantaged type's mixed strategy must satisfy Equation 3.

The characterization of the Advantaged type's mixed strategy is similar. Recall from Lemma 9 that $Eu_m(\bar{s}_D) = Eu_m(\underline{s}_A)$. As Eu_m is strictly increasing on S, this implies $\underline{s}_A = \bar{s}_D = p_D / c_D$. Now, to derive the expression for Advantaged candidates' mixed strategy, take any $s \in \operatorname{supp} \sigma_A$. We have $\lambda(s) = p_D + p_A F_A(s)$, again because Eu_m is strictly increasing on S and neither type's mixed strategy contains any mass points. By continuity of Eu_A at s and the indifference condition of equilibrium, we have

$$Eu_A(s) = p_D + p_A F_A(s) - c_A s = p_D - c_A \frac{p_D}{c_D} = Eu_A(\underline{s}_A).$$

Rearranging terms gives

$$F_A(s) = \frac{c_A}{p_A} \left[s - \frac{p_D}{c_D} \right].$$

Then, from $F_A(\bar{s}_A) = 1$, we yield $\bar{s}_A = p_D / c_D + p_A / c_A$. Therefore, the Advantaged type's mixed strategy must satisfy Equation 4. $\square$

Proposition 1. *If Advantaged candidates are centrist, then there is an equilibrium (σ^*, μ^*) that is essentially unique under D1 in which Disadvantaged candidates employ a mixed strategy whose CDF is*

$$F_D^*(s) = \begin{cases} 0 & s < 0, \\ c_D s / p_D & 0 \leq s \leq \bar{s}_D^*, \\ 1 & s > \bar{s}_D^*, \end{cases} \tag{3}$$

and Advantaged candidates employ a mixed strategy whose CDF is

$$F_A^*(s) = \begin{cases} 0 & s < \bar{s}_D^*, \\ c_A(s - \bar{s}_D^*)/p_A & \bar{s}_D^* \leq s \leq \bar{s}_A^*, \\ 1 & s > \bar{s}_A^*, \end{cases} \tag{4}$$

where $\bar{s}_D^* = p_D/c_D$ *and* $\bar{s}_A^* = \bar{s}_D^* + p_A/c_A$*. The electorate's beliefs are*

$$\mu^*(s) = \begin{cases} 0 & s < \bar{s}_D^*, \\ p_A & s = \bar{s}_D^*, \\ 1 & s > \bar{s}_D^*. \end{cases} \tag{5}$$

Proof. The first task is to confirm that (σ^*, μ^*) is an equilibrium, which requires confirming that there are no profitable deviations available and that the given beliefs are consistent with the candidates' strategies. Because centrists are Advantaged and μ^* is weakly increasing, the median voter's expected utility Eu_m is strictly increasing. Consequently, as neither type's strategy contains any mass points, the probability of victory by a candidate who spends s is $\lambda(s) = p_A F_A^*(s) + p_D F_D^*(s)$. Each type's expected utility is continuous in s regardless of the median voter's behavior when indifferent, so the choice of sharing rule is immaterial.

I begin by checking for profitable deviations. For every point in the support of a Disadvantaged type's mixed strategy, $s \in [0, \bar{s}_D^*]$, we have

$$Eu_D(s) = p_D F_D^*(s) - c_D s = 0,$$

confirming the indifference condition for Disadvantaged types. It is not profitable for a Disadvantaged candidate to mimic an Advantaged one, because for any $s \in (\bar{s}_D^*, \bar{s}_A^*]$ we have

$$Eu_D(s) = p_D + p_A F_A^*(s) - c_D s = (c_A - c_D)(s - \bar{s}_D^*) < 0.$$

Similarly, for an Advantaged candidate, for all $s \in [\bar{s}_D^*, \bar{s}_A^*]$, we have

$$Eu_A(s) = p_D + p_A F_A^*(s) - c_A s = p_D - c_A \bar{s}_D^*,$$

confirming the indifference condition for Advantaged types. It is not profitable for an Advantaged candidate to deviate to spending less, because for any $s < \bar{s}_D^*$ we have

$$Eu_A(s) = p_D F_D^*(s) - s = (c_D - c_A)s < p_D - c_A \bar{s}_D^*.$$

Finally, it is not profitable for either type to deviate to spending $s > \bar{s}_A^*$, because doing so yields the same chance of victory as spending $\bar{s}_A^*$ but at greater cost.

Next, I confirm that the electorate's beliefs are consistent with the application of Bayes' rule on the path of play and that the off-the-path beliefs survive D1. It is obvious that the on-the-path beliefs are consistent with Bayes' rule. Then, notice that the cutpoint for beliefs under D1 is $\hat{s} = (U_A - U_D)/(c_D - c_A) = p_D/c_D = \bar{s}_D^*$, so the equilibrium survives D1, per Lemma 6. The final claim of the proposition, that the equilibrium is essentially unique under D1, follows from Lemma 10. □

4 Equilibrium with Advantaged Non-Centrists

Lemma 11. *Let (σ, μ) be an equilibrium that survives D1. If non-centrists are Advantaged, then the Disadvantaged type's strategy places positive probability on $\hat{s}$, and there exists $\delta > 0$ such that neither type's strategy places positive probability on $(\hat{s}, \hat{s} + \delta)$.*

Proof. Suppose $\alpha > 0$, so non-centrists are Advantaged. I begin by showing that the Disadvantaged type's strategy places positive probability on $\hat{s}$. For the sake of contradiction, suppose not, so $\sigma_D(\{\hat{s}\}) = 0$. Because $\hat{s}$ is the only possible mass point of either type's strategy, per Lemma 7(a), this means σ_D contains no mass points. Moreover, as non-centrists are Advantaged, the lack of a mass point in σ_D implies there is none in σ_A either, per Lemma 7(c). Because neither type t's mixed strategy contains a mass point, the support of each is an interval $[\underline{s}_t, \bar{s}_t]$, per Lemma 8. From Lemma 5, we have $\bar{s}_D \leq \hat{s} \leq \underline{s}_A$.

To show that the lack of a mass point in σ_D yields a contradiction, there are two cases to consider. First, suppose $\bar{s}_D = \underline{s}_A = \hat{s}$. By Bayes' rule, $\mu(s) = 0$ for all $s \in [\underline{s}_D, \hat{s})$ and $\mu(s) = 1$ for all $s \in (\hat{s}, \bar{s}_A]$. Taking the left- and right-hand limits of the median voter's expected utility at $\hat{s}$ gives

$$\lim_{s \to \hat{s}^+} Eu_m(s) = \hat{s} - \alpha < \hat{s} = \lim_{s \to \hat{s}^-} Eu_m(s).$$

Therefore, it would be profitable for an Advantaged candidate to deviate to spending slightly less than $\hat{s}$, contradicting the assumption of equilibrium. Second, suppose $\bar{s}_D < \underline{s}_A$. In this case, we have $\mu(s) = 1$ for all $s \in [\underline{s}_A, \bar{s}_A]$. In addition, because neither type's mixed strategy contains any mass points, the probability of victory λ is continuous on this interval. Therefore, by Lemma 2, $U_A = Eu_A(\underline{s}_A)$. However, recall that $Eu_m(\underline{s}_A) = Eu_m(\bar{s}_D)$ in any equilibrium that survives D1, per Lemma 9. Because $\bar{s}_D < \underline{s}_A$, it would thus be profitable for the Advantaged type to deviate to spending $\bar{s}_D$, again contradicting the assumption of equilibrium. As both cases yield a contradiction, we can conclude that σ_D places positive mass on $\hat{s}$.

The last step is to prove that there exists $\delta > 0$ such that $(\hat{s}, \hat{s} + \delta)$ lies outside the support of both types' strategies. Because the Disadvantaged type's mixed strategy places positive mass on $\hat{s}$, Bayes' rule implies $\mu(\hat{s}) < 1$. Because the equilibrium survives D1, we have $\mu(s) = 1$ for all $s \in (\hat{s}, \bar{s}_A]$, per Lemma 6. Similar to before, taking the right-hand limit of the median voter's utility at $\hat{s}$ gives

$$\lim_{s \to \hat{s}^+} Eu_m(s) = \hat{s} - \alpha < \hat{s} - \mu(\hat{s})\alpha = Eu_m(\hat{s}).$$

Any candidate would be strictly better off spending $\hat{s}$ than any amount just above it, so there must exist $\delta > 0$ such that $\sigma_t((\hat{s}, \hat{s} + \delta)) = 0$ for each type t. □

Lemma 12. *Suppose $\alpha > p_D / 2c_A$. In any equilibrium that survives D1,*

(a) Disadvantaged types spend 0 for certain.

(b) Advantaged types spend 0 with probability $\pi^* > 0$, where

$$\pi^* = \min\left\{ \frac{\sqrt{2\alpha c_A p_D} - p_D}{p_A}, 1 \right\}. \tag{6}$$

Proof. Throughout the proof, let π denote the probability that an Advantaged candidate spends 0, so $\pi = \sigma_A(\{0\})$.

To prove claim (a), it will suffice to show that $\pi > 0$, as then Lemma 7(c) gives $\sigma_D(\{0\}) = 1$. For a proof by contradiction, suppose there is an equilibrium (σ, μ) that survives D1 in which $\pi = 0$. We know from Lemma 7(c) that the Advantaged type's strategy cannot contain any mass point besides 0. Consequently, the support of the Advantaged type's strategy is an interval, $\operatorname{supp} \sigma_A = [\underline{s}_A, \bar{s}_A]$, per Lemma 8. On the other hand, as non-centrists are Advantaged, we have from Lemma 11 that the Disadvantaged type's strategy places positive mass on $\hat{s} < \underline{s}_A$. We then have from Lemma 9 that $Eu_m(\underline{s}_A) = Eu_m(\hat{s})$. Bayes' rule gives $\mu(\underline{s}_A) = 1$ and $\mu(\bar{s}_D) = 0$, so we have $\underline{s}_A = \hat{s} + \alpha$. Because the Advantaged type's strategy contains no mass points, $\lambda(s) \to p_D$ as $s \to \underline{s}_A$ from the right. In addition, because a candidate who spends $\hat{s}$ either defeats or ties any Disadvantaged opponent, $\lambda(\hat{s}) \geq p_D/2$. Combining these with the fact that $\alpha > p_D/2c_A$ gives

$$\begin{aligned} U_A &= \lim_{s \to \underline{s}_A^+} Eu_A(s) \\ &= p_D - c_A(\hat{s} + \alpha) \\ &< \frac{p_D}{2} - c_A\hat{s} \\ &\leq Eu_A(\hat{s}). \end{aligned}$$

Therefore, it would be profitable for the Advantaged type to deviate to spending $\hat{s}$, contradicting the assumption of equilibrium. We conclude that $\pi > 0$ in any equilibrium that survives D1, which in turn implies that Disadvantaged candidates spend 0 for certain.

Before moving on to the proof of claim (b), it is worth noting two implications of the results just derived. First, the electorate's beliefs about a candidate who spends 0 are

$$\mu(0) = \frac{\pi p_A}{\pi p_A + p_D}. \tag{7}$$

Second, we know from Lemma 3 that on the equilibrium path, a candidate who spends nothing has zero probability of defeating a candidate who spends more. Consequently, the chance of victory for a candidate who spends 0 is

$$\lambda(0) = \frac{1}{2}\left(\pi p_A + p_D\right). \tag{8}$$

The proof of claim (b) consists of two steps. First, assume $\alpha \geq 1/2c_A p_D$, so $\pi^* = 1$. For a proof by contradiction, suppose there is an equilibrium (σ, μ) that survives D1 in which $\pi < 1$. We can then write the support of the Advantaged type's strategy as $\operatorname{supp} \sigma_A = \{0\} \cup [\tilde{s}_A, \bar{s}_A]$, where $0 < \tilde{s}_A < \bar{s}_A$. Because off-the-path beliefs must satisfy D1, Lemma 6 gives $\mu(s) = 1$ for all $s \in (0, \bar{s}_A]$. Then, applying the same line of argument as in the proof of Lemma 9, we must have $Eu_m(\tilde{s}_A) = Eu_m(0)$, which implies $\tilde{s}_A = (1 - \mu(0))\alpha$. A candidate who spends slightly more than $\tilde{s}_A$ thereby defeats those who spend 0 rather than tying. Taking the limit of the Advantaged type's utility as it approaches $\tilde{s}_A$ from the right gives

$$\begin{aligned}\lim_{s\to\tilde{s}_A^+} Eu_A(s) &= \pi p_A + p_D - c_A \frac{p_D}{\pi p_A + p_D}\alpha \\ &\leq \pi p_A + p_D - \frac{1}{2}\frac{1}{\pi p_A + p_D} \\ &< \frac{1}{2}(\pi p_A + p_D) \\ &= Eu_A(0),\end{aligned}$$

where the last inequality holds because $\pi < 1$ implies $\pi p_A + p_D < 1 < 1/(\pi p_A + p_D)$. Therefore, an Advantaged candidate is better off spending 0 than any amount just above $\tilde{s}_A$, which contradicts the assumption of equilibrium. We conclude that $\pi = 1$ in any equilibrium surviving D1 if $\alpha \geq 1/2c_A p_D$.

Second, assume $p_D/2c_A < \alpha < 1/2c_A p_D$, so $\pi^* < 1$. For a proof by contradiction, suppose there is an equilibrium (σ, μ) that survives D1 in which $\pi = 1$. Because both candidates spend 0 for certain, we have $U_A = U_D = \lambda(0) = 1/2$ and $\mu(0) = p_A$. A candidate could assure victory by spending $s > p_D\alpha$, as

$$Eu_m(s) > p_D\alpha - \alpha = -p_A\alpha = Eu_m(0).$$

Therefore, for any $s \in (p_D\alpha, 1/2c_A)$ (where $p_D\alpha < 1/2c_A$ because $\alpha < 1/2c_A p_D$), we have

$$Eu_A(s) = 1 - c_A s > \frac{1}{2} = U_A,$$

contradicting the assumption of equilibrium. We conclude that $\pi < 1$ in any equilibrium surviving D1 if $p_D/2c_A < \alpha < 1/2c_A p_D$. Moreover, as in the previous part of the proof, we have $\operatorname{supp}\sigma_A = \{0\} \cup [\tilde{s}_A, \bar{s}_A]$, where $\tilde{s}_A = (1 - \mu(0))\alpha > 0$. Taking limits as the Advantaged type's utility approaches $\tilde{s}_A$ from the right, the indifference condition of equilibrium gives

$$\lim_{s\to\tilde{s}_A^+} Eu_A(s) = \pi p_A + p_D - c_A \frac{p_D}{\pi p_A + p_D}\alpha = \frac{1}{2}(\pi p_A + p_D) = Eu_A(0).$$

Multiplying both sides by $\pi p_A + p_D$ and rearranging terms gives $\pi = (\sqrt{2\alpha c_A p_D} - p_D)/p_A$, as claimed. □

Proposition 2. *If $\alpha \geq 1/2c_A p_D$, then there is an equilibrium (σ^*, μ^*) that is essentially unique under D1 in which both types of candidates spend 0 for certain. The electorate's beliefs are $\mu^*(0) = p_A$ and $\mu^*(s) = 1$ for all $s > 0$.*

Proof. As in the proof of Proposition 1, we must confirm that there are no profitable deviations available and that the given beliefs are consistent with the candidates' strategies. In the proposed equilibrium, the election always ends in a tie with both candidates spending nothing, so each type's utility is $U_A = U_D = \lambda(0) = 1/2$. A candidate who spent $s \in (0, p_D\alpha)$ would lose to a candidate who spent 0, so neither type has an incentive to deviate to spending such an amount. Given his beliefs, the median voter is indifferent between a candidate who spends 0 and one who spends $p_D\alpha$. Regardless of the sharing rule employed in case of indifference, a candidate who spent $s \geq p_D\alpha$ would receive a payoff of

$$Eu_t(s) \leq 1 - c_A s \leq 1 - c_A p_D \alpha \leq \frac{1}{2},$$

so such a deviation would also be unprofitable. It is obvious that the on-the-path beliefs (namely, $\mu(0) = p_A$) are consistent with Bayes' rule, so the proposed assessment is an equilibrium. In addition, the cutpoint for beliefs under D1 is $\hat{s} = (U_A - U_D)/(c_D - c_A) = 0$, so the equilibrium survives D1, per Lemma 6. Its essential uniqueness under D1 follows from Lemma 12. □

Lemma 13. *If $p_D/2c_A < \alpha < 1/2c_A p_D$, then in any equilibrium that survives D1, the Disadvantaged type spends 0 for certain, and the CDF of the Advantaged type's mixed strategy is given by Equation 9.*

Proof. Assume $p_D/2c_A < \alpha < 1/2c_A p_D$, and let (σ, μ) be an equilibrium that survives D1. We already have from Lemma 12 that Disadvantaged candidates spend 0 for certain and that Advantaged candidates spend 0 with probability $\pi^* = (\sqrt{2\alpha c_A p_D} - p_D)/p_A > 0$. In addition, using the same logic as in the final part of the proof of Lemma 12, we can derive that the support of the Advantaged type's strategy is $\{0\} \cup [\tilde{s}_A, \bar{s}_A]$, where

$$\tilde{s}_A = \frac{p_D}{\pi^* p_A + p_D}\alpha = \frac{1}{2c_A}(\pi^* p_A + p_D).$$

All that remains is to derive $\bar{s}_A$ and the probability distribution of the Advantaged type's strategy on $[\tilde{s}_A, \bar{s}_A]$. Under D1, we have from Lemma 6 that $\mu(s) = 1$ for all $s \in (0, \bar{s}_A]$, so the median voter's expected utility is strictly increasing on this interval. Because $Eu_m(\tilde{s}_A) = Eu_m(0)$, this implies $\lambda(s) = p_D + p_A F_A(s)$ for all $s \in (\tilde{s}_A, \bar{s}_A]$. Moreover, because neither candidate's strategy contains any mass points besides 0, the probability of victory λ is continuous on $(\tilde{s}_A, \bar{s}_A]$. By the indifference condition of equilibrium,

$$Eu_A(s) = p_D + p_A F_A(s) - c_A s = \frac{1}{2}(\pi^* p_A + p_D) = Eu_A(0)$$

for all $s \in (\tilde{s}_A, \bar{s}_A]$. A rearrangement of terms gives

$$\begin{aligned} F_A(s) &= \pi^* + \frac{1}{p_A}\left(c_A s - \frac{1}{2}(\pi^* p_A + p_D)\right) \\ &= \pi^* + \frac{c_A}{p_A}(s - \tilde{s}_A), \end{aligned}$$

as claimed. Finally, setting $F_A(\bar{s}_A) = 1$ gives $\bar{s}_A = \tilde{s}_A + p_A(1 - \pi^*)/c_A$, as claimed. □

Proposition 3. *If $p_D/2c_A < \alpha < 1/2c_A p_D$, then there is an equilibrium (σ^*, μ^*) that is essentially unique under D1 in which Disadvantaged candidates spend 0 for certain and Advantaged candidates employ a mixed strategy whose CDF is*

$$F_A^*(s) = \begin{cases} 0 & s < 0, \\ \pi^* & 0 \leq s \leq \tilde{s}_A^*, \\ \pi^* + c_A(s - \tilde{s}_A^*)/p_A & \tilde{s}_A^* < s < \bar{s}_A^*, \\ 1 & s \geq \bar{s}_A^*, \end{cases} \tag{9}$$

where $\pi^* = (\sqrt{2\alpha c_A p_D} - p_D)/p_A$, $\tilde{s}_A^* = (\pi^* p_A + p_D)/2c_A$, *and* $\bar{s}_A^* = \tilde{s}_A^* + p_A(1-\pi^*)/c_A$. *The electorate's beliefs are* $\mu^*(0) = \pi^* p_A/(\pi^* p_A + p_D)$ *and* $\mu^*(s) = 1$ *for all* $s > 0$.

Proof. As in the proofs of the previous propositions, we must confirm that there are no profitable deviations available and that the given beliefs are consistent with the candidates' strategies. In the proposed equilibrium, a candidate who spends 0 ties with probability $\pi^* p_A + p_D$ and loses otherwise, so each type's utility is $U_A = U_D = \lambda(0) = (\pi^* p_A + p_D)/2$. A candidate who deviated to an off-the-path amount $s \in (0, \tilde{s}_A^*)$ would not defeat one who spent 0, as

$$Eu_m(s) = s - \alpha \leq \tilde{s}_A^* - \alpha = -\frac{\pi^* p_A}{\pi^* p_A + p_D}\alpha = Eu_m(0),$$

so such a deviation cannot be profitable. The median voter is indifferent between a candidate who spends 0 and one who spends $\tilde{s}_A^*$. The only sharing rule that makes the candidates' expected utility functions upper semicontinuous in s is for the median voter to elect a candidate who spends $\tilde{s}_A^*$ over one who spends 0. Then, the probability of victory for a candidate who spends $s \in [\tilde{s}_A^*, \bar{s}_A^*]$ is $\lambda(s) = p_A F_A^*(s) + p_D$. The payoff to an Advantaged type for spending such an amount is

$$\begin{aligned} Eu_A(s) &= p_A F_A^*(s) + p_D - s \\ &= \pi^* p_A + p_D - c_A \tilde{s}_A^* \\ &= \frac{\pi^* p_A + p_D}{2} \\ &= U_A, \end{aligned}$$

confirming the indifference condition for the Advantaged types. This also proves that Disadvantaged types have no incentive to deviate to an amount in this range, as $Eu_A \geq Eu_D$. Finally, neither type has an incentive to deviate to spending $s > \bar{s}_A^*$, as doing so yields the same chance of victory as spending $\bar{s}_A^*$ at strictly greater cost. It is obvious that the beliefs are consistent with Bayes' rule for spending amounts on the path, $s \in \{0\} \cup [\tilde{s}_A^*, \bar{s}_A^*]$, so the proposed assessment is an equilibrium. In addition, the cutpoint for beliefs under D1 is $\hat{s} = (U_A - U_D)/(c_D - c_A) = 0$, so the equilibrium survives D1, per Lemma 6. Its essential uniqueness under D1 follows from Lemmas 12 and 13. □

Lemma 14. *If* $0 < \alpha < p_D/2c_A$, *then in any equilibrium that survives D1, the CDF of the Disadvantaged type's mixed strategy is given by Equation 10, and the CDF of the Advantaged type's mixed strategy is given by Equation 11.*

Proof. Assume $0 < \alpha < p_D/2c_A$, and let (σ, μ) be an equilibrium that survives D1. We know from Lemma 11 that the Disadvantaged type's strategy places probability $\rho > 0$ on $\hat{s}$. If the Advantaged type's strategy contained a mass point, that would entail placing probability $\pi^* > 0$ on 0, as shown in the proof of Lemma 12. But $\alpha < p_D/2c_A$ implies $\pi^* < 0$, so σ_A must not have any mass points. We therefore have $\operatorname{supp} \sigma_A = [\underline{s}_A, \bar{s}_A]$, where $\bar{s}_A > \underline{s}_A > \hat{s}$. Bayes' rule then gives $\mu(\underline{s}_A) = 1$ and $\mu(\hat{s}) = 0$. The median voter must be indifferent between candidates spending $\underline{s}_A$ and $\hat{s}$, per Lemma 9, so we have $\underline{s}_A = \hat{s} + \alpha$.

I begin by ruling out the possibility that the Disadvantaged type employs a pure strategy. For a proof by contradiction, suppose $\rho = 1$. Then the chance of victory by a candidate who spends $\hat{s}$ is $p_D/2$, and the Disadvantaged type's equilibrium payoff is $U_D = p_D/2 - c_D\hat{s}$. Because $Eu_m(\underline{s}_A) = Eu_m(\hat{s})$, any candidate spending more than $\underline{s}_A$ defeats all Disadvantaged candidates. The Advantaged type's equilibrium utility is thus

$$U_A = \lim_{s \to \underline{s}_A^+} Eu_A(s) = p_D - c_A(\hat{s} + \alpha).$$

Substituting each type's equilibrium utility into Equation 2, the definition of $\hat{s}$, gives

$$\begin{aligned} \hat{s} &= \frac{U_A - U_D}{c_D - c_A} \\ &= \frac{p_D - c_A\hat{s} - c_A\alpha - p_D/2 + c_D\hat{s}}{c_S - c_A} \\ &> \frac{p_D - c_A\hat{s} - p_D + c_D\hat{s}}{c_S - c_A} \\ &= \hat{s}, \end{aligned}$$

where the inequality follows from $\alpha < p_D/2c_A$. This is a contradiction, so we conclude that $\rho < 1$.

Next, I characterize the Disadvantaged type's mixed strategy. Because the strategy places probability $\rho \in (0, 1)$ on $\hat{s}$, we may write its support as $\operatorname{supp} \sigma_D = [\underline{s}_D, \tilde{s}_D] \cup \{\hat{s}\}$, by Lemma 8. Under D1, we have $\mu(s) = 0$ for all $s \in [0, \hat{s}]$, by Lemma 6. Then, as the Advantaged type's strategy does not contain any mass points, the probability of victory λ is continuous on $[0, \hat{s})$. This implies $\underline{s}_D = 0$, as otherwise we have

$$U_D = Eu_D(\underline{s}_D) = -c_D\underline{s}_D < 0 = Eu_D(0),$$

contradicting the assumption of equilibrium. As a result, $U_D = Eu_D(0) = 0$. The Advantaged type's equilibrium utility is

$$U_A = \lim_{s \to \underline{s}_A^+} Eu_A(s) = p_D - c_A(\hat{s} + \alpha),$$

so Equation 2, the definition of $\hat{s}$, gives

$$\hat{s} = \frac{U_A - U_D}{c_D - c_A} = \frac{p_D - c_A(\hat{s} + \alpha)}{c_D - c_A}.$$

Rearranging terms yields $\hat{s} = (p_D - c_A\alpha)/c_D$. Substituting this into the Disadvantaged type's expected utility from spending $\hat{s}$ gives

$$Eu_D(\hat{s}) = \left(1 - \frac{\rho}{2}\right)p_D - c_D\hat{s} = c_A\alpha - \frac{\rho p_D}{2}.$$

By the indifference condition of equilibrium, $Eu_D(\hat{s}) = Eu_D(0) = 0$, so the above implies $\rho = 2c_A\alpha/p_D$. (The conditions on α imply $0 < \rho < 1$, as required.) Because candidates spending $\hat{s}$ tie with positive probability, there is a discrete upward jump in the Disadvantaged type's expected utility at $\hat{s}$. Therefore, by the assumption of equilibrium, $\hat{s} > \tilde{s}_D$; otherwise, it would be profitable to deviate from spending just less than $\tilde{s}_D$. Consequently, for $s \in [0, \tilde{s}_D]$, the indifference condition of equilibrium gives

$$Eu_D(s) = p_D F_D(s) - c_D s = 0 = Eu_D(0),$$

and thereby $F_D(s) = c_D s / p_D$. Lastly, setting $F_D(\tilde{s}_D) = 1 - \rho$ gives $\tilde{s}_D = (p_D - 2c_A\alpha)/c_D$. We therefore yield Equation 10 as the expression for F_D.

To conclude the proof, we must derive the CDF of the Advantaged type's mixed strategy. We already have that

$$\underline{s}_A = \hat{s} + \alpha = \frac{p_D + (c_D - c_A)\alpha}{c_D}$$

and that

$$U_A = \lim_{s \to \underline{s}_A^+} Eu_A(s) = p_D - c_A \underline{s}_A.$$

By continuity of λ on $(\underline{s}_A, \bar{s}_A]$ and the indifference condition of equilibrium, for $s \in (\underline{s}_A, \bar{s}_A]$ we have

$$Eu_A(s) = p_D + p_A F_A(s) - c_A s = p_D - c_A \underline{s}_A = U_A,$$

and therefore

$$F_A(s) = \frac{c_A(s - \underline{s}_A)}{p_A}.$$

Setting $F_A(\bar{s}_A) = 1$ gives

$$\bar{s}_A = \underline{s}_A + \frac{p_A}{c_A}.$$

We therefore yield Equation 11 as the expression for F_A. □

Proposition 4. *If $0 < \alpha \le p_D/2c_A$, then there is an equilibrium (σ^*, μ^*) that survives D1 in which Disadvantaged candidates employ a mixed strategy whose CDF is*

$$F_D^*(s) = \begin{cases} 0 & s < 0, \\ c_D s/p_D & 0 \le s \le \tilde{s}_D^*, \\ c_D \tilde{s}_D^*/p_D & \tilde{s}_D^* < s < \bar{s}_D^*, \\ 1 & s \ge \bar{s}_D^*, \end{cases} \tag{10}$$

and Advantaged candidates employ a mixed strategy whose CDF is

$$F_A^*(s) = \begin{cases} 0 & s < \underline{s}_A^*, \\ c_A(s - \underline{s}_A^*)/p_A & \underline{s}_A^* \le s \le \bar{s}_A^*, \\ 1 & s > \bar{s}_A^*, \end{cases} \tag{11}$$

where $\tilde{s}_D^* = (p_D - 2c_A\alpha)/c_D$, $\bar{s}_D^* = (p_D - c_A\alpha)/c_D$, $\underline{s}_A^* = \bar{s}_D^* + \alpha$, and $\bar{s}_A^* = \underline{s}_A^* + p_A/c_A$. The electorate's beliefs are $\mu^*(s) = 0$ for all $s \le \bar{s}_D^*$ and $\mu^*(s) = 1$ for all $s > \bar{s}_D^*$. If $0 < \alpha < p_D/2c_A$, this equilibrium is essentially unique under D1.

Proof. As in the proofs of the previous propositions, we must confirm that there are no profitable deviations available and that the given beliefs are consistent with the candidates' strategies. To begin, I will characterize the probability of victory for each potential level of spending. The median voter is indifferent between a candidate who spends $\bar{s}_D^*$ and one who spends $\underline{s}_A^*$, as

$$Eu_m(\bar{s}_D^*) = \bar{s}_D^* = \underline{s}_A^* - \alpha = Eu_m(\underline{s}_A^*).$$

The only sharing rule that makes the candidates' expected utility functions upper semicontinuous in s is for the median voter to elect a candidate who spends $\underline{s}_A^*$ over one who spends $\bar{s}_D^*$. We therefore have $\lambda(s) = p_D F_D^*(s)$ for all $s < \bar{s}_D^*$ and $\lambda(s) = p_D + p_A F_A^*(s)$ for all $s \geq \underline{s}_A^*$. For the off-the-path values $s \in (\bar{s}_D^*, \underline{s}_A^*)$, we have $\lambda(s) = \lambda(\max\{0, s-\alpha\})$. Finally, because Disadvantaged candidates spend $\bar{s}_D^*$ with probability

$$\rho = 1 - F_D^*(\tilde{s}_D^*) = \frac{2c_A\alpha}{p_D},$$

we have $\lambda(\bar{s}_D) = p_D(1-\rho/2) = p_D - c_A\alpha$.

To rule out a profitable deviation for Disadvantaged types, notice that their expected utility from spending $s \in [0, \tilde{s}_D^*]$ is

$$Eu_D(s) = p_D F_D^*(s) - c_D s = 0,$$

so their equilibrium payoff is $U_D = 0$. At the mass point $\bar{s}_D^*$, we have

$$Eu_D(\bar{s}_D^*) = (p_D - c_A\alpha) - c_D\bar{s}_D^* = 0,$$

confirming the Disadvantaged type's indifference condition. To mimic an Advantaged candidate by spending $s \in [\underline{s}_A^*, \bar{s}_A^*]$ would yield a payoff of

$$\begin{aligned} Eu_D(s) &= p_D + p_A F_A^*(s) - c_D s \\ &= p_D - c_A\underline{s}_A^* + (c_A - c_D)s \\ &< p_D - c_D\underline{s}_A^* \\ &= (c_A - c_D)\alpha \\ &< 0, \end{aligned}$$

and would thus be unprofitable. Finally, it is obviously unprofitable to deviate to any value $s \in (\tilde{s}_D^*, \bar{s}_D^*) \cup (\bar{s}_D^*, \underline{s}_A^*) \cup (\bar{s}_A^*, \infty)$, as for any such value it is possible to attain the same chance of victory at strictly less cost.

To rule out a profitable deviation for Advantaged types, notice that their expected utility from spending $s \in [\underline{s}_A^*, \bar{s}_A^*]$ is

$$Eu_A(s) = p_D + p_A F_A^*(s) - c_A s = p_D - c_A\underline{s}_A^*,$$

so their equilibrium payoff is

$$U_A = p_D - c_A\underline{s}_A^* = (c_D - c_A)\bar{s}_D^*.$$

To mimic a Disadvantaged candidate by spending $s \in [0, \tilde{s}_D^*]$ would yield a payoff of

$$\begin{aligned} Eu_A(s) &= p_D F_D^*(s) - c_A s \\ &= (c_D - c_A)s \\ &< (c_D - c_A)\bar{s}_D^* \\ &= U_A, \end{aligned}$$

so such a deviation would be unprofitable. Similarly, deviating to the mass point $\bar{s}_D^*$ would yield a payoff of

$$Eu_A(\bar{s}_D^*) = p_D - \alpha - c_A \bar{s}_D^* = (c_D - c_A)\bar{s}_D^* = U_A,$$

so it is also unprofitable. Finally, just as with Disadvantaged candidates, there cannot be an incentive for an Advantaged candidate to deviate to any $s \in (\tilde{s}_D^*, \bar{s}_D^*) \cup (\bar{s}_D^*, \underline{s}_A^*) \cup (\bar{s}_A^*, \infty)$.

Because the assessment is fully separating, it is obvious that the beliefs on the path are consistent with Bayes' rule, so the assessment is an equilibrium. To confirm that it survives D1, per Lemma 6, notice that the cutpoint for off-the-path beliefs is

$$\hat{s} = \frac{U_A - U_D}{c_D - c_A} = \frac{(c_D - c_A)\bar{s}_D^* - 0}{c_D - c_A} = \bar{s}_D^*.$$

Lastly, essential uniqueness when $\alpha < p_D/2c_A$ follows from Lemma 14. □

5 Equalizing Reform

Here I outline the argument that a marginal increase in c_A has a weakly negative effect on the median voter's *ex ante* expected utility.

Centrists Advantaged. In the parameter region covered by Proposition 1, a marginal increase in c_A does not affect the distribution over the winning candidate's type, nor does it affect the distribution of Disadvantaged candidates' spending. Because Advantaged candidates mix uniformly over $[p_D/c_D + p_A/c_A]$, a marginal increase in c_A reduces spending by Advantaged candidates, thereby reducing the median voter's *ex ante* expected utility. Therefore, the overall effect of the increase is a reduction in the median voter's *ex ante* expected utility.

Non-Centrists Advantaged, Full Concealment. In the parameter region covered by Proposition 2, a marginal increase in c_A does not affect the distribution over the winning candidates' type or either types' spending strategy. Therefore, there is no effect on the median voter's *ex ante* expected utility.

Non-Centrists Advantaged, Partial Concealment. In the parameter region covered by Proposition 3, a marginal increase in c_A affects the distribution over the winning candidates' types and the Advantaged type's spending strategy. Since these effects may offset each other in the median voter's utility, I now explicitly demonstrate that the total effect is negative.

In case both candidates are Disadvantaged, the election ends with no spending and the median voter's expected utility is 0. If one candidate is Advantaged and the other is Disadvantaged, the median voter's expected utility is

$$\pi^*(c_A)\left[\frac{1}{2}(0)+\frac{1}{2}(-\alpha)\right]+(1-\pi^*(c_A))\left[\frac{\tilde{s}_A^*+\bar{s}_A^*}{2}-\alpha\right]=\frac{1-\pi^*(c_A)}{2c_A}-\alpha\left[1-\frac{\pi^*(c_A)}{2}\right],$$

where π^* is written as a function of c_A because below we differentiate with respect to c_A. Finally, using the fact that the expected value of the maximum of two i.i.d. random variables distributed $U[a,b]$ is $a+2(b-a)/3$ (Casella and Berger 1990, 235), the median voter's expected utility in case both candidates are Advantaged is

$$(1-\pi^*(c_A)^2)\left[\tilde{s}_A^*+\frac{2}{3}\left(\bar{s}_A^*-\tilde{s}_A^*\right)\right]-\alpha=(1-\pi^*(c_A)^2)\frac{4-p_D-\pi^*(c_A)p_A}{6c_A}-\alpha.$$

Altogether, the median voter's *ex ante* expected utility as a function of c_A is

$$\begin{aligned}U_m(c_A)=&\,p_D^2[0]+2p_Ap_D\left[\frac{1-\pi^*(c_A)}{2c_A}-\alpha\left(1-\frac{\pi^*(c_A)}{2}\right)\right]\\&+p_A^2\left[(1-\pi^*(c_A)^2)\frac{4-p_D-\pi^*(c_A)p_A}{6c_A}-\alpha\right]\\=&\,p_A\Bigg[-\alpha(2p_D+p_A)+\frac{4+2p_D-p_Ap_D}{6c_A}\\&+\pi^*(c_A)\left(\alpha p_D-\frac{6p_D+p_A^2}{6c_A}\right)-\pi^*(c_A)^2p_A\left(\frac{4-p_D-\pi^*(c_A)p_A}{6c_A}\right)\Bigg].\end{aligned}$$

Differentiating, factoring, and substituting $p_A=1-p_D$ yields

$$\begin{aligned}\frac{dU_m(c_A)}{dc_A}=&\,\frac{1}{12c_A^2}\big[(2p_A^2p_D-8p_A-4p_Ap_D)\\&+[\upsilon(3\upsilon^2+6p_D+p_A^2)-12p_D^2-2p_Dp_A^2]+[\upsilon(\upsilon^2-8p_D-p_D^2)+8p_D^2]\big]\\\propto&-8+4p_D+\upsilon(4\upsilon^2+1-4p_D),\end{aligned}$$

where $\upsilon=\sqrt{2\alpha c_Ap_D}$. Under the conditions of Proposition 3, $p_D<\upsilon<1$, so we have

$$\begin{aligned}\frac{dU_m(c_A)}{dc_A}&\propto-8+4p_D+\upsilon(4\upsilon^2+1-4p_D)\\&<-8+4p_D+\upsilon(5-4p_D)\\&<-3.\end{aligned}$$

Therefore, a marginal increase in c_A strictly decreases the median voter's expected utility.

Non-Centrists Advantaged, Full Separation. In the parameter region covered by Proposition 4, a marginal increase in c_A does not affect the distribution over the winning candidate's type. It unambiguously decreases spending by Advantaged candidates, leading to a reduction in the median voter's expected utility in case either candidate is Advantaged. For the Disadvantaged types, a marginal increase in c_A shrinks the continuum over which they mix and reduces the location of the mass point, but increases the mass placed on the mass point. We therefore must explicitly confirm that the total effect is negative when both candidates are Disadvantaged. The median voter's expected utility in this case is

$$(1-\rho)^2\left[\frac{2}{3}\tilde{s}_D^*\right]+(\rho^2+2\rho(1-\rho))\bar{s}_D^*=\frac{2}{3c_D p_D^2}\left[p_D^3-2\alpha^3 c_A^3\right],$$

which is strictly decreasing in c_A.

6 Public Financing

Let $\mu(\emptyset)$ denote the electorate's updated belief about a candidate who chooses public finance, and let $Eu_m(\emptyset)=\ell-\mu(\emptyset)\alpha$ denote his utility from electing such a candidate.

Proposition 5. *If $\ell \geq 1/2c_A - p_D\alpha$, there is an equilibrium of the game with public finance in which all candidates select public finance.*

Proof. Suppose $\ell \geq 1/2c_A - p_D\alpha$, and consider the following assessment.

- Along the path of play, both candidates (regardless of type) choose public finance. The median voter infers that each candidate is Advantaged with probability p_A and randomizes uniformly between them.
- If either candidate deviates by foregoing public finance, the median voter and the other candidate infer she is Advantaged with probability one. Because $Eu_m(\emptyset)=\ell-p_A\alpha$, a deviant must spend $s'=\max\{0,\ell+p_D\alpha\}$ to make the median voter indifferent. The only sharing rule that averts an open-set problem in this subgame is for the median voter to elect a deviant who spends s' with probability one over a publicly financed opponent.
- In the subgame where one candidate deviates, she employs a pure strategy drawn from $\text{argmax}_{s\in\{0,s'\}}\left\{\mathbf{1}\{s=s'\}-c_{t_i}s\right\}$, which is sequentially rational by construction.
- In the subgame where both deviate, the median voter believes both are Advantaged for sure and consequently elects whichever spends most. Advantaged candidates mix uniformly over $[0,1/c_A]$, which is sequentially rational given the median voter's strategy and the fact that each candidate believes the other is Advantaged (see Meirowitz 2008, Proposition 2). Consequently, the best response for a Disadvantaged candidate in this subgame is to spend nothing.

Under this assessment, $U_A=U_D=1/2$. Consider a unilateral deviation by an Advantaged candidate. If the Advantaged type's strategy in the consequent subgame is to spend 0 and $0<s'$, then she loses the election and receives a payoff of 0, so the deviation is not profitable. If her

strategy is to spend s', thereby winning the election, her utility from the deviation is

$$\begin{aligned}1 - c_A s' &\leq 1 - c_A(\ell + p_D \alpha) \\ &\leq 1 - c_A\left(\frac{1}{2c_A} - p_D\alpha + p_D\alpha\right) \\ &= \frac{1}{2} \\ &= U_A,\end{aligned}$$

so the deviation is not profitable. Nor would such a deviation be profitable for a Disadvantaged type, whose marginal cost of fundraising is even greater. Therefore, because beliefs along the path of play are consistent with the application of Bayes' rule, this assessment is an equilibrium. □

Proposition 6. *If $\alpha \leq 0$ and $\ell \leq 1/c_D - p_A\alpha$, there is an equilibrium of the game with public finance that is outcome-equivalent to the equilibrium in Proposition 1, with no candidate selecting public finance.*

Proof. Suppose $\alpha \leq 0$ and $\ell \leq 1/c_D - p_A\alpha$, and consider the following assessment.

- Along the path of play, both candidates (regardless of type) forego public finance. After a candidate selects to forego public finance, the electorate and the other candidate infer that she is Advantaged with probability p_A. The candidates then employ the same spending strategies, and the electorate updates its beliefs according to the same system, as in Proposition 1. This constitutes an equilibrium of the subgame, per Proposition 1.
- If either candidate deviates by selecting public finance, the electorate infers she is Disadvantaged with probability one. The median voter's utility from such a deviant is $Eu_m(\emptyset) = \ell$. The sharing rule must be the same as in Proposition 5, with the median voter selecting the non-publicly funded candidate when indifferent.
- In the subgame where one candidate deviates to public financing, her opponent spends $s' = \max\{0, \ell + p_A\alpha\}$ regardless of type. The median voter infers that the non-deviant is Advantaged with probability p_A regardless of her spending choice. The median voter is thus indifferent; consequently, under the sharing rule above, the non-deviant wins the election. The non-deviant's payoff in this subgame is

$$\begin{aligned}1 - c_{t_i} s' &\geq 1 - c_D(\ell + p_A\alpha) \\ &\geq 1 - c_D\left(\frac{1}{c_D} - p_A\alpha + p_A\alpha\right) \\ &\geq 0,\end{aligned}$$

  so her choice of s' is sequentially rational.
- In the subgame where both candidates deviate to public financing, the median voter randomizes uniformly between them.

Because the strategies in each subgame are sequentially rational and the electorate's beliefs are consistent with the application of Bayes' rule whenever possible, all that remains is to confirm that neither candidate has an incentive to deviate to taking public financing. A candidate who does so loses the election for sure, receiving a payoff of zero. But we have $U_A \geq U_D = 0$ along the path of play, so such a deviation is not profitable for either type. □

7 Correlated Types

For any pair of spending choices (s_1, s_2), denote the median voter's beliefs

$$\mu_{AA}(s_1, s_2) = \Pr(t_1 = A, t_2 = A \mid s_1, s_2),$$

and so on for the other possible type pairings.

Proposition 7. *Let $\alpha \leq 0$.*

(a) If $q \geq c_A/(c_A + c_D)$, there exists an equilibrium of the game with correlated types in which an Advantaged candidate defeats a Disadvantaged opponent with probability one.

(b) If $q < c_A/(c_A + c_D)$, there exists an equilibrium of the game with correlated types in which an Advantaged candidate defeats a Disadvantaged opponent with probability

$$\frac{1}{2}\left(1 + \frac{c_D - c_A}{(1-q)c_D - q c_A}\right), \tag{12}$$

which is decreasing in c_A.

Proof of part (a). Suppose $\alpha \leq 0$ and $q \geq c_A/(c_A + c_D)$. I claim that the following assessment constitutes an equilibrium. The mixed strategy profile is given by the CDFs

$$F_D(s) = \begin{cases} 0 & s < 0, \\ c_D s/q & 0 \leq s \leq \bar{s}_D, \\ 1 & s > \bar{s}_D, \end{cases}$$

$$F_A(s) = \begin{cases} 0 & s < \bar{s}_D, \\ c_A(s - \bar{s}_D)/q & \bar{s}_D \leq s \leq \bar{s}_A, \\ 1 & s > \bar{s}_A, \end{cases}$$

where $\bar{s}_D = q/c_D$ and $\bar{s}_A = \bar{s}_D + q/c_A$. The electorate's updated beliefs are

$$\mu_{DD}(s_1, s_2) = \begin{cases} 1 & s_1 \leq \bar{s}_D, s_2 \leq \bar{s}_D, \\ 0 & \text{otherwise}, \end{cases}$$

$$\mu_{DA}(s_1, s_2) = \begin{cases} 1 & s_1 \leq \bar{s}_D, s_2 > \bar{s}_D, \\ 0 & \text{otherwise}, \end{cases}$$

$$\mu_{AD}(s_1, s_2) = \begin{cases} 1 & s_1 > \bar{s}_D, s_2 \leq \bar{s}_D, \\ 0 & \text{otherwise}, \end{cases}$$

$$\mu_{AA}(s_1, s_2) = 1 - \mu_{DD}(s_1, s_2) - \mu_{DA}(s_1, s_2) - \mu_{AD}(s_1, s_2).$$

Given these beliefs and the fact that $\alpha \leq 0$, the median voter is never indifferent between two candidates who spend different amounts; he always strictly prefers whichever spends more. Because neither type's mixed strategy contains a mass point, spending the same amount as one's opponent is a zero-probability event, so the choice of sharing rule for these cases is immaterial. It follows that in this assessment, an Advantaged candidate defeats a Disadvantaged opponent with probability one.

I begin by proving there are no profitable deviations for Disadvantaged candidates. For any $s \in [0, \bar{s}_D]$, we have

$$Eu_D(s) = qF_D(s) - c_D s = 0,$$

which confirms D's indifference condition and implies $U_D = 0$. For any $s \in (\bar{s}_D, \bar{s}_A]$, we have

$$Eu_D(s) = q + (1-q)F_A(s) - c_D s$$

and thus

$$Eu_D'(s) = \frac{(1-q)c_A}{q} - c_D \leq 0;$$

therefore, $Eu_D(s) \leq Eu_D(\bar{s}_D) = U_D$. Finally, it cannot be profitable to deviate to $s > \bar{s}_A$, as doing so yields the same probability of victory as spending $\bar{s}_A$ at strictly greater cost.

To prove that there are no profitable deviations for Advantaged candidates, first observe that for any $s \in [\bar{s}_D, \bar{s}_A]$,

$$Eu_A(s) = (1-q) + qF_A(s) - c_A s = 1 - q - c_A \bar{s}_D.$$

This confirms A's indifference condition and implies $U_A = 1 - q - c_A \bar{s}_D$. For any $s \in [0, \bar{s}_D)$, we have

$$Eu_A(s) = (1-q)F_D(s) - c_A s$$

and thus

$$Eu_A'(s) = \frac{(1-q)c_D}{q} - c_A.$$

From $q \leq 1/2$ we have $(1-q)/q \geq 1$ and thus $Eu_A'(s) \geq c_D - c_A > 0$; therefore, $Eu_A(s) < Eu_A(\bar{s}_D) = U_A$. Finally, as before, it cannot be profitable to deviate to $s > \bar{s}_A$.

Given the candidates' strategies, the median voter's beliefs are consistent with the application of Bayes' rule wherever possible. Therefore, the assessment constitutes an equilibrium. □

Proof of part (b). Suppose $\alpha \leq 0$ and $q > c_A/(c_A + c_D)$. I claim that the following assessment constitutes an equilibrium. The mixed strategy profile is given by the CDFs

$$F_D(s) = \begin{cases} 0 & s < 0, \\ c_D s/q & 0 \leq s \leq \underline{s}_A, \\ c_D \underline{s}_A/q + k_D(s - \underline{s}_A) & \underline{s}_A \leq s \leq \bar{s}, \\ 1 & s > \bar{s}, \end{cases}$$

$$F_A(s) = \begin{cases} 0 & s < \underline{s}_A, \\ k_A(s - \underline{s}_A) & \underline{s}_A \leq s \leq \bar{s}, \\ 1 & s > \bar{s}, \end{cases}$$

where

$$
\begin{aligned}
\underline{s}_A &= \frac{1-c_A/c_D}{(1-q)c_D/q-c_A},\\
\bar{s} &= \frac{1}{c_D},\\
k_A &= \frac{(1-q)c_D-qc_A}{1-2q},\\
k_D &= \frac{(1-q)c_A-qc_D}{1-2q}.
\end{aligned}
$$

The electorate's updated beliefs are

$$
\mu_{DD}(s_1,s_2)=\begin{cases}1 & s_1\le \underline{s}_A, s_2\le \underline{s}_A,\\ (q-\Phi q)/(1-\Phi q) & s_1\le \underline{s}_A, s_2>\underline{s}_A,\\ (q-\Phi q)/(1-\Phi q) & s_1>\underline{s}_A, s_2\le \underline{s}_A,\\ (1-\Phi)^2q/(2-2\Phi+\Phi^2 q) & s_1>\underline{s}_A, s_2>\underline{s}_A,\end{cases}
$$

$$
\mu_{DA}(s_1,s_2)=\begin{cases}0 & s_2\le \underline{s}_A,\\ (1-q)/(1-\Phi q) & s_1\le \underline{s}_A, s_2>\underline{s}_A,\\ (1-\Phi)(1-q)/(2-2\Phi+\Phi^2 q) & s_1>\underline{s}_A, s_2>\underline{s}_A,\end{cases}
$$

$$
\mu_{AD}(s_1,s_2)=\begin{cases}0 & s_1\le \underline{s}_A,\\ (1-q)/(1-\Phi q) & s_1> \underline{s}_A, s_2\le\underline{s}_A,\\ (1-\Phi)(1-q)/(2-2\Phi+\Phi^2 q) & s_1>\underline{s}_A, s_2>\underline{s}_A,\end{cases}
$$

$$
\mu_{AA}(s_1,s_2)=\begin{cases}q/(2-2\Phi+\Phi^2 q) & s_1>\underline{s}_A, s_2>\underline{s}_A,\\ 0 & \text{otherwise,}\end{cases}
$$

where $\Phi = F_D(\underline{s}_A)$. Given these beliefs and the fact that $\alpha \le 0$, the median voter is never indifferent between two candidates who spend different amounts; he always strictly prefers whichever spends more. (To confirm this, notice that μ_{AD} and μ_{AA} are increasing in s_1, and μ_{DA} and μ_{AA} are increasing in s_2.) Because neither type's mixed strategy contains a mass point, spending the same amount as one's opponent is a zero-probability event, so the choice of sharing rule for these cases is immaterial.

First I will confirm that there are no profitable deviations for Disadvantaged candidates. For $s \in [0, \underline{s}_A]$ we have

$$
Eu_D(s) = qF_D(s) - c_D s = 0,
$$

which implies $U_D = 0$ and confirms D's indifference across this range. For $s \in [\underline{s}_A, \bar{s}]$, we have

$$
\begin{aligned}
Eu_D(s) &= qF_D(s) + (1-q)F_A(s) - c_D s\\
&= c_D\underline{s}_A + [qk_D + (1-q)k_A](s-\underline{s}_A) - c_D s\\
&= c_D\underline{s}_A + c_D(s-\underline{s}_A) - c_D s\\
&= 0
\end{aligned}
$$

$$= U_D,$$

confirming D's indifference condition for this range. It cannot be profitable for D to deviate to $s > \bar{s}$, as doing so yields the same probability of victory as spending $\bar{s}$ for strictly greater cost.

Next I will confirm that there are no profitable deviations for Advantaged candidates. For $s \in [\underline{s}_A, \bar{s}]$ we have

$$\begin{aligned} Eu_A(s) &= (1-q)F_D(s) + qF_A(s) - c_A s \\ &= \frac{(1-q)c_D \underline{s}_A}{q} + [(1-q)k_D + qk_A](s - \underline{s}_A) - c_A s \\ &= \frac{(1-q)c_D \underline{s}_A}{q} + c_A(s - \underline{s}_A) - c_A s \\ &= \left[\frac{(1-q)c_D}{q} - c_A\right]\underline{s}_A \\ &= 1 - \frac{c_A}{c_D}, \end{aligned}$$

which implies $U_A = 1 - c_A/c_D$ and confirms A's indifference across this range. For $s \in [0, \underline{s}_A)$ we have

$$\begin{aligned} Eu_A'(s) &= (1-q)F_D'(s) - c_A \\ &= \frac{(1-q)c_D}{q} - c_A \\ &\geq c_D - c_A \\ &> 0, \end{aligned}$$

so $Eu_A(s) < Eu_A(\underline{s}_A) = U_A$, meaning it is not profitable for A to deviate to such s. Finally, as before, it also cannot be profitable for A to deviate to $s > \bar{s}$.

For the median voter's beliefs, observe that

$$\begin{aligned} \Pr(s_i \leq \underline{s}_A \,|\, t_i = D) &= \Phi, \\ \Pr(s_i > \underline{s}_A \,|\, t_i = D) &= 1 - \Phi, \\ \Pr(s_i \leq \underline{s}_A \,|\, t_i = A) &= 0, \\ \Pr(s_i > \underline{s}_A \,|\, t_i = A) &= 1, \end{aligned}$$

where I denote $\Phi = F_D(\underline{s}_A)$. Moreover, s_1 and s_2 are conditionally independent given (t_1, t_2). Therefore, in case both candidates spend no more than $\underline{s}_A$, we have

$$\Pr(t_1 = D, t_2 = D \,|\, s_1 \leq \underline{s}_A, s_2 \leq \underline{s}_A) = 1.$$

In case 1 spends no more than $\underline{s}_A$ and 2 spends more, the electorate infers for sure that $t_1 = D$, so we have

$$\Pr(t_1 = D, t_2 = D \,|\, s_1 \leq \underline{s}_A, s_2 > \underline{s}_A) = \frac{\Phi(1-\Phi)q/2}{\Phi(1-\Phi)q/2 + \Phi(1-q)/2}$$

$$
\begin{aligned}
&= \frac{q - \Phi q}{1 - \Phi q}, \\
\Pr(t_1 = D, t_2 = A \,|\, s_1 \le \underline{s}_A, s_2 > \underline{s}_A) &= 1 - \Pr(t_1 = D, t_2 = D \,|\, s_1 \le \underline{s}_A, s_2 > \underline{s}_A) \\
&= \frac{1 - q}{1 - \Phi q}.
\end{aligned}
$$

Then, by symmetry,

$$
\begin{aligned}
\Pr(t_1 = D, t_2 = D \,|\, s_1 > \underline{s}_A, s_2 \le \underline{s}_A) &= \frac{q - \Phi q}{1 - \Phi q}, \\
\Pr(t_1 = A, t_2 = D \,|\, s_1 > \underline{s}_A, s_2 \le \underline{s}_A) &= \frac{1 - q}{1 - \Phi q}.
\end{aligned}
$$

Finally, consider the case where both candidates spend more than $\underline{s}_A$. The probability that this occurs is

$$
\begin{aligned}
\Pr(s_1 > \underline{s}_A, s_2 > \underline{s}_A) &= (1 - \Phi)^2 \Pr(t_1 = D, t_2 = D) \\
&\quad + (1 - \Phi)[\Pr(t_1 = D, t_2 = A) + \Pr(t_1 = A, t_2 = D)] \\
&\quad + \Pr(t_1 = A, t_2 = A) \\
&= \frac{(1 - \Phi)^2 q}{2} + 2(1 - \Phi)\left[\frac{1 - q}{2}\right] + \frac{q}{2} \\
&= 1 - \Phi + \frac{\Phi^2 q}{2}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\Pr(t_1 = D, t_2 = D \,|\, s_1 > \underline{s}_A, s_2 > \underline{s}_A) &= \frac{(1 - \Phi)^2 q/2}{1 - \Phi + \Phi^2 q/2}, \\
\Pr(t_1 = D, t_2 = A \,|\, s_1 > \underline{s}_A, s_2 > \underline{s}_A) &= \frac{(1 - \Phi)(1 - q)/2}{1 - \Phi + \Phi^2 q/2}, \\
\Pr(t_1 = A, t_2 = D \,|\, s_1 > \underline{s}_A, s_2 > \underline{s}_A) &= \frac{(1 - \Phi)(1 - q)/2}{1 - \Phi + \Phi^2 q/2}, \\
\Pr(t_1 = A, t_2 = A \,|\, s_1 > \underline{s}_A, s_2 > \underline{s}_A) &= \frac{q/2}{1 - \Phi + \Phi^2 q/2}.
\end{aligned}
$$

The given beliefs are consistent with these conditional probabilities.

I have confirmed that the given assessment is an equilibrium. In equilibrium, in an election between an Advantaged and a Disadvantaged candidate, the Advantaged candidate is guaranteed to win if D spends $s < \underline{s}_A$ and wins with probability $1/2$ if D spends $s \in [\underline{s}_A, \bar{s}]$. Therefore, the probability of victory by an Advantaged candidate is

$$
\begin{aligned}
\Phi + \frac{1 - \Phi}{2} &= \frac{1 + \Phi}{2} \\
&= \frac{1 + c_D \underline{s}_A / q}{2}
\end{aligned}
$$

$$= \frac{1}{2}\left(1 + \frac{c_D - c_A}{(1-q)c_D - qc_A}\right),$$

as claimed in Equation 12. Differentiating with respect to c_A confirms that this probability decreases with c_A. □

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