Online Appendix

Omitted Proofs

Before the proofs, it is useful to list conditions for a player to win the election. Suppose candidates i and j are two front-runners such that $i < j \leq g$. Let candidate i's vote share be v_i and candidate j's be v_j. Because ties are broken in favor of the incumbent, candidate j beats candidate i (and hence wins the election) if and only if $v_j \geq v_i$. For candidate i to win, it must be $v_i > v_j$.

Proof of Lemma 1. Suppose at least two challengers enter the election in an equilibrium. Denote the winner as w. Let one of the other entering challengers be $i_{(1)}$. Because $i_{(1)}$ is not the winner of the election, $i_{(1)}$ enters the election to change the identity of the winner. Now, I prove this cannot be the case. I here only consider the case of $i_{(1)} < w$ because the logic for $i_{(1)} > w$ is the same.

If $i_{(1)}$ drops out, the winner is another candidate, say $i_{(2)}$, and it must be that $-|i_{(2)} - i_{(1)}| < -|w - i_{(1)}|$ for $i_{(1)}$ to be willing to enter. It obvious that $i_{(1)} < i_{(2)} < w$ cannot be the case. Otherwise, it is easy to verify $-|i_{(2)} - i_{(1)}| > -|w - i_{(1)}|$. To show $i_{(2)} < i_{(1)} < w$ cannot be the case either, I suppose otherwise. By $i_{(2)} < i_{(1)} < w$ and $-|i_{(2)} - i_{(1)}| < -|w - i_{(1)}|$, one has $i_{(1)} > (i_{(2)} + w)/2$. Now I consider $i_{(2)}$'s choice. Similar to $i_{(1)}$, $i_{(2)}$ enters the election to make w the winner because $i_{(2)}$ is not the winner of the election. Again, if $i_{(2)}$ drops out, the winner is $i_{(3)}$, and it must be $-|i_{(3)} - i_{(2)}| < -|w - i_{(2)}|$. Through the same argument, one has $i_{(3)} < i_{(2)} < w$ and $i_{(2)} > (i_{(3)} + w)/2$. By repeating this process, one can construct a sequence of candidates $i_{(1)}, i_{(2)}, \dots$ such that $i_{(l)} > i_{(l+1)}$ and $i_{(l)} > (i_{(l+1)} + w)/2$ for all $l = 1, 2, \dots$. Because the sequence is bounded below, its infimum, denoted by i_{-}, is its limit. In particular, it must be $i_{-} < w$. However, taking the limit on both sides of $i_{(l)} > (i_{(l+1)} + w)/2$ gives $i_{-} \geq w$, which is a contradiction. $\square$

Lemma 2 is obvious; I omit its proof.

Proof of Lemma 3. First I claim $c_w = 0$. Otherwise, the incumbent can set $c_w = 0$ and w is still willing to run. Second, to show $c_i = 0$ for all i such that $w < i < g$, I suppose $c_i > 0$ for some i. By Assumption 2 and Lemma 1, it must be the case that i can enter the election and win if $c_i = 0$. However, because $w < i < g$, it is strictly better for the incumbent to let i enter and win the election. Therefore, it must be the case that $c_i = 0$ for all $i \in (w, g)$.

For candidate $i < w$ who is eliminated, her payoff from running is $\beta - c_i$, while her payoff from not entering is $-(w-i)$. To keep i from running, it must be $c_i \geq w-i+\beta$. By Assumption 1, $c_i = w - i + \beta$.

If $i < w$ is not eliminated, then $c_i = 0$. Also, it implies i cannot win the election even if she enters, i.e., $(i+w)/2 \leq \max\{(g-i)/2, 1-(w+g)/2\}$ when $w < g$, and $(i+g)/2 \leq 1-(i+g)/2$ when $w = g$. The former is equivalent to

$$\begin{cases} i \leq \frac{g-w}{2} & \text{if } i \leq w - 2(1-g) \\ i \leq 2(1-w) - g & \text{otherwise} \end{cases}, \tag{A1}$$

and the latter can be rearranged to

$$i \leq 1 - g. \tag{A2}$$

For the proof of contradiction, suppose $c_j > 0$ for citizen $j < i$. Then it must be the case that j can enter and win the election. That is $(j + w)/2 > \max\{(g - j)/2, 1 - (w + g)/2\}$ if $w < g$, and $(j + g)/2 > 1 - (j + g)/2$ when $w = g$. Analogously, one can write the former as $j > (g - w)/2$ and $j > 2(1 - w) - g$ and the latter as $j > 1 - g$. Because $i > j$, one must have $\{i > (g - w)/2, i > 2(1 - w) - g\}$ or $i > 1 - g$, which contradicts Equations A1 and A2. $\square$

Proof of Lemma 4. If $w_s \leq (2 - g)/3$, by Assumption 4, the incumbent can deviate to the free and fair election profitably. Therefore, it must be that $w_s \in ((2 - g)/3, g)$ in a partially manipulated election.

In equilibrium, the incumbent needs to eliminate all other challengers that can beat w_s. By the definition of candidate selection, we only need to consider $i < w_s$. For $i < w_s$ to win the election, it must be $(i+w_s)/2 > \max\{(g-i)/2, 1-(w_s+g)/2\}$, which translates to

$$\begin{cases} i > \frac{g-w_s}{2} & \text{if } i < w_s - 2(1-g) \\ i > 2(1-w_s) - g & \text{otherwise} \end{cases}. \tag{A3}$$

For the first line to hold, a necessary condition is $(g-w_s)/2 < w_s-2(1-g)$, which is equivalent to $3w_s+3g > 4$. Moreover, when $3w_s+3g > 4$, we have $w_s-2(1-g) > (g-w_s)/2 > 2(1-w_s)-g$. The converse is also true. Therefore, when $3w_s+3g > 4$, $i \geq w_s-2(1-g)$ implies $i > 2(1-w_s)-g$ and when $3w_s+3g \leq 4$, $i > 2(1-w_s)-g$ implies $i > w_s-2(1-g)$. Using them, one can rearrange Equation A3 to

$$\begin{cases} i > \frac{g-w_s}{2} & \text{if } 3w_s + 3g > 4 \\ i > 2(1-w_s) - g & \text{otherwise} \end{cases}.$$

That is, when $w_s \leq 4/3 - g$, any candidate in $(2(1-w_s)-g, w_s)$ can enter and win if she faces no constraint of running. Similarly, any citizen in $((g-w_s)/2, w_s)$ may enter and win if $w_s > 4/3 - g$. Therefore, the incumbent needs to eliminate all candidates in $(2(1-w_s)-g, w_s)$ if $w_s \leq 4/3 - g$ and all citizens in $((g-w_s)/2, w_s)$ if otherwise. □

Proof of Corollary 1. In a free and fair election, the best case for the incumbent is candidate $(2-g)/3$ entering. Therefore, the incumbent can at most get

$$U_{(2-g)/3} = \frac{2}{3} - \frac{4}{3}g$$

from a free and fair election. Because $U_{w_1} - U_{(2-g)/3} = (\beta - 1/3)^2/2$, we have $U_{w_1} > U_{(2-g)/3}$ for all $\beta < 1/3$. From the proof of Proposition 3, we have

$$U_{w_2} - U_{(2-g)/3} = \frac{1}{2}\left(\frac{2}{3} - \beta\right)^2 - \frac{2}{3}(1-g). \tag{A4}$$

When $(g, \beta) \in \mathcal{P}^2_{w_2}$, one has $U_{w_2} \geq U_{(2-g)/3}$ by the proof of Proposition 3. When $(g, \beta) \in \mathcal{P}^1_{w_2}$, one has $2(1-g) + \beta \leq 1/2$. Now consider the constrained minimization problem:

$$\min_{g,\beta} \frac{1}{2}\left(\frac{2}{3} - \beta\right)^2 - \frac{2}{3}(1-g) \tag{A5}$$

subject to

$$\begin{aligned} 2(1-g) + \beta &\leq \frac{1}{2}, \\ \frac{2}{3} \leq g &\leq 1, \\ 0 \leq \beta &\leq \frac{1}{3}. \end{aligned}$$

Its solution is $g = 11/12$ and $\beta = 1/3$. Plugging them into Equation A5, one may conclude that $(2/3 - \beta)^2/2 - (2/3)(1-g) \geq 0$. In sum, the utility from a partially manipulated election is larger than that from any free and fair election for both w_1 and w_2. $\square$

Proof of Corollary 2. From the proof of Proposition 3, the incumbent's payoff is decreasing on $(2/3 - g, g)$ if $\{g \leq 2/3, \beta \geq 1/3\}$ or $\{\beta \geq 1/3, 2(1-g) + \beta \geq 2/3\}$. Also, the incumbent's payoff is decreasing on $w \in (2/3 - g/3, 4/3 - g]$ and $U_{w_2} < U_{(2-g)/3}$ if $1/3 \leq \beta < 2/3$ and $(3/2)(2/3 - \beta)^2 < 2(1-g) < 2/3 - \beta$.

Hence, selecting any citizen in $((2-g)/3, g)$ yields a lower payoff than the free and fair election with citizen $(2-g)/3$ running if (g, β) falls into one of the following sets:

$$\begin{aligned} \mathcal{F}^1 &= \left\{ (g, \beta) \middle| g \leq \frac{2}{3}, \beta \geq \frac{1}{3} \right\} \\ \mathcal{F}^2 &= \left\{ (g, \beta) \middle| g > \frac{2}{3}, \beta \geq \frac{1}{3}, 2(1-g) \geq \frac{2}{3} - \beta \right\} \\ \mathcal{F}^3 &= \left\{ (g, \beta) \middle| g > \frac{2}{3}, \frac{1}{3} \leq \beta < \frac{2}{3}, \frac{3}{2}\left(\frac{2}{3} - \beta\right)^2 < 2(1-g) < \frac{2}{3} - \beta \right\} \end{aligned}$$

In the statement of Corollary 2, $\mathcal{F}$ is a simplified version of $\mathcal{F}^1 \cup \mathcal{F}^2 \cup \mathcal{F}^3$. $\square$

Proof of Corollary 3. When $\beta > 2/3$, $(g, \beta) \in \mathcal{F}$ for all $g > 1/2$. Because $2(g-1/2)(g+\beta-7/6)$ is strictly increasing with respect to β for all $g > 1/2$, for any β_g one can find β^* such that $2(g-1/2)(g+\beta-7/6) > \beta_g$ for all $\beta \geq \beta^*$. $\square$

Proof of Lemma 5. Given that w is the winner of the election, the utilitarian voter welfare is $\int_0^w -(w-j)\,\mathrm{d}j + \int_w^1 -(j-w)\,\mathrm{d}j = -(w-1/2)^2 - 1/4$, which is a positive transformation of the median voter's utility function. $\square$

Supplementary Results on Strategic Voting

To win the election, the incumbent has to eliminate all challengers who can beat him, i.e., those who are more moderate than the incumbent. The next proposition formalizes this idea.

Proposition A1. *In a fully manipulated election, the incumbent eliminates all challengers in the set $(1-g, g)$. In particular, if challenger i is eliminated, then $c_i = (g-i) + \beta$.*

Proof. By the distribution of the voters, the median voter's ideal point is $1/2$. The rest follows the logic in Proposition 1. $\square$

The following proposition characterizes free and fair elections.

Proposition A2. *Suppose all challengers have free entry. For any challenger $i \in (1-g, g)$, a corresponding equilibrium exists such that i is the only entrant and thus the winner of the election.*

Proof. For any i such that $i \in (1-g, g)$, the median voter strictly prefers i over the incumbent. Because voters do not use weakly dominated voting strategies, candidate i wins the election if she enters and she is the only entrant. In such a case, voters whose ideal points are smaller than $(i+g)/2$ vote for i and voters whose ideal points are weakly greater than $(i+g)/2$ vote for the incumbent in equilibrium. Given these equilibrium voting strategies, no other candidate has any incentive to enter. Suppose otherwise, for any voter with an ideal point that is smaller than

$(i+g)/2$, holding other citizens' voting strategies fixed, voting for the third candidate either make the incumbent win or leave the result of the election unchanged. Therefore, this voter cannot profitably deviate. For any voter whose ideal point is weakly greater than $(i+g)/2$, voting for the third candidate cannot change the result of the election. Therefore, given the voting equilibrium, a third entrant cannot get any vote and thus does not enter in the first place. $\square$

The next lemma shows there is always an entrant for any elimination set E.

Lemma A1. *For all $E \subsetneqq (1-g, g)$, a corresponding challenger ϖ_E enters in equilibrium.*

Proof. Suppose for some $E \subsetneqq (1-g, g)$, no challenger enters in equilibrium if the incumbent eliminates E. In other words, the incumbent wins the election by eliminating E. Now pick an arbitrary challenger in $(1-g, g) \setminus E$ and call her ϖ_E. If ϖ_E enters the election instead, she can beat the incumbent, which is a profitable deviation for her. $\square$

The following proposition describes the equilibrium of the game with strategic voting.

Proposition A3. *When $(g, \beta, \varpi_\emptyset) \in \overline{\mathcal{P}}_{w_3}$ and $\beta_g < (2g+\beta-3)^2/8 + (\varpi_\emptyset - 1/2)$, the incumbent holds a partially manipulated election and selects w_3 as his successor. When $(g, \beta, \varpi_\emptyset) \in \overline{\mathcal{P}}_{w_4}$ and $\beta_g < (2g+\beta-2)^2/2$, the incumbent organizes a partially manipulated election and selects w_4 as his successor. When $(g, \beta, \varpi_\emptyset) \notin \overline{\mathcal{P}}_{w_3} \cup \overline{\mathcal{P}}_{w_4}$ and $\beta_g < 2(g-1/2)(g+\beta-1)+(\varpi_\emptyset-1/2)$, the incumbent runs the free and fair election. In other cases, the incumbent organizes a fully manipulated election.*

Proof. When $w_s \to \varpi_\emptyset$, the incumbent's payoff converges to the free and fair case continuously. Therefore, the incumbent is better off by selecting w_3 (w_4) when $(g, \beta, \varpi_\emptyset) \in \overline{\mathcal{P}}_{w_3}$ ($(g, \beta, \varpi_\emptyset) \in \overline{\mathcal{P}}_{w_4}$) than organizing the free and fair election. For the incumbent to choose w_1 over the fully manipulated election, we require the incumbent's payoff from selecting w_3,

$$\frac{(2\beta-1)^2}{8} + \varpi_\emptyset - g,$$

to be greater than the payoff from running a fully manipulated election,

$$\beta_g - 2\left(g - \frac{1}{2}\right)\left(g + \beta - \frac{1}{2}\right).$$

After rearranging, one can show that the incumbent runs a partially manipulated election and selects w_3 when $(g, \beta, \varpi_\emptyset) \in \overline{\mathcal{P}}_{w_3}$ and $\beta_g < (2g+\beta-3)^2/8 + (\varpi_\emptyset - 1/2)$. The logic for selecting w_4 is analogous.

When $(g, \beta, \varpi_\emptyset) \notin \overline{\mathcal{P}}_{w_3} \cup \overline{\mathcal{P}}_{w_4}$, the incumbent needs to make a decision between the free and fair election and the fully manipulated one. Because the payoff from the free and fair election is $-(g - \varpi_\emptyset)$, one can easily find that the payoff from the free and fair election is greater if $\beta_g < 2(g - 1/2)(g + \beta - 1) + (\varpi_\emptyset - 1/2)$. □

Discussion on the Deferential Challenger Assumption

If one can arbitrarily assign entry behavior for each citizen when all citizens have free entry, one may cook up equilibria considered unrealistic. For example, when $g = 0.6$ and $c_i = 0$ for all i, the challenger with ideal point 0 entering the election constitutes an equilibrium. In this equilibrium, the incumbent wins the election and candidate 0 loses. Thus, she is indifferent between entering and staying not. However, her entry deters many other candidates who have winning potentials to run. If candidate 0 drops out, then any challenger in the set $(0.4, 0.6)$ can enter and win the election, which, not only benefits moderate voters, but also benefits candidate 0. In other words, candidate 0's entry makes her worse off.

For another example, consider the case when $g > 1/2$ and $c_i = 0$ for all i. In this case, one can find a sufficiently small ε such that all citizens except for those in the set $(g - \varepsilon, g + \varepsilon)$ entering supports an equilibrium. In this equilibrium, the incumbent wins the election. Holding other players' strategies fixed, each candidate is indifferent between losing and dropping out. Also, the incumbent is indifferent between them entering or not. However, if all losing candidates except for, say, the median citizen drop out, then the median citizen can beat the incumbent, which benefits the majority of citizens. In other words, the majority of candidates enter and make themselves worse off.

The above two cases share a common property: a losing candidate's entry creates a negative externality on other survivable challengers. Should this losing candidate drop out, a survivable challenger could enter and make a difference. Therefore, it is reasonable to assume losing candidates do not enter in the first place.

Election-Day Fraud

Suppose in addition to pre-electoral candidate elimination, the incumbent has access to election-day fraud. Typical election-day fraud includes ballot stuffing, vote misrecording, ballot invalidation, or even blatant result falsification. To model election-day fraud in the current framework, I assume the incumbent, after candidate entry, can boost a candidate's (this candidate may or may not be the incumbent himself) vote by α. Substantively, one can think that the incumbent stuffs the ballot box with α votes. For simplicity, I assume the cost of election-day fraud is $C(\alpha) = \alpha^2/2 + \gamma\alpha$, where $\gamma \geq 0$.

Now a free and fair election requires two conditions: $c_i = 0$ for all i and $\alpha = 0$. The analysis of the free and fair election is the same as in the main model. For the other two classes of equilibria, depending on the value of γ, election-day fraud may or may not partially substitute pre-electoral candidate elimination. However, even if election-day fraud substitutes candidate elimination, the qualitative results from the model still hold. In other words, election-day fraud and pre-electoral candidate elimination serve the same end here. Whichever gets picked only depends on their relative marginal cost. The rest of this section substantiates this claim.

First I consider the full manipulated election. Because of α, the incumbent now only needs to eliminate a smaller set of citizens, which is $(1 + \alpha - g, g)$. Therefore, the incumbent's payoff is:

$$\beta_g - \int_{1+\alpha-g}^{g} (g - i + \beta)\, \mathrm{d}i - \frac{\alpha^2}{2} - \gamma\alpha.$$

Its FOC gives $\alpha = (g - 1/2) + (\beta - \gamma)/2$. For this α to be greater than zero, it must be that $\gamma < \beta + 2(g-1/2)$. When $\gamma \geq \beta + 2(g-1/2)$, the cost of election-day fraud is too high compared to candidate elimination and the incumbent does not use election-day fraud in equilibrium.

Second, I examine a partially manipulated election in which $w_s \in ((2-g)/3, g)$ enters. Given α, we first need to calculate the set of citizens that need to be eliminated. For $i < w_s$ to win the election, it must be

$$\frac{i + w_s}{2} > \max\left\{\frac{g-i}{2}, 1 - \frac{w_s + g}{2}\right\} + \alpha,$$

which translates to

$$\begin{cases} i > \frac{g-w_s}{2} + \alpha & \text{if } i < w_s - 2(1-g) \\ i > 2(1-w_s) - g + 2\alpha & \text{if } i \geq w_s - 2(1-g) \end{cases}.$$

As in the proof of Lemma 4, $w_s - 2(1-g) > (g-w_s)/2 + \alpha > 2(1-w_s) - g + 2\alpha$ if $3w_s + 3g > 4 + 2\alpha$. Using them, one has

$$\begin{cases} i > \frac{g-w_s}{2} + \alpha & \text{if } 3w_s + 3g > 4 + 2\alpha \\ i > 2(1-w_s) - g + 2\alpha & \text{if } 3w_s + 3g \leq 4 + 2\alpha \end{cases}.$$

Hence, the total cost of repression in a partially manipulated election is

$$\begin{cases} \int_{2(1-w_s)-g+2\alpha}^{w_s} (w_s - i + \beta)\,\mathrm{d}i + \frac{\alpha^2}{2} + \gamma\alpha & \text{if } w_s \in \left(\frac{2-g}{3}, \frac{4}{3} - g\right] \\ \int_{\frac{g-w_s}{2}+\alpha}^{w_s} (w_s - i + \beta)\,\mathrm{d}i + \frac{\alpha^2}{2} + \gamma\alpha & \text{if } w_s \in \left(\frac{4}{3} - g, g\right) \end{cases}.$$

If $g > 2/3$ and $w_s \in ((2-g)/3, 4/3 - g]$ (or $g \leq 2/3$), the incumbent's payoff from a partially manipulated election is

$$-(g - w_s) - \int_{2(1-w_s)-g+2\alpha}^{w_s} (w_s - i + \beta)\,\mathrm{d}i - \frac{\alpha^2}{2} - \gamma\alpha. \tag{A6}$$

When $\gamma < 2/3$ and $\beta - 2(2/3 - \gamma) < 1/3 \leq 2(1 - g) + (\beta - 2(2/3 - \gamma))$ (or $\beta - 2(2/3 - \gamma) < 1/3 < 4(g - 1/2) + (\beta - 2(2/3 - \gamma))$ if $g \leq 2/3$), the FOC of Equation A6 has an interior solution, which is

$$\begin{aligned} w_1 &= \frac{7}{9} - \frac{g}{3} - \frac{\beta - 2\left(\frac{2}{3} - \gamma\right)}{3}, \\ \alpha &= \frac{2}{3} - \gamma. \end{aligned}$$

When $\gamma \geq 2/3$, the optimal α is zero. This implies the incumbent only uses candidate elimination.

If $g > 2/3$ and $w_s \in (4/3 - g, g)$, the incumbent's payoff from a partially manipulated election is

$$-\left(g - w_s\right) - \int_{\frac{g - w_s}{2} + \alpha}^{w_s} \left(w_s - i + \beta\right) \mathrm{d}i - \frac{\alpha^2}{2} - \gamma\alpha. \tag{A7}$$

When $\gamma < 2/3$ and $2(1 - g) + (\beta - (2/3 - \gamma)) < 2/3 \leq g + (\beta - (2/3 - \gamma))$, the FOC of Equation A7 has an interior solution, which is

$$\begin{aligned} w_2 &= \frac{4}{9} + \frac{g}{3} - \frac{2}{3}\left[\beta - \left(\frac{2}{3} - \gamma\right)\right], \\ \alpha &= \frac{2}{3} - \gamma. \end{aligned}$$

Therefore, the incumbent only uses candidate elimination when $\gamma \geq 2/3$ as well under this case.

In sum, election-day fraud is a substitute for pre-electoral candidate elimination in the model, and the incumbent may use this type of fraud if it is sufficiently cheap. However, it should be evident at this point that adding election-day fraud cannot contribute to new findings.