Online Appendix for Protest Puzzles

Uncorrelated Signals

The difficulties in ensuring the existence of monotone equilibria in our setting raise the question of why we do not consider a setting with uncorrelated private signals, where we know that the best response to a monotone strategy is monotone. One answer is that such a setting is unnatural because costs must reflect some common factor, in which case a citizen's cost realization contains some information about the costs of others. Moreover, as we will see, this setting offers a less natural resolution of the Tullock's Paradox.

Our setting is the same as before except that now, $c_i \sim iid\ F$, where $F(\cdot)$ has full support on $\mathbb{R}$. The best response to a monotone strategy is clearly monotone: Higher c_i only reduces i's incentive to revolt without changing his beliefs about others' behavior. The equilibria are characterized by the indifference condition:

$$u(N)\binom{N}{qN}\ F(c^*)^{qN}\ (1-F(c^*))^{(1-q)N} = c^*. \tag{1}$$

It is beneficial to do a change of variables $z^* = F(c^*)$, so that (1) becomes:

$$u(N)\binom{N}{qN}\ [z^*]^{qN}\ [1-z^*]^{(1-q)N} = F^{-1}(z^*), \text{ with } \ z^* \in [0,1]. \tag{2}$$

A key simple observation is that as N increases, the maximum of $[z^*]^{qN}\ [1-z^*]^{(1-q)N}$ becomes very sharp, even though the whole expression approaches zero. In fact, using the Stirling approximation, one can identify the rate of convergence as $N \to \infty$:

$$\binom{N}{qN}\ z^{qN}\ (1-z)^{(1-q)N} \approx \frac{1}{\sqrt{\pi N}}\ \frac{1}{\sqrt{2q(1-q)}}\ \left(\frac{z}{q}\right)^{qN}\ \left(\frac{1-z}{1-q}\right)^{(1-q)N}. \tag{3}$$

Because Stirling approximation is close even when N is small, (3) provides a good approximation even for small N. The maximum of the estimated probability of pivotality (the left hand side of the indifference condition (2)), which happens at $z = q$, approaches:

$$\lim_{N\to\infty} \max\left\{\binom{N}{qN}\ z^{qN}\ (1-z)^{(1-q)N}\right\} = \lim_{N\to\infty} \frac{1}{\sqrt{N}}\ \frac{1}{\sqrt{2\pi q(1-q)}}. \tag{4}$$

If $u(N)$ does not depend on N or grows with N at a rate smaller than $N^{1/2}$, then in the limit, there is a unique equilibrium with $\lim_{N\to\infty} c^*(N) = 0$. Moreover, if $u(N)$ grows with N at the rate N, then in the limit, there are three equilibria: There is an equilibrium, in which $\lim_{N\to\infty} c^*(N) = 0$, and there are two equilibria in which $\lim_{N\to\infty} F(c^*(N)) = q$, one from below and one from above. These observations follow from equations (2) to (4). Our

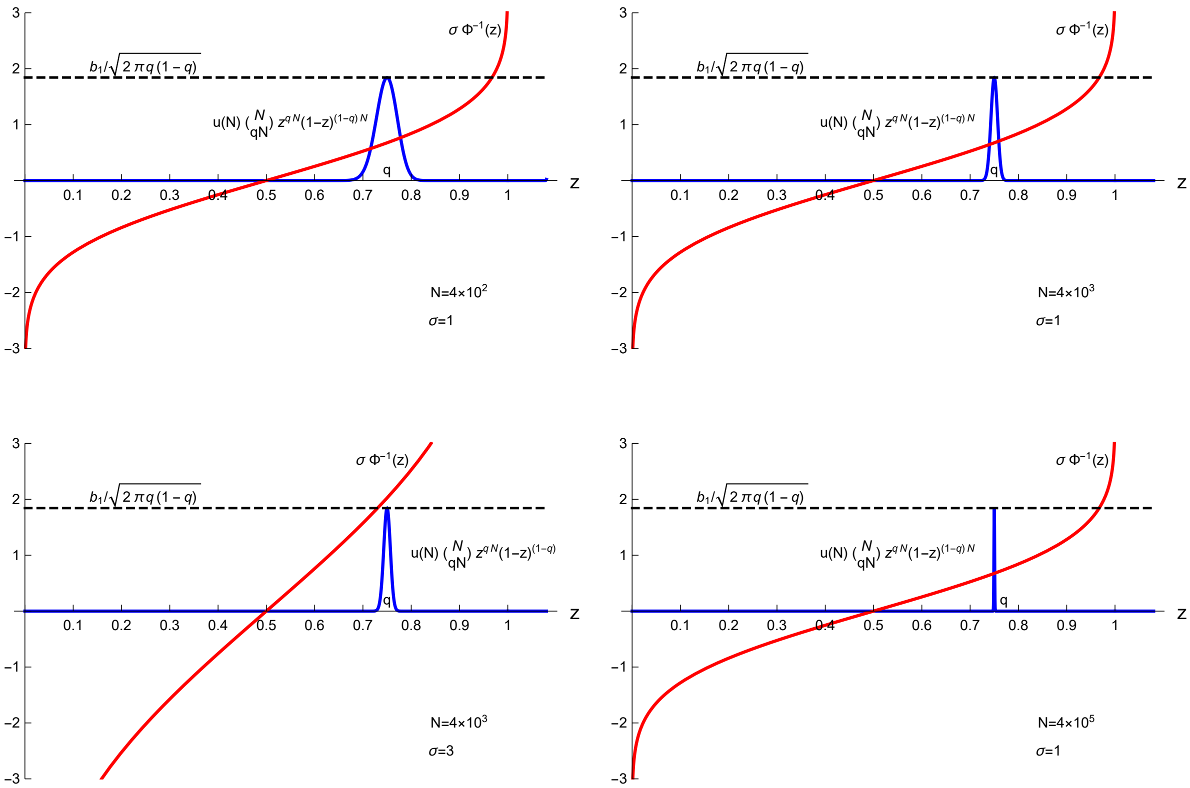

Figure 1: The unidomal curve is the left hand side of the indifference equation (1), and the increasing curve is its right hand side. The dashed line is (4). Parameters: $c_i \sim N(0, \sigma)$, $q = 0.75$, $b_1 = 2$, $b_0 = 0$, N and σ are shown on the graph.

simulations with Normal distribution suggest that when $\lim_{N\to\infty} F(c^*(N)) = q^+$, the likelihood of success approaches 1, and when $\lim_{N\to\infty} F(c^*(N)) = q^-$, the likelihood of success approaches 0. Next, suppose $u(N)$ increases at the rate $N^{1/2}$. From (3),

$$u(N) = b_0 + b_1\sqrt{N} \quad \Rightarrow \quad \lim_{N\to\infty} u(N) \binom{N}{qN} [z^*]^{qN} \; [1 - z^*]^{(1-q)N} = \begin{cases} \frac{b_1}{\sqrt{2\pi q(1-q)}} & ; z^* = q \\ \\ 0 & ; z^* \neq q \end{cases}$$

Figure 1 illustrates the left and right hand sides of equation (1) for a few cases of N, when $q = 0.75$, and F is the Normal distribution. Clearly, as long as N is moderately large, there is always an equilibrium with $c^* \approx 0$. In addition, because $u(N) \binom{N}{qN} [z^*]^{qN} \; [1 - z^*]^{(1-q)N}$ remains single-peaked, when σ is not too large, there are two equilibria, in which $F(c^*(N))$ approaches q as N grows. When σ is larger, for large N, any equilibrium with $c^* > 0$ disappears, and we are left with $c^* \approx 0$.

Alternative Models of Revolution

Another class of games used in the literature on revolutions contains uncertainty about the revolution payoff that is received when there is a regime change (Bueno de Mesquita 2010; Shadmehr and Bernhardt 2011).

Common Value Payoffs. Consider the game in Figure 2 with a continuum of players, indexed by $i \in [0,1]$. The revolution succeeds whenever the fraction of revolters exceeds a threshold $q \in (0,1)$. The status quo payoff is 0. If the revolution succeeds, everyone gets θ, and those who participated in a successful revolution, get an additional $\alpha\theta$, with $\alpha \in (0,1)$. As before, a citizen i receives private signals $x_i = \theta + \sigma\ \epsilon_i$, where θ and ϵ_is are independent. Citizens share an improper prior that θ is distributed uniformly on $\mathbb{R}$, and $\epsilon_i \sim F$ with full support on $\mathbb{R}$. There is always an equilibrium in which no one revolts. We focus on finite-cutoff strategies, where i revolts if and only if $x_i > x^*$. Then, the regime collapses if and only if $\theta > \theta^*$, where

$$Pr(x_i > x^*|\theta^*) = 1 - F\left(\frac{x^* - \theta^*}{\sigma}\right) = q, \text{ so that } x^* = \theta^* + \sigma\ F^{-1}(1-q). \qquad (5)$$

		outcome	
		$n > q$	$n \leq q$
citizen i	*revolt*	$(1+\alpha)\theta - c$	$-c$
	no revolt	θ	0

Figure 2: A common value version of the revolution model of Bueno de Mesquita (2010).

The indifference condition is:

$$\begin{aligned}
\frac{c}{\alpha} &= Pr(\theta > \theta^*|x_i = x^*)E[\theta|x_i = x^*, \theta > \theta^*] \\
&= \int_{\theta^*}^{\infty} \theta\ pdf(\theta|x^*)\ d\theta \\
&= \int_{\theta^*}^{\infty} \theta\ \frac{1}{\sigma}\ f\left(\frac{x^* - \theta}{\sigma}\right)\ d\theta \quad \text{(because the prior is uniform)} \\
&= \int_{-\infty}^{z^* \equiv z(\theta = \theta^*)} (x^* - \sigma\ z)\ f(z)\ dz, \quad z = \frac{x^* - \theta}{\sigma} \\
&= \int_{-\infty}^{F^{-1}(1-q)} (x^* - \sigma\ z)\ f(z)\ dz \quad \text{(from equation (5))} \\
&= x^*\ F(F^{-1}(1-q)) - \sigma\ F(F^{-1}(1-q))\ E[\epsilon_i|\epsilon_i < F^{-1}(1-q)] \\
&= (1-q)\ (x^* - \sigma\ E[\epsilon_i|\epsilon_i < F^{-1}(1-q)]).
\end{aligned}$$

Thus,

$$\begin{aligned} x^* &= \frac{c}{\alpha} \frac{1}{(1-q)} + \sigma \; E[\epsilon_i | \epsilon_i < F^{-1}(1-q)]. \\ \theta^* &= \frac{c}{\alpha} \frac{1}{(1-q)} + \sigma \; \{E[\epsilon_i | \epsilon_i < F^{-1}(1-q)] - F^{-1}(1-q)\}. \end{aligned} \tag{6}$$

The term $\sigma \; E[\epsilon_i | \epsilon_i < F^{-1}(1-q)]$ is decreasing in q, indicating a force that increases the citizens' incentives to revolt when the regime is stronger. This force stems from learning-in-equilibrium incentives generated by common value payoffs: When the regime becomes stronger so that citizens become more hesitant to revolt, the information content of their actions is a better news of θ, and hence the expected revolution payoff conditional on regime change is higher. However, when F is logconcave (An 1998, p. 357), the curly bracket in θ^* is increasing in q.[1] Thus, as the regime becomes stronger (q increases), θ^* increases. The analysis is far simpler in a private value setting, where a citizen's payoff is his signal x_i rather than the uncertain fundamental θ. Then, the indifference condition is:

$$\begin{aligned} c &= Pr(\theta > \theta^* | x_i = x^*) \; \alpha \; x^* \\ &= [1 - Pr(x_i > x^* | \theta^*)] \; \alpha \; x^* \\ &= (1-q) \; \alpha \; x^*. \qquad \text{(from equation (5))} \end{aligned}$$

Thus,

$$x^* = \frac{c}{\alpha} \frac{1}{1-q} \quad \text{and} \quad \theta^* = x^* - \sigma F^{-1}(1-q) = \frac{c}{\alpha} \frac{1}{1-q} - \sigma F^{-1}(1-q) \tag{7}$$

Proposition 1 *The equilibria in finite-cutoff strategies are characterized by (x^*, θ^*), so that a citizen revolts whenever his signal is above x^* and the regime collapses whenever $\theta > \theta^*$. When the prior is uniform or the noise in private signals approaches zero, the equilibrium is unique and is given by (6) for the common value and by (7) for the private value setting. In both settings, as the regime becomes stronger (q increases), the revolution is less likely.*[2]

Pivotality. Now, consider the setting with $N+1$ players, which features the logic of pivotality. We show that for large N, with strictly unimodal distributions like Normal, the best response to a finite-cutoff strategy is not a finite-cutoff strategy. Because revolting is costly, i only revolts if he is pivotal, i.e., only if the number of revolters is qN. Thus, i's net expected payoff from revolting versus not revolting is:

$$Pr(piv|x_i) \; E[u(\theta, N)|x_i, piv] - c = \int_{\theta=-\infty}^{\infty} Pr(piv|\theta) \; u(\theta, N) \; pdf(\theta|x_i) \; d\theta - c,$$

[1] An, Mark Yuying. 1998. "Logconcavity versus Logconvexity: A Complete Characterization." *Journal of Economic Theory* 80: 350-69.

[2] In the common value setting, when the prior is uniform, but the noise is not vanishingly small, we also require that F be logconcave as a sufficient condition.

where piv denotes the event of i being pivotal. We focus on symmetric monotone strategies, so that a citizen revolts if and only of his signal exceeds a threshold: $x_i > x^*$. If the best response to a monotone strategy was also a monotone strategy, then the equilibrium would be characterized by the indifference condition of the marginal player whose signal is the exact cutoff:

$$Pr(piv|x_i = x^*) \cdot E[u(\theta, N)|piv, x_i = x^*] = \int_{\theta=-\infty}^{\infty} Pr(piv|\theta)\ u(\theta, N)\ pdf(\theta|x_i = x^*)\ d\theta = c.$$

$$Pr(piv|\theta) \quad = \quad \binom{N}{qN} \left[1 - F\left(\frac{x^* - \theta}{\sigma}\right)\right]^{qN} \left[F\left(\frac{x^* - \theta}{\sigma}\right)\right]^{(1-q)N}.$$

Thus, focusing on $u(\theta, N) = (b_0 + b_1 N)\theta$ to match the standard games of the literature, the indifference condition that characterizes the equilibrium cutoffs is:

$$\begin{aligned}
c \quad &= \quad \int_{\theta=-\infty}^{\infty} u(\theta, N) \binom{N}{qN} \left[1 - F\left(\frac{x^* - \theta}{\sigma}\right)\right]^{qN} \left[F\left(\frac{x^* - \theta}{\sigma}\right)\right]^{(1-q)N} \frac{1}{\sigma}\, f\left(\frac{x^* - \theta}{\sigma}\right)\, d\theta \\
&= \quad \int_{z=0}^{1} u(x^* - \sigma\ F^{-1}(1-z), N) \binom{N}{qN} z^{qN}\ (1-z)^{(1-q)N}\ dz \\
&= \quad (b_0 + b_1 N) \int_{z=0}^{1} (x^* - \sigma\ F^{-1}(1-z)) \binom{N}{qN} z^{qN}\ (1-z)^{(1-q)N}\ dz \\
&\overbrace{=}^{large\ N} \quad b_1\ (x^* - \sigma\ F^{-1}(1-q)) \quad \text{(from Chamberlain and Rothschild (1981)).}
\end{aligned}$$

Thus,

$$x^* = \frac{c}{b_1} + \sigma\ F^{-1}(1-q).$$

Ignoring the directs costs of revolting, the net expected payoffs from revolting versus not revolting for a citizen i with signal x_i is:

$$\begin{aligned}
&\int_{\theta=-\infty}^{\infty} u(\theta, N) \binom{N}{qN} \left[1 - F\left(\frac{x^* - \theta}{\sigma}\right)\right]^{qN} \left[F\left(\frac{x^* - \theta}{\sigma}\right)\right]^{(1-q)N} \frac{1}{\sigma}\, f\left(\frac{x_i - \theta}{\sigma}\right)\, d\theta \\
= \quad &\int_{z=0}^{1} u(x^* - \sigma\ F^{-1}(1-z), N) \binom{N}{qN} z^{qN}\ (1-z)^{(1-q)N}\ \frac{f\left(\frac{x_i - x^*}{\sigma} + F^{-1}(1-z)\right)}{f(F^{-1}(1-z))}\ dz \\
= \quad &(b_0 + b_1 N) \int_{z=0}^{1} (x^* - \sigma\ F^{-1}(1-z)) \binom{N}{qN} z^{qN}\ (1-z)^{(1-q)N}\ \frac{f\left(\frac{x_i - x^*}{\sigma} + F^{-1}(1-z)\right)}{f(F^{-1}(1-z))}\ dz \\
\overbrace{=}^{large\ N} \quad &b_1\ (x^* - \sigma\ F^{-1}(1-q))\ \frac{f\left(\frac{x_i - x^*}{\sigma} + F^{-1}(1-q)\right)}{f(F^{-1}(1-q))}. \\
\overbrace{=}^{in\ equilibrium} \quad &c\ \frac{f\left(\frac{x_i - c/b_1}{\sigma}\right)}{f(F^{-1}(1-q))}.
\end{aligned}$$

In sum, we have established that if the best response to a cutoff strategy is indeed a cutoff strategy, then there is a unique equilibrium with x^* given above. Now, given this x^* that characterizes the strategies of other citizens, the net expected payoff from revolting versus not revolting for a citizen i with signal x_i is:

$$c \times \left(\frac{f\left(\frac{x_i - c/\alpha}{\sigma}\right)}{f(F^{-1}(1-q))} - 1 \right),$$

implying that i revolts if and only if

$$f\left(\frac{x_i - c/\alpha}{\sigma}\right) > f(F^{-1}(1-q)).$$

When f is strictly unimodal (e.g., Normal distribution), this expression does *not* have a single-crossing property: Either there is no crossing and i never revolts, or it has two crossings and i's best response is non-monotone.

Private Value Payoffs. Now, consider a private value payoff structure, so that a citizen with signal x_i receives $u(x_i, N)$. Then, mirroring the calculations for the common value case, we have:

$$x^* = \frac{c}{b_1} \quad \text{and} \quad \theta^* = x^* - \sigma\ F^{-1}(1-q) = \frac{c}{b_1} - \sigma\ F^{-1}(1-q), \tag{8}$$

where we recognize that, similar to our setting in the text, the fraction of citizens who participate in a revolution does not change with the regime's strength. Again, mirroring the calculations for the common value case, we have:

$$\begin{aligned}
B(x_i; x^*) &= \int_{\theta=-\infty}^{\infty} u(x_i, N) \binom{N}{qN} \left[1 - F\left(\frac{x^* - \theta}{\sigma}\right)\right]^{qN} \left[F\left(\frac{x^* - \theta}{\sigma}\right)\right]^{(1-q)N} \frac{1}{\sigma} f\left(\frac{x_i - \theta}{\sigma}\right) d\theta \\
&= \int_{z=0}^{1} u(x_i, N) \binom{N}{qN} z^{qN} (1-z)^{(1-q)N} \frac{f\left(\frac{x_i - x^*}{\sigma} + F^{-1}(1-z)\right)}{f(F^{-1}(1-z))} dz \\
&= (b_0 + b_1 N) \int_{z=0}^{1} x_i \binom{N}{qN} z^{qN} (1-z)^{(1-q)N} \frac{f\left(\frac{x_i - x^*}{\sigma} + F^{-1}(1-z)\right)}{f(F^{-1}(1-z))} dz \\
&\overbrace{=}^{large\ N} b_1\ x_i \frac{f\left(\frac{x_i - x^*}{\sigma} + F^{-1}(1-q)\right)}{f(F^{-1}(1-q))}. \\
&= b_1\ x_i \frac{f\left(\frac{x_i - c/b_1}{\sigma} + F^{-1}(1-q)\right)}{f(F^{-1}(1-q))} \quad \text{(in equilibrium, from (8)).}
\end{aligned}$$

Recall that given other citizens' cutoff strategy with associated cutoff x^*, citizen i with signal x_i revolts if and only if $B(x_i; x^*) > c$. Next, observe that $B(0; x^*) = \lim_{x_i \to \infty} B(x_i; x^*) = 0$. Thus, the best response to a finite-cutoff strategy is not a finite-cutoff strategy.