Appendix

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A Campaign Equilibria with one interest group

A.1 Proof of Proposition 1

Proof. We prove each result separately.

(a) Suppose there exists a positive advertising campaign equilibrium such that unadvertised challengers are winning. That is, suppose there exists an equilibrium strategy $\gamma(\theta)$ such that $\gamma(\theta) > 0$ for some $\theta \in (-z, z)$, and that $V_{\text{uninf}} \geq \frac{1}{2}$. Since $V_{\text{uninf}} \geq \frac{1}{2}$, we have that $\bar{x} \in (z, 0)$. Consider a challenger type $\theta' \in (-z, z)$ such that $\gamma(\theta') > 0$. Suppose the interest group deviates to $\gamma'(\theta') = 0$. Then the vote share of challenger type θ' is V_{uninf} which is weakly greater than $\frac{1}{2}$, and so, challenger type θ' wins the election against the incumbent. In fact, for all challenger types $\theta \in (-z, z)$ such that $\gamma(\theta) > 0$, the interest group can deviate to spending nothing. These challenger types would still

win, at no cost to the interest group. Thus, $\gamma(\theta)$ cannot constitute a PBE. Therefore, if $\gamma(\theta)$ is a positive advertising equilibrium strategy for the interest group, then $V_{\text{uninf}} < \frac{1}{2}$. And so, by the definition of V_{uninf}, we have that either $\bar{x} < z$ or $\bar{x} > 0$.

(b) Similarly, suppose that there exists a negative advertising campaign equilibrium such that unadvertised challengers are losing. That is, suppose there exists an equilibrium strategy $\gamma(\theta)$ such that $\gamma(\theta) > 0$ for some $\theta \in [-1, z) \cup (-z, 1]$, and that $V_{\text{uninf}} < \frac{1}{2}$. Since $V_{\text{uninf}} < \frac{1}{2}$, we have that either $\bar{x} < z$ or $\bar{x} > 0$. Consider a challenger type $\theta' \in [-1, z) \cup (-z, 1]$, such that $\gamma(\theta') > 0$. There are two possible cases, either $\theta' \in [-1, z)$ or $\theta' \in (-z, 1]$.

Case 1: $\theta' \in [-1, z)$. The interest group prefers that these challenger types lose to the incumbent. Suppose that the interest group deviates to $\gamma(\theta') = 0$. Then, the vote share of this challenger type is V_{uninf} which is strictly less than $\frac{1}{2}$, and so, challenger type θ' loses the election against the incumbent. For all challenger types $\theta \in [-1, z)$ with $\gamma(\theta) = 0$, the interest group can deviate to $\gamma(\theta) = 0$, and these challenger types would continue to lose against the incumbent, at no cost to the interest group.

Case 2: $\theta' \in (-z, 1]$. The interest group prefers that these challenger types win against the incumbent, but the median voter prefers the incumbent over these challenger types. Since the median voter prefers the incumbent z to θ', we have that $V_{\text{inf}}(\theta') < \frac{1}{2}$. As $V_{\text{uninf}} < \frac{1}{2}$, no matter what $\gamma(\theta')$, challenger θ' will lose to the incumbent. Thus, the interest group can deviate to $\gamma(\theta') = 0$. For all challenger types $\theta \in (-z, 1]$, the interest group can thus deviate to $\gamma(\theta) = 0$, because they will lose to the incumbent z no matter what.

Thus, $\gamma(\theta)$ cannot constitute a PBE. Therefore, if $\gamma(\theta)$ is a negative advertising campaign equilibrium strategy for the interest group, then $V_{\text{uninf}} \geq \frac{1}{2}$. And, as a result, we have that $\bar{x} \in (z, 0)$.

(c) Since V_{uninf} cannot be both weakly greater than and less than $\frac{1}{2}$, and because of parts *(a)* and *(b)*, we cannot have an equilibrium which exhibits both positive and negative advertising.

(d) By part *(a)*, in a positive advertising equilibrium, unadvertised challengers lose. Thus, it is not optimal for the interest group to advertise challengers that it prefers to lose the election ($\theta \in [-1, z)$) or challengers that it prefers to the incumbent but would lose the election if their ideal points were advertised for ($\theta \in (-z, 1]$). Thus, in a positive advertising equilibrium, if $\gamma(\theta) > 0$, then $\theta \in (z, -z]$, and by optimality of γ, equation

Equation 8 holds. If moreover $\theta \in (z, -z)$, then Equation 8) and part (a) imply that $V_{\inf}(\theta) > \frac{1}{2}$.

(e) By part *(b)*, in a negative advertising campaign equilibrium, unadvertised challengers win. Thus it is not optimal for the interest group to spend money on advertising any challengers that it prefers to the incumbent ($\theta \in (z, 1)$). Thus, $\gamma(\theta) > 0$ implies that $\theta < z$. Since θ is to the left of z, it is worse for the median than the incumbent is. And so, $V_{\inf}(\theta) < \frac{1}{2}$. □

A.2 Positive advertising equilibria

A.2.1 Proof of Lemma 1

Proof. By Proposition part (a) of Proposition1, the challenger receives a vote share of less than one half among uninformed voters. Thus, either $\bar{x} > 0$ or $\bar{x} < z$. In case $\bar{x} > 0$, we have for all $\theta \in (z, -z]$,

$$\Gamma(\theta) \quad = \quad \frac{G(\bar{x}) - \frac{1}{2}}{G(\bar{x}) - G\left(\frac{\theta+z}{2}\right)}.$$

Let $\mathfrak{R}_<$ denote the strictly negative real numbers. Define $A\colon \mathfrak{R}_< \to \mathfrak{R}$ and $B\colon [z, -z] \to \mathfrak{R}$ by

$$A(x) = \frac{G(\bar{x}) - \frac{1}{2}}{-x} \quad \text{and} \quad B(x) = G\left(\frac{x+z}{2}\right) - G(\bar{x}),$$

and note that we can write $\Gamma(\theta) = A(B(\theta))$. Clearly, using $\bar{x} > 0$, the function $A(\cdot)$ is strictly increasing and strictly convex. By Condition 1, $g(\cdot)$ is weakly single-peaked at zero, so that $g\left(\frac{\theta+z}{2}\right)$ is weakly increasing on $(z, -z]$, and thus $G\left(\frac{\theta+z}{2}\right)$ is weakly convex on $(z, -z]$. Moreover, as $g(\cdot)$ is positive on $(z, 0]$, it follows that $G\left(\frac{\theta+z}{2}\right)$ is strictly increasing on $(z, -z]$. Combining these observations, the function $B(\cdot)$ is strictly increasing and weakly convex. We conclude that Γ, viewed as the composition $A(B(\cdot))$, is strictly convex on $(z, -z]$.

In case $\bar{x} < z$, we have for all $\theta \in (z, -z]$,

$$\Gamma(\theta) \quad = \quad \frac{\frac{1}{2} - G(\bar{x})}{1 - G\left(\frac{\theta+z}{2}\right) - G(\bar{x})}.$$

Now define $A\colon \mathfrak{R}_< \to \mathfrak{R}$ and $B\colon (z, -z] \to \mathfrak{R}$ by

$$A(x) = \frac{\frac{1}{2} - G(\bar{x})}{-x} \quad \text{and} \quad B(x) = G\left(\frac{x+z}{2}\right) + G(\bar{x}) - 1,$$

so that again $\Gamma(\theta) = A(B(\theta))$ for all $\theta \in (z, -z]$. Again, using $\bar{x} < z < 0$, the function $A(\cdot)$ is strictly increasing and strictly convex, and the function $B(\cdot)$ is weakly convex and strictly increasing. Thus, Γ is strictly convex on $(z, -z]$, as required. □

A.2.2 Proof of Theorem 2

Proof. Since γ is positive advertising, either $\bar{x} < z$ or $\bar{x} > 0$. Note that

$$f(\theta|\text{uninf}) = \begin{cases} f(\theta)/\sigma & \text{if } \theta \notin [\underline{\theta}, \bar{\theta}], \\ f(\theta)(1-\gamma(\theta))/\sigma & \text{else,} \end{cases}$$

where $\sigma \leq 1$ is a scale factor defined by

$$\sigma = F(\underline{\theta}) + \int_{\underline{\theta}}^{\bar{\theta}} f(\theta)(1-\gamma(\theta))d\theta + (1 - F(\bar{\theta})).$$

Therefore, using the fact that $[\underline{\theta}, \bar{\theta}] \subseteq [z, -z]$, we have

$$\begin{aligned} \mathbb{E}[\theta|\text{uninf}] &= \int_{-1}^{1} \theta f(\theta|\text{uninf})d\theta \\ &= \frac{1}{\sigma}\left(\int_{-1}^{z} \theta f(\theta)d\theta + \int_{-z}^{1} \theta f(\theta)d\theta\right) + \int_{z}^{-z} \theta f(\theta|\text{uninf})d\theta. \end{aligned}$$

Letting $\tau = \frac{F(z)+1-F(-z)}{\sigma} \in [0,1]$, we can write

$$\mathbb{E}[\theta|\text{uninf}] = \tau\mathbb{E}[\theta|\theta \notin [z, -z]] + \int_{z}^{-z} \theta f(\theta|\text{uninf})d\theta. \tag{A.1}$$

In fact, it can be checked that $\int_{z}^{-z} f(\theta|\text{uninf})d\theta = 1 - \tau$. Then we can define a new density function $\tilde{f}(\cdot)$ by

$$\tilde{f}(\theta) = \begin{cases} \frac{f(\theta|\text{uninf})}{1-\tau} & \text{if } \theta \in [z, -z], \\ 0 & \text{else,} \end{cases}$$

and we can rewrite Equation A.1 as

$$\mathbb{E}[\theta|\text{uninf}] = \tau\mathbb{E}[\theta|\theta \notin [z, -z]] + (1-\tau)\mathbb{E}_{\tilde{f}}[\theta].$$

Clearly, we have $\mathbb{E}_{\tilde{f}}[\theta] > z$, and by assumption of the theorem, we have $\mathbb{E}[\theta|\theta \notin [z, -z]] > z$, and therefore $\mathbb{E}[\theta|\text{uninf}] > z$. This implies $\bar{x} > z$. Since either $\bar{x} < z$ or $\bar{x} > 0$, we conclude that $\bar{x} > 0$, as required. □

A.2.3 Proof of Theorem 3

Proof. The proof proceeds in four steps. First, we define the spending decisions of the interest group as a function of the position of the uninformed indifferent voter, $\bar{x}$. Second, we define the posterior expectation and variance of the challenger's location, conditional on a voter not observing an advertisement. Third, we use these moments to update the location of the uninformed indifferent voter. This gives us a continuous mapping $\psi\colon [0,1] \to [0,1]$, and by Brouwer's fixed point theorem, it admits a fixed point, $\bar{x}^*$. Finally, we show that $\bar{x}^*$ corresponds to the location of the indifferent uninformed voter in a campaign equilibrium, and we show that if α is sufficiently small, the equilibrium exhibits positive advertising.

First, to solve the optimal spending problem of the interest group, we define the cost function $\Gamma\colon [0,1] \times [z, -z) \to \mathfrak{R}$ by

$$\Gamma(\bar{x}|\theta) \quad = \quad \min\left\{1, \frac{G(\bar{x}) - \frac{1}{2}}{G(\bar{x}) - G\left(\frac{\theta+z}{2}\right)}\right\},$$

which gives the minimum level of spending required to ensure that the informed vote share of a challenger type $\theta \in [z, -z)$ is sufficiently high to secure the election for the challenger. Note that for all $\theta \in [z, -z)$, $\Gamma(\bar{x}|\theta)$ is continuous in $\bar{x}$. Given this cost function, define $\gamma\colon [0,1] \times [z, -z] \to \mathfrak{R}$ by

$$\gamma(\bar{x}|\theta) \quad = \quad \begin{cases} \Gamma(\bar{x}|\theta) & \text{if } \theta > -z \text{ and } \beta(\theta) \geq \alpha\Gamma(\bar{x}|\theta), \\ 0 & \text{else,} \end{cases}$$

which gives the optimal spending of the interest group in response to $\bar{x}$. Since β is strictly concave and $\Gamma(\bar{x}|\theta)$ is convex in θ, it follows that for almost all θ, $\gamma(\bar{x}|\theta)$ is continuous in $\bar{x}$.

Second, given the spending decisions of the interest group based on $\bar{x}$, we define the posterior mean and variance of an uninformed voter's beliefs about the challenger's location by

$$\mathbb{E}[\theta|\bar{x}] = \int_{-1}^{1} \theta f(\theta|\bar{x}) d\theta \tag{A.2}$$

$$\mathbb{V}[\theta|\bar{x}] = \int_{-1}^{1} (\theta - \mathbb{E}[\theta|\bar{x}])^2 f(\theta|\bar{x}) d\theta, \tag{A.3}$$

where the conditional density in Equation A.2 and Equation A.3 is defined as

$$f(\theta|\bar{x}) = \begin{cases} \dfrac{f(\theta)}{D(\bar{x})} & \text{if } \theta \in [-1, z) \cup (-z, 1] \\ \dfrac{(1-\gamma(\bar{x}|\theta))f(\theta)}{D(\bar{x})} & \text{if } \theta \in [z, -z], \end{cases} \tag{A.4}$$

and the denominator in Equation A.4 is defined as

$$D(\bar{x}) = \int_{-1}^{z} f(\theta)d\theta + \int_{z}^{-z} f(\theta)(1-\gamma(\bar{x}|\theta))d\theta + \int_{-z}^{1} f(\theta)d\theta.$$

Third, we use these moments to update the location of the uninformed indifferent voter according to a mapping $\psi: [0,1] \to [0,1]$, so that given any $\bar{x}$, the indifferent uninformed voter is located at $\psi(\bar{x})$. To define this function, we consider the analogue of Equation 5, namely,

$$x^* = \frac{\mathbb{E}[\theta'|\bar{x}] + z}{2} + \frac{\mathbb{V}[\theta'|\bar{x}}{2(\mathbb{E}[\theta'|\bar{x}] - z)},$$

and we set $\psi(\bar{x}) = \max\{-1, \min\{1, x^*\}\}$. Note that $\gamma(\bar{x}|\theta)$ is continuous in $\bar{x}$ for almost all $\theta \in [z, -z]$. And so, by an elementary application of Lebesgue's dominated convergence theorem, $D(\bar{x})$ is continuous in $\bar{x}$ for almost all θ. Similarly, another application of Lebesgue's dominated convergence theorem implies that the conditional density function $f(\theta|\bar{x})$ is continuous in $\bar{x}$ for almost all θ. Thus the conditional mean and variance functions are continuous, as is the above expression for x^*. We conclude that the mapping ψ is also continuous.

Therefore, since ψ is a continuous mapping on a set that maps into itself, Brouwer's fixed point theorem implies that ψ admits a fixed point, i.e., $\bar{x}^* = \psi(\bar{x}^*)$ for some cutpoint $\bar{x}^* \in [0,1]$. To show that $\bar{x}^*$ indeed corresponds to the uninformed indifferent voter, we prove that $\bar{x}^* > 0$. Suppose toward a contradiction that $\bar{x}^* = 0$. Then for all $\theta \in [z, -z)$, we have $\Gamma(0|\theta) = 0$, and thus $\gamma(0|\theta) = 0$ for all challenger types. This implies that uninformed voters do not update, i.e., $f(\theta|0) = f(\theta)$ for all $\theta \neq -z$, and we deduce that

$$u_0(z) \;>\; \int_{-1}^{1} u_0(\theta)f(\theta)d\theta \;=\; \int_{-1}^{1} u_0(\theta)f(\theta|\bar{x}^*)d\theta, \tag{A.5}$$

where the first inequality above follows from Condition 3. This implies that an uninformed voter at the median voter's location strictly prefers the incumbent, but then $\bar{x}^* > 0$, a contradiction. Therefore, $\bar{x}^* > 0$, as desired.

Finally, if the set of challenger types that are advertised, i.e.,

$$\Phi^* \quad = \quad \{\theta \in [z, -z] \mid \beta(\theta) \geq \alpha\Gamma(\bar{x}^*|\theta)\},$$

is nonempty, then let $\underline{\theta}^*$ and $\bar{\theta}^*$ be the endpoints of the interval, so that $\Phi^* = [\underline{\theta}^*, \bar{\theta}^*]$, and define the campaign spending strategy γ by $\gamma(\theta) = \Gamma(\theta|\bar{x}^*)$ for all $\theta \in [\underline{\theta}^*, \bar{\theta}^*]$ and $\gamma(\theta) = 0$ for all $\theta \notin [\underline{\theta}^*, \bar{\theta}^*]$. Otherwise, if Φ^* is empty, then set $\gamma \equiv 0$. By construction, this specification satisfies Equation 7, and therefore γ is a campaign equilibrium. Let $\tilde{x}$ solve $\max_{x \in [0,1]} \Gamma(x|0)$, and assume α is sufficiently small that $\alpha\Gamma(\tilde{x}|0) < u_1(0) - u_1(z)$. It follows that

$$\alpha\Gamma(\bar{x}^*|0) \;\leq\; \alpha\Gamma(\tilde{x}|0) \;<\; u_1(0) - u_1(z),$$

and we have $\underline{\theta}^* < \bar{\theta}^*$. Thus, the equilibrium exhibits positive advertising, as required. □

A.3 Negative advertising equilibria

A.3.1 Proof of Lemma 2

Proof. By part (b) of Proposition 1, the challenger receives a vote share of greater than one half among uninformed voters, so that $\bar{x} \in (z, 0)$. Thus, we have for all $\theta \in [-1, z)$,

$$\Gamma(\theta) \quad = \quad \frac{\frac{1}{2} - G(\bar{x})}{1 - G\left(\frac{\theta+z}{2}\right) - G(\bar{x})}.$$

Since G is strictly increasing on $\left[\frac{-1+z}{2}, z\right)$, by Condition 4, it follows immediately that Γ is strictly increasing on $[-1, z)$. Following the proof of Lemma 1, define $A\colon \Re_< \to \Re$ and $B\colon [-1, z] \to \Re$ by

$$A(x) = \frac{\frac{1}{2} - G(\bar{x})}{-x} \quad \text{and} \quad B(x) = G\left(\frac{x+z}{2}\right) + G(\bar{x}) - 1,$$

and note that $\Gamma(\theta) = A(B(\theta))$ for all $\theta \in [-1, z)$. Clearly, $A(\cdot)$ is strictly increasing and strictly convex. By Condition 4, $B(\cdot)$ is strictly increasing and weakly convex. We conclude that Γ, viewed as the composition $A(B(\cdot))$, is strictly convex on $[-1, z)$, as required. □

A.3.2 Proof of Theorem 5

Proof. Assume Condition 5, and suppose γ is a negative advertising equilibrium. Then $z \leq \bar{x} \leq 0$ and $\bar{\theta} < z$. Note that

$$f(\theta|\text{uninf}) \quad = \quad \begin{cases} f(\theta)/\sigma & \text{if } \theta > \bar{\theta} \\ f(\theta)(1-\gamma(\theta))/\sigma & \text{else,} \end{cases}$$

where $\sigma \leq 1$ is a scale factor defined by

$$\sigma = \int_{-1}^{\bar{\theta}} f(\theta)(1-\gamma(\theta))d\theta + (1-F(\bar{\theta})).$$

Therefore, we have

$$\begin{aligned}
\mathbb{E}[u_0(\theta)|\text{uninf}] &= \int_{-1}^{1} u_0(\theta) f(\theta|\text{uninf}) d\theta \\
&= \int_{-1}^{z} u_0(\theta) f(\theta|\text{uninf}) d\theta + \frac{1}{\sigma}\int_{z}^{1} u_0(\theta) f(\theta) d\theta \\
&\leq \left(\int_{-1}^{z} f(\theta|\text{uninf}) d\theta\right) u_0(z) + \frac{1}{\sigma}\int_{z}^{1} u_0(\theta) f(\theta) d\theta \\
&= \left(\int_{-1}^{z} f(\theta|\text{uninf}) d\theta\right) u_0(z) + \left(\frac{1-F(z)}{\sigma}\right)\mathbb{E}[u_0(\theta)|\theta \geq z],
\end{aligned}$$

where the inequality follows from $u_0(z) \geq u_0(\theta)$ for all $\theta \in [-1, z]$. Note that $\sigma \geq 1 - F(\bar{\theta}) \geq 1 - F(z)$, so we can write

$$\mathbb{E}[u_0(\theta)|\text{uninf}] \leq (1-\tau)u_0(z) + \tau\mathbb{E}[u_0(\theta)|\theta \geq z]$$

for $\tau = \frac{1-F(z)}{\sigma} \in [0,1]$. Finally, by Condition 5, we conclude that $\mathbb{E}[u_0(\theta)|\theta \geq z] < u_0(z)$, which contradicts $z < \bar{x} \leq 0$. □

A.3.3 Proof of Theorem 6

Proof. We define a fixed point mapping $\psi\colon [z,0] \to [z,0]$ in two steps. First, define $\psi^1\colon [z,0] \to [-1,z]$ so that given any cutpoint $\bar{x} \in [z,0]$, such that uninformed voters vote for the challenger if and only if $\theta > \bar{x}$, the interest group is indifferent between targeting the type $\psi^1(\bar{x})$ challenger or spending zero. To be more precise, we must consider the size of the campaign spending needed to secure victory for the incumbent, assuming the behavior of uninformed voters is given by the cutpoint $\bar{x}$. To this end, define the mapping $\Gamma\colon [-1,z] \times [z,0] \to \Re$ by

$$\Gamma(\theta|\bar{x}) = \frac{G(\bar{x}) - \frac{1}{2}}{G\left(\frac{\theta+z}{2}\right) + G(\bar{x}) - 1}.$$

Note that: $\alpha\Gamma(z|\bar{x}) > 0$, and when α is sufficiently small, $\alpha\Gamma(-1|\bar{x}) < u_1(z) - u_1(-1)$. If $\alpha\Gamma(z|\bar{x}) < u_1(z) - u_1(-1)$, then set $\bar{\theta} = z$. Otherwise, if $\alpha\Gamma(z|\bar{x}) \geq u_1(z) - u_1(-1)$, then since $\alpha\Gamma(\theta|\bar{x}) + u_1(\theta)$ is a continuous and strictly increasing function of $\theta \in [-1, z]$, it follows that there is a unique cutpoint $\bar{\theta}$ such that $\alpha\Gamma(\bar{\theta}|\bar{x}) = u_1(z) - u_1(\bar{\theta})$ holds, and we set $\psi^1(\bar{x}) = \bar{\theta}$. Moreover, since $\Gamma(\theta|\bar{x})$ is jointly continuous in θ and $\bar{x}$, it follows that $\psi^1(\bar{x})$ is a continuous function of $\bar{x}$.

Second, define $\psi^2 \colon [z, 0] \times [-1, z] \to [z, 0]$ so that given any pair $(\bar{x}, \bar{\theta}) \in [z, 0] \times [-1, z]$ of cutpoints, such that uninformed voters use $\bar{x}$ and the interest group targets a type θ challenger if and only if $\theta \leq \bar{\theta}$, the indifferent uninformed voter is located at $\psi^2(\bar{x}, \bar{\theta})$. To define this precisely, we must consider the updating of beliefs by uninformed voters, given the cutpoint. Assuming that the interest group spends just enough to defeat the challenger for any type $\theta \leq \bar{\theta}$, the uninformed voters' posterior beliefs are given by the density $f(\cdot|\bar{x}, \bar{\theta})$ defined as:

$$f(\theta|\bar{x}, \bar{\theta}) \;=\; \begin{cases} \frac{f(\theta)(1-\Gamma(\theta|\bar{x}))}{\int_{-1}^{\bar{\theta}} f(\theta')(1-\Gamma(\theta'|\bar{x}))d\theta' + (1-F(\bar{\theta}))} & \text{if } \theta \leq \bar{\theta}, \\ \frac{f(\theta)}{\int_{-1}^{\bar{\theta}} f(\theta')(1-\Gamma(\theta'|\bar{x}))d\theta' + (1-F(\bar{\theta}))} & \text{else.} \end{cases}$$

Denote the mean of this density by $\mathbb{E}[\theta|\bar{x}, \bar{\theta}]$ and the variance by $\mathbb{V}[\theta|\bar{x}, \bar{\theta}]$. Define

$$x^* \;=\; \frac{\mathbb{E}[\theta|\bar{x}, \bar{\theta}] + z}{2} + \frac{\mathbb{V}[\theta|\bar{x}, \bar{\theta}]}{2(\mathbb{E}[\theta|\bar{x}, \bar{\theta}] - z)},$$

and set $\psi^2(\bar{x}, \bar{\theta}) = \max\{z, \min\{0, x^*\}\}$. Note that $\psi^2(\bar{x}, \bar{\theta})$ is a jointly continuous function of $\bar{x}$ and $\bar{\theta}$.

Next, we define $\psi(\bar{x})$ as $\psi(\bar{x}) = \psi^2(\bar{x}, \psi^1(\bar{x}))$ for all $\bar{x} \in [z, 0]$. Since ψ^1 and ψ^2 are continuous, it follows that $\psi(\cdot)$ is a continuous function from $[z, 0]$ into itself, and then Brouwer's fixed point theorem implies that it possesses a fixed point, i.e., $\bar{x}^* = \psi(\bar{x}^*)$ for some cutpoint $\bar{x}^* \in [z, 0]$. Set $\bar{\theta}^* = \psi^1(\bar{x}^*)$. We claim that

$$\psi(\bar{x}^*) \;=\; \frac{\mathbb{E}[\theta|\bar{x}^*, \bar{\theta}^*] + z}{2} + \frac{\mathbb{V}[\theta|\bar{x}^*, \bar{\theta}^*]}{2(\mathbb{E}[\theta|\bar{x}^*, \bar{\theta}^*] - z)} \tag{A.6}$$

holds, so that $\bar{x}^*$ indeed corresponds to the uninformed indifferent voter. To this end, we prove that $\bar{x}^* \in (z, 0)$. Indeed, to see that $\psi(\bar{x}^*) > z$, we simply note that an uninformed voter located at z trivially prefers the incumbent to the challenger. Now, suppose toward a contradiction that $\bar{x}^* = 0$. Then for all $\theta \in [0, z]$, we have $\Gamma(\theta|z) = 0$. This implies that uninformed voters do not update, i.e., we have $f(\theta|\bar{x}^*, \bar{\theta}^*) = f(\theta)$ for all θ, and we deduce that

$$u_0(z) \;<\; \int_{-1}^{1} u_0(\theta)f(\theta)d\theta \;=\; \int_{-1}^{1} u_0(\theta)f(\theta|\bar{x}^*, \bar{\theta}^*)d\theta, \tag{A.7}$$

where the first inequality above follows from Condition 6. This implies that the uninformed voter at the median voter's location strictly prefers the challenger, but then $\bar{x}^* < 0$, a contradiction. Therefore, $\bar{x}^* < 0$, as desired.

Finally, define the campaign spending strategy γ by $\gamma(\theta) = \Gamma(\theta|\bar{x}^*)$ for all $\theta \in [-1, \bar{\theta}^*]$ and $\gamma(\theta) = 0$ for all $\theta > \bar{\theta}^*$. By construction, this specification satisfies Equation 7, and therefore γ is a campaign equilibrium. Let $\tilde{x}$ solve $\max_{x \in [z,0]} \Gamma(-1|x)$, and assume α is sufficiently small that $\alpha\Gamma(-1|\tilde{x}) < u_1(z) - u_1(-1)$. It follows that

$$\alpha\Gamma(-1|\bar{x}^*) \;\leq\; \alpha\Gamma(-1|\tilde{x}) \;<\; u_1(z) - u_1(-1),$$

and since $\alpha\Gamma(\theta|\bar{x}^*) + u_1(\theta)$ is increasing in $\theta \in [-1, z]$, we have $\bar{\theta}^* > -1$. Thus, the equilibrium exhibits negative advertising, as required. □

A.4 Proof of Corollary 1

Proof. For a proof by contradiction, suppose there is a sequence $\{z^m\}$ that converges to zero such that for each m, there is a campaign equilibrium γ^m such that for some θ, we have $\gamma(\theta) > 0$. Because $u_0(0) > \mathbb{E}[u_0(\theta)|\theta \geq 0]$, it follows that for sufficiently high m, Condition 5 holds, and by Theorem 5, with Proposition 1, it follows that γ^m is a positive advertising equilibrium. For such m, there exists θ^m such that $\beta(\theta^m) > \alpha\Gamma^m(\theta^m)$, where $\Gamma^m(\cdot)$ is the minimum spending to ensure victory of the challenger in the model with incumbent at z^m. By Theorem 1, we have $\theta^m \in (z^m, -z^m]$, and thus $\theta^m \to 0$ and $\beta(\theta^m) \to 0$.

Note that as the spending interval shrinks to $\{0\}$, the distribution of the challenger, conditional on being uninformed converges to the prior distribution. Using $\mathbb{E}^m[\theta|\text{uninf}]$ and $\mathbb{V}^m[\theta|\text{uninf}]$ to denote the mean and variance, conditional on being uninformed, in equilibrium γ^m, we then have $\mathbb{E}^m[\theta|\text{uninf}] \to \mathbb{E}[\theta]$ and $\mathbb{V}^m[\theta|\text{uninf}] \to \mathbb{V}[\theta]$. Since $\mathbb{E}[\theta] > 0$, it follows that $\mathbb{E}^m[\theta|\text{uninf}] > z^m$ for high enough m, and thus Condition 2 is satisfied. By Theorem 2, we conclude that $\bar{x}^m > 0$ for high enough m.

Going to a convergent subsequence, if needed, we can assume $\bar{x}^m \to \bar{x}$. Next, we claim that $\bar{x} = 0$, for suppose toward a contradiction that $\bar{x} > 0$. For all m, we have

$$\Gamma^m(\theta^m) \;=\; \frac{G(\bar{x}^m) - \frac{1}{2}}{G\left(\frac{\theta^m + z^m}{2}\right) + G(\bar{x}^m) - 1}.$$

Taking limits, we have

$$\lim_{m\to\infty} \Gamma^m(\theta^m) \;=\; \frac{G(\bar{x}) - \frac{1}{2}}{G(0) + G(\bar{x}) - 1} \;=\; \frac{G(\bar{x}) - \frac{1}{2}}{G(\bar{x}) - \frac{1}{2}} \;=\; 1,$$

and we deduce $\beta(\theta^m) < \alpha\Gamma^m(\theta^m)$ for high enough m, a contradiction. Thus, the claim holds.

Finally, note that for all m, the location of the indifferent uninformed voter is defined by

$$\bar{x}^m = \frac{\mathbb{E}^m[\theta|\text{uninf}] + z^m}{2} + \frac{\mathbb{V}^m[\theta|\text{uninf}]}{2(\mathbb{E}^m[\theta|\text{uninf}] - z^m)}.$$

Taking limits and multiplying both sides by $2\mathbb{E}[\theta]$, we deduce that

$$0 = (\mathbb{E}[\theta])^2 + \mathbb{V}[\theta] > 0,$$

a contradiction. We conclude that there is no such sequence $\{z^m\}$, and that when the incumbent is sufficiently moderate, campaign spending is zero in equilibrium. $\square$

B Campaign Equilibria with competing interest groups

B.1 Proof of Proposition 2

Proof. Let γ be a campaign equilibrium.

(a) Suppose toward a contradiction that $V_{\text{uninf}} = \frac{1}{2}$, and note that the indifferent uninformed voter is thus located at the median, i.e., $u_0(z) = \mathbb{E}[u_0(\theta)|\text{uninf}]$. Consider any challenger type $\theta \notin \{z, -z\}$. Then $V_{\text{inf}}(\theta) \neq \frac{1}{2}$. If $\theta \in (z, -z)$, then $V_{\text{inf}}(\theta) > \frac{1}{2}$. If interest group 1 spends any positive amount $\epsilon > 0$, then we have

$$V_{\text{tot}}(\epsilon, \gamma_{-1}(\theta)|\theta) = (\epsilon + \gamma_{-1}(\theta) - \epsilon\gamma_{-1}(\theta))\big(V_{\text{inf}}(\theta)\big) + (1-\epsilon)(1-\gamma_{-1}(\theta))\left(\frac{1}{2}\right) \;>\; \frac{1}{2},$$

so that the challenger wins. Since group 1 can ensure the challenger's victory at arbitrarily small cost, it follows that interest group 1 (along with group 2) in fact spends zero, and the challenger wins. By a similar logic, if $\theta \notin [z, -z]$, then again both groups spend zero, and the incumbent wins. In particular, if $V_{\text{uninf}} = \frac{1}{2}$, then for a set of challenger types with probability one, neither group advertises. But then voters do not update after failing to see and advertisement, so the posterior density over challengers equals the prior, i.e., $f(\cdot|\text{uninf}) = f(\cdot)$. But then $u_0(z) = \mathbb{E}[u_0(\theta)|\text{uninf}] = \mathbb{E}[u_0(\theta)]$, contradicting our assumption that the median voter is not ex ante indifferent between the candidates.

(b) Consider any challenger type $\theta \notin \{z, -z\}$, and assume without loss of generality that $\theta > z$. Suppose toward a contradiction that both groups spend a positive amount, i.e., $\gamma_1(\theta) > 0$ and $\gamma_{-1}(\theta) > 0$. There are three cases. First, assume that the challenger wins an outright majority, so that $V_{\text{tot}}(\gamma_1(\theta), \gamma_{-1}(\theta)|\theta) > \frac{1}{2}$. But then the outcome of the election is the worst candidate for group -1, and the group could deviate profitably by spending zero. Second, assume that the incumbent wins an outright majority, so that $V_{\text{tot}}(\gamma_1(\theta), \gamma_{-1}(\theta)|\theta) < \frac{1}{2}$. But then group 1 could profit by deviating to spending zero. In the third case, the election is tied, i.e., $V_{\text{tot}}(\gamma_1(\theta), \gamma_{-1}(\theta)|\theta) = \frac{1}{2}$, and the challenger wins with some probability $p = p(\gamma_1(\theta), \gamma_2(\theta), \theta)$.

In this case, note that if some interest group i spends $\gamma_i(\theta) = 1$, then we have

$$V_{\text{inf}}(\theta) \;=\; V_{\text{tot}}(\gamma_1(\theta), \gamma_{-1}(\theta)|\theta) \;=\; \frac{1}{2},$$

so that exactly half of informed voters prefer the challenger, but this implies $\theta = -z$, a contradiction. Thus, both candidates spend less than one. Next, if $V_{\text{uninf}} > \frac{1}{2}$, then we must have $V_{\text{inf}}(\theta) < \frac{1}{2}$. But then interest group 1 could deviate to spending zero, and the challenger's total vote share would be

$$\gamma_{-1}(\theta)V_{\text{inf}}(\theta)+(1-\gamma_{-1}(\theta))V_{\text{uninf}} \quad > \quad V_{\text{tot}}(\gamma_1(\theta),\gamma_{-1}(\theta)|\theta)$$

so that the challenger wins. This deviation would be profitable, and since this is impossible in equilibrium, we conclude that $V_{\text{uninf}} \leq \frac{1}{2}$. By an analogous argument, applied to interest group -1, we deduce the opposite inequality. Therefore, $V_{\text{uninf}} = \frac{1}{2}$. But since both groups spend less than one, then it follows that $V_{\text{inf}}(\theta) = \frac{1}{2}$, so again $\theta = -z$, a contradiction. $\square$

B.2 Positive advertising equilibria

B.2.1 Proof of Theorem 7

Proof. We prove the most interesting direction. Consider any campaign equilibrium γ such that interest group i uses a positive advertising strategy, so there exists $\theta \in (z,-z)$ such that $\gamma_i(\theta) > 0$. Note that $V_{\text{inf}}(\theta) > \frac{1}{2}$. By Proposition 2, we then have $\gamma_{-i}(\theta) = 0$. Since group i's spending is optimal, it follows that the payoff from spending $\gamma_i(\theta) > 0$ is at least equal to the payoff from spending zero, and thus the positive spending must strictly increase the probability that group i's preferred candidate wins. If i spends zero, then the total spending is zero, so all voters are uninformed, and the challengers vote share is V_{uninf}. By Proposition 2, we have $V_{\text{uninf}} \neq \frac{1}{2}$, and we conclude that i's preferred candidate wins with probability zero if i spends zero; in particular, $V_{\text{uninf}} < \frac{1}{2}$. If i spends $\gamma_i(\theta)$, then since $V_{\text{inf}}(\theta) > \frac{1}{2}$, the challenger's vote share is strictly greater than V_{uninf}, so the challenger's probability of victory cannot decrease as a result of the spending. Combining these observations, it follows that the preferred candidate of the group is the challenger, and thus $i = 1$.

Suppose toward a contradiction that there is a challenger type θ such that $\gamma_{-1}(\theta) > 0$. By Proposition 2, $\gamma_1(\theta) = 0$. Again, compared to spending zero, campaign expenditure $\gamma_{-1}(\theta)$ must strictly increase the probability that group -1's preferred candidate wins. If the group spent zero, the vote share of the challenger would be $V_{\text{uninf}} < \frac{1}{2}$, so the incumbent would win. Thus, group -1's preferred candidate is the challenger, i.e., $\theta < z$. But then $V_{\text{inf}}(\theta) < \frac{1}{2}$, and the campaign expenditure $\gamma_{-1}(\theta)$ cannot lead to a tie or victory for the challenger, i.e.,

$$V_{\text{tot}}(\gamma_i(\theta),0|\theta) \quad = \quad \gamma_i(\theta)V_{\text{inf}}(\theta)+(1-\gamma_i(\theta))V_{\text{uninf}} \; < \; \frac{1}{2},$$

a contradiction. Therefore, interest group −1 always spends zero, i.e., $\gamma_{-1} \equiv 0$.

Therefore, uninformed voters update as in Equation 2, i.e.,

$$f(\theta|\text{uninf}) = \frac{f(\theta)(1-\gamma_1(\theta))}{\int_0^1 f(\theta')(1-\gamma_1(\theta'))d\theta'}, \tag{B.1}$$

and the challenger's vote share among uninformed voters is as in Equation 6. We conclude that the optimal spending of interest group 1, given by γ_1, is as in Equation 7, and therefore γ_1 is a campaign equilibrium in the model with a single interest group. □

B.3 Negative advertising equilibria

B.3.1 Proof of Lemma 3

Proof. The first part is proved as in the proof of Lemma 1. For the second part, note that G is strictly increasing on $\left[0, \frac{1-z}{2}\right]$, by Condition 7, and since the equilibrium is negative advertising, we have $\bar{x} \in (z, 0)$. It follows that Γ_{-1} is strictly decreasing on $[-z, 1]$. Following the proof of Lemma 1, define $A\colon \mathfrak{R}_< \to \mathfrak{R}$ and $B\colon [-z, 1] \to \mathfrak{R}$ by

$$A(x) = \frac{\frac{1}{2} - G(\bar{x})}{-x} \quad \text{and} \quad B(x) = G(\bar{x}) - G\left(\frac{x+z}{2}\right),$$

and note that $\Gamma_{-1}(\theta) = A(B(\theta))$ for all $\theta \in [-z, 1]$. Clearly, $A(\cdot)$ is strictly increasing and strictly convex. By Condition 4, $B(\cdot)$ is strictly decreasing and weakly convex. We conclude that Γ_{-1}, viewed as the composition $A(B(\cdot))$, is strictly convex on $[-z, 1]$, as required. □

B.3.2 Proof of Theorem 8

Proof. We define a fixed point mapping $\psi\colon [z, 0] \to [z, 0]$ in two steps. First, we update the campaign spending strategies of the interest groups. As in the proof of Theorem 6, define $\psi_1^1\colon [z, 0] \to [-1, z]$ so that given any cutpoint $\bar{x} \in [z, 0]$, such that uninformed voters vote for the challenger if and only if $\theta > \bar{x}$, interest group 1 is indifferent between targeting the type $\psi_1^1(\bar{x})$ challenger or spending zero. To be more precise, define the mapping $\Gamma_1\colon [-1, z] \times [z, 0] \to \mathfrak{R}$ by

$$\Gamma_1(\theta|\bar{x}) = \frac{G(\bar{x}) - \frac{1}{2}}{G\left(\frac{\theta+z}{2}\right) + G(\bar{x}) - 1}.$$

As before, $\alpha\Gamma_1(z|\bar{x}) > 0$, and when α is sufficiently small, $\alpha\Gamma_1(-1|\bar{x}) < u_1(z) - u_1(-1)$. If $\alpha\Gamma_1(z|\bar{x}) < u_1(z) - u_1(-1)$, then set $\underline{\theta} = z$; and otherwise, if $\alpha\Gamma_1(z|\bar{x}) \geq u_1(z) - u_1(-1)$, then there is a unique cutpoint $\underline{\theta}$ such that $\alpha\Gamma_1(\underline{\theta}|\bar{x}) = u_1(z) - u_1(\underline{\theta})$ holds, and we set $\psi_1^1(\bar{x}) = \underline{\theta}$. Moreover, since $\Gamma_1(\theta|\bar{x})$ is jointly continuous in θ and $\bar{x}$, it follows that $\psi_1^1(\bar{x})$ is a continuous function of $\bar{x}$.

Similarly, define $\psi_{-1}^1 : [z, 0] \to [-z, 1]$ so that given any cutpoint $\bar{x} \in [z, 0]$, such that uninformed voters vote for the challenger if and only if $\theta > \bar{x}$, interest group -1 is indifferent between targeting the type $\psi_{-1}^1(\bar{x})$ challenger or spending zero. To be more precise, define the mapping $\Gamma_{-1} : [-z, 1] \times [z, 0] \to \mathfrak{R}$ by

$$\Gamma_{-1}(\theta|\bar{x}) \quad = \quad \frac{G(\bar{x}) - \frac{1}{2}}{G(\bar{x}) - G\left(\frac{\theta+z}{2}\right)}.$$

As above, $\alpha\Gamma_{-1}(z|\bar{x}) > 0$, and when α is sufficiently small, $\alpha\Gamma_{-1}(1|\bar{x}) < u_{-1}(z) - u_{-1}(1)$. If $\alpha\Gamma_{-1}(-z|\bar{x}) < u_{-1}(z) - u_{-1}(1)$, then set $\bar{\theta} = -z$; and otherwise, if $\alpha\Gamma_{-1}(-z|\bar{x}) \geq u_{-1}(z) - u_{-1}(1)$, then there is a unique cutpoint $\bar{\theta}$ such that $\alpha\Gamma_{-1}(\bar{\theta}|\bar{x}) = u_1(z) - u_1(\bar{\theta})$ holds, and we set $\psi_{-1}^1(\bar{x}) = \bar{\theta}$. Moreover, since $\Gamma_{-1}(\theta|\bar{x})$ is jointly continuous in θ and $\bar{x}$, it follows that $\psi_{-1}^1(\bar{x})$ is a continuous function of $\bar{x}$.

Second, define $\psi^2 : [z, 0] \times [-1, z] \times [-z, 1] \to [z, 0]$ so that given any triple $(\bar{x}, \underline{\theta}, \bar{\theta}) \in [z, 0] \times [-1, z] \in [-z, 1]$ of cutpoints, such that uninformed voters use $\bar{x}$, interest group 1 targets a type θ challenger if and only if $\theta \leq \underline{\theta}$, and interest group -1 targets a type θ challenger if and only if $\theta \geq \bar{\theta}$, the indifferent uninformed voter is located at $\psi^2(\bar{x}, \underline{\theta}, \bar{\theta})$. To define this precisely, we must consider the updating of beliefs by uninformed voters, given the cutpoint. The uninformed voters' posterior beliefs are given by the density $f(\cdot|\bar{x}, \underline{\theta}, \bar{\theta})$ defined as:

$$f(\theta|\bar{x}, \underline{\theta}, \bar{\theta}) \quad = \quad \begin{cases} \frac{f(\theta)(1-\Gamma_1(\theta|\bar{x}))}{\int_{-1}^{\underline{\theta}} f(\theta')(1-\Gamma_1(\theta'|\bar{x}))d\theta' + (F(\bar{\theta}) - F(\underline{\theta}) + \int_{\bar{\theta}}^{1} f(\theta')(1-\Gamma_{-1}(\theta'|\bar{x}))d\theta'} & \text{if } \theta \leq \underline{\theta}, \\ \frac{f(\theta)(1-\Gamma_{-1}(\theta|\bar{x}))}{\int_{-1}^{\underline{\theta}} f(\theta')(1-\Gamma_1(\theta'|\bar{x}))d\theta' + (F(\bar{\theta}) - F(\underline{\theta}) + \int_{\bar{\theta}}^{1} f(\theta')(1-\Gamma_{-1}(\theta'|\bar{x}))d\theta'} & \text{if } \theta \geq \bar{\theta}, \\ \frac{f(\theta)}{\int_{-1}^{\underline{\theta}} f(\theta')(1-\Gamma_1(\theta'|\bar{x}))d\theta' + (F(\bar{\theta}) - F(\underline{\theta}) + \int_{\bar{\theta}}^{1} f(\theta')(1-\Gamma_{-1}(\theta'|\bar{x}))d\theta'} & \text{else.} \end{cases}$$

Denote the mean of this density by $\mathbb{E}[\theta|\bar{x}, \underline{\theta}, \bar{\theta}]$ and the variance by $\mathbb{V}[\theta|\bar{x}, \underline{\theta}, \bar{\theta}]$. Define

$$x^* \quad = \quad \frac{\mathbb{E}[\theta|\bar{x}, \underline{\theta}, \bar{\theta}] + z}{2} + \frac{\mathbb{V}[\theta|\bar{x}, \underline{\theta}, \bar{\theta}]}{2(\mathbb{E}[\theta|\bar{x}, \underline{\theta}, \bar{\theta}] - z)},$$

and set $\psi^2(\bar{x}, \underline{\theta}, \bar{\theta}) = \max\{z, \min\{0, x^*\}\}$. Note that $\psi^2(\bar{x}, \underline{\theta}, \bar{\theta})$ is a jointly continuous function of $\bar{x}$, $\underline{\theta}$, and $\bar{\theta}$.

Next, we define $\psi(\bar{x})$ as $\psi(\bar{x}) = \psi^2(\bar{x}, \psi^1_{-1}(\bar{x}), \psi^1_{-1}(\bar{x}))$ for all $\bar{x} \in [z, 0]$. Since ψ^1_1, ψ^1_{-1}, and ψ^2 are continuous, it follows that $\psi(\cdot)$ is a continuous function from $[z, 0]$ into itself, and then Brouwer's fixed point theorem implies that it possesses a fixed point, i.e., $\bar{x}^* = \psi(\bar{x}^*)$ for some cutpoint $\bar{x}^* \in [z, 0]$. Set $\underline{\theta}^* = \psi^1_1(\bar{x}^*)$ and $\bar{\theta}^* = \psi^1_{-1}(\bar{x}^*)$. We claim that

$$\psi(\bar{x}^*) \;=\; \frac{\mathbb{E}[\theta|\bar{x}^*, \underline{\theta}^*, \bar{\theta}^*] + z}{2} + \frac{\mathbb{V}[\theta|\bar{x}^*, \underline{\theta}^*, \bar{\theta}^*]}{2(\mathbb{E}[\theta|\bar{x}^*, \underline{\theta}^*, \bar{\theta}^*] - z)} \tag{B.2}$$

holds, so that $\bar{x}^*$ indeed corresponds to the uninformed indifferent voter. To this end, we prove that $\bar{x}^* \in (z, 0)$. Indeed, to see that $\psi(\bar{x}^*) > z$, we simply note that an uninformed voter located at z trivially prefers the incumbent to the challenger. Now, suppose toward a contradiction that $\bar{x}^* = 0$. Then for all $\theta \in [0, z]$, we have $\Gamma_1(\theta|z) = \Gamma_{-1}(\theta|z) = 0$. This implies that uninformed voters do not update, i.e., we have $f(\theta|\bar{x}^*, \underline{\theta}^*, \bar{\theta}^*) = f(\theta)$ for all θ, and we deduce that

$$u_0(z) \;<\; \int_{-1}^{1} u_0(\theta) f(\theta) d\theta \;=\; \int_{-1}^{1} u_0(\theta) f(\theta|\bar{x}^*, \underline{\theta}^*, \bar{\theta}^*) d\theta, \tag{B.3}$$

where the first inequality above follows from Condition 6. This implies that the uninformed voter at the median voter's location strictly prefers the challenger, but then $\bar{x}^* < 0$, a contradiction. Therefore, $\bar{x}^* < 0$, as desired.

Finally, define the campaign spending strategy γ_1 by $\gamma_1(\theta) = \Gamma_1(\theta|\bar{x}^*)$ for all $\theta \in [-1, \underline{\theta}^*]$ and $\gamma_1(\theta) = 0$ for all $\theta > \underline{\theta}^*$; and define γ_{-1} by $\gamma_{-1}(\theta) = \Gamma_{-1}(\theta|\bar{x}^*)$ for all $\theta \in [\bar{\theta}^*, 1]$ and $\gamma_{-1}(\theta) = 0$ for all $\theta < \bar{\theta}^*$. By construction, this specification of $\gamma = (\gamma_1, \gamma_{-1})$ is a campaign equilibrium. Let $\tilde{x}$ solve $\max_{x \in [z,0]} \max\{\Gamma_1(-1|x), \Gamma_{-1}(1|x)\}$, and assume α is sufficiently small that $\alpha\Gamma_1(-1|\tilde{x}) < u_1(z) - u_1(-1)$ and $\alpha\Gamma_{-1}(1|\tilde{x}) < u_{-1}(z) - u_{-1}(1)$. It follows that

$$\alpha\Gamma_1(-1|\bar{x}^*) \;\leq\; \alpha\Gamma_1(-1|\tilde{x}) \;<\; u_1(z) - u_1(-1),$$

and

$$\alpha\Gamma_{-1}(1|\bar{x}^*) \;\leq\; \alpha\Gamma_{-1}(1|\tilde{x}) \;<\; u_{-1}(z) - u_{-1}(1).$$

Since $\alpha\Gamma_1(\theta|\bar{x}^*) + u_1(\theta)$ is increasing in $\theta \in [-1, z]$, and $\alpha\Gamma_{-1}(\theta|\bar{x}^*) + u_{-1}(\theta)$ is decreasing in $\theta \in [-z, 1]$, we have $\underline{\theta}^* > -1$ and $\bar{\theta} < 1$. Thus, the equilibrium campaign spending strategies of both groups exhibit negative advertising. The form of the campaign spending strategies described in (a)–(d) then follows from properties of Γ established in Lemma 3, as required. □

C Low Cost of Advertising

C.1 Proof of Theorem 9

For existence of a campaign equilibrium, note that either $\mathbb{E}[u_0(\theta)] < u_0(z)$, in which case Condition 3 holds, or $\mathbb{E}[u_0(\theta)] > u_0(z)$, in which case Condition 6 holds. Then Theorems 3 and 6 yield the existence result. Now, to prove that campaign equilibria exhibit positive or negative advertising when α is sufficiently small, suppose toward a contradiction that there is a sequence $\{\alpha_n\}$ of cost parameters such that $\alpha_n \to 0$, and such that for each n, there is a campaign equilibrium γ^n that exhibits neither positive nor negative advertising. Then for all n, the interest groups do not contribute for any challenger types. Thus, conditional on being uninformed, the posterior beliefs of the voters coincide with their priors, and in particular, $\mathbb{E}[u_0(\theta)|\text{uninf}] = \mathbb{E}[u_0(\theta)]$. For all n, either (i) $\mathbb{E}[u_0(\theta)] < u_0(z)$ or (ii) $\mathbb{E}[u_0(\theta)] > u_0(z)$ holds.

First, suppose there is a subsequence (still indexed by n) such that (i) holds. It follows that the median voter has a strict preference for the incumbent, and thus the incumbent always wins in equilibrium. But then for realizations θ close to zero, interest group 1 will have an incentive to advertise the challenger's position when n is high enough. Formally, let $\epsilon > 0$ be such that $z < -\epsilon$. For each $\theta \in (-\epsilon, \epsilon)$, the minimum level of campaign advertising sufficient to ensure victory of the challenger is

$$\Gamma_{>z}(\theta) \;=\; \frac{G(\bar{x}) - \frac{1}{2}}{G(\bar{x}) - G\left(\frac{\theta+z}{2}\right)} \;<\; \frac{G(\bar{x}) - \frac{1}{2}}{G(\bar{x}) - G\left(\frac{z+\epsilon}{2}\right)},$$

where we use the fact that $\bar{x} > 0$, owing to the assumption that the median voter strictly prefers the incumbent along the subsequence. Then the net benefit to group 1 from spending on advertising a challenger $\theta \in (-\epsilon, \epsilon)$ is at least equal to

$$u_1(-\epsilon) - u_1(z) - \alpha_n \left(\frac{G(\bar{x}) - \frac{1}{2}}{G(\bar{x}) - G\left(\frac{z+\epsilon}{2}\right)} \right),$$

which is positive for α_n sufficiently small. But then for high enough n, interest group 1 has a profitable deviation from γ_1^n, a contradiction.

Next, suppose there is a subsequence (still indexed by n) such that (ii) holds. It follows that the median voter has a strict preference for the challenger, and thus the challenger always wins in equilibrium. But then for realizations θ close to -1, interest group 1 will have an incentive to advertise the challenger's position when n is high. Formally, let $\epsilon \in (0, 1)$. For each $\theta \in (-1, -\epsilon)$, the minimum level of campaign advertising sufficient to secure victory for the incumbent is

$$\Gamma_{<z}(\theta) \;=\; \frac{G(\bar{x})-\frac{1}{2}}{G\left(\frac{\theta+z}{2}\right)+G(\bar{x})-1} \;<\; \frac{G(\bar{x})-\frac{1}{2}}{G\left(\frac{z-\epsilon}{2}\right)+G(\bar{x})-1},$$

where we use the fact that $\bar{x} \in (z,0)$, owing to the assumption that the median voter strictly prefers the challenger along the subsequence. Then the net benefit to group 1 from advertising the position of a challenger $\theta \in (-1,-\epsilon)$ is at least equal to

$$u_1(z)-u_1(-\epsilon)-\alpha_n\left(\frac{G(\bar{x})-\frac{1}{2}}{G\left(\frac{z-\epsilon}{2}\right)+G(\bar{x})-1},\right)$$

which is positive for α_n sufficiently small. But then for high enough n, interest group 1 has a profitable deviation from γ_1^n, a contradiction. We conclude that when the cost of advertising is sufficiently small, all equilibria exhibit either positive or negative advertising, as required.

To prove that Pr(error) $\to 0$ as $\alpha \to 0$, consider any sequence $\{\alpha_n\}$ of cost parameters with $\alpha_n \to 0$ and any sequence $\{\gamma^n\}$ of campaign equilibria such that $\gamma^n = (\gamma_1^n, \gamma_2^n)$ is an equilibrium in the game parameterized by α_n. By the foregoing, it suffices to consider two kinds of subsequences. First, consider a subsequence (still indexed by n) such that γ^n is a positive advertising equilibrium for all n. By Theorems 1 and 7, a challenger's position is advertised if located in the set $\Theta^n = \{\theta : \beta(\theta) > \alpha_n \Gamma^n(\theta)\}$, where Γ^n is indexed by n, as the indifferent uninformed voter may depend on α_n. Then the probability of error is

$$\Pr(\text{error}) \;=\; \int_{[z,-z]\setminus\Theta^n} f(\theta)d\theta.$$

Defining $g_n : \mathfrak{R} \to \mathfrak{R}$ by

$$g_n(\theta) \;=\; \begin{cases} f(\theta) & \text{if } \theta \in [z,-z]\setminus\Theta^n \\ 0 & \text{else,} \end{cases}$$

we then have $\Pr(\text{error}) = \int g_n(\theta)d\theta$. For each $\theta \in (z,-z)$, we have $\beta(\theta) > 0$, and thus for high enough n, we have $\theta \in \Theta^n$. Letting $g_0 \equiv 0$ be the zero function, it follows that for almost every θ, we have $g_n(\theta) \to g_0(\theta)$. The family $\{g_n\}$ is bounded in absolute value by $|f|$, and thus Lebesgue's dominated convergence theorem implies that

$$\Pr(\text{error}) \;=\; \int g_n(\theta)d\theta \;\to\; \int g_0(\theta)d\theta \;=\; 0,$$

so that the probability of error goes to zero.

Finally, consider a subsequence (still indexed by n) such that γ^n is a negative advertising equilibrium for all n. Define $\beta_1(\theta):[-1,1]\rightarrow\Re$ as follows

$$\beta_1(\theta)=(1-z)^2-(1-\theta)^2,$$

and $\beta_{-1}(\theta):[-1,1]\rightarrow\Re$ as

$$\beta_{-1}(\theta)=(-1-z)^2-(-1-\theta)^2.$$

Thus, $\beta_1(\theta)$ is the utility differential for interest group 1 from the challenger winning the election relative to the incumbent, and $\beta_{-1}(\theta)$ is the corresponding utility differential for interest group -1. By Theorems 6 and 8, a challenger's position is advertised if located in the set

$$\Theta^n=\{\theta:-\beta_1(\theta)>\alpha_n\Gamma_1^n(\theta)\}\cup\{\theta:-\beta_{-1}(\theta)>\alpha_n\Gamma_{-1}^n(\theta)\},$$

where Γ_1^n and Γ_{-1}^n are indexed by n, as the indifferent uninformed voter may depend on α_n. Then the probability of error is

$$\Pr(\text{error}) \;=\; \int_{[-1,z]\cup[-z,1]\setminus\Theta^n} f(\theta)d\theta.$$

As before, defining $g_n:\Re\rightarrow\Re$ by

$$g_n(\theta) \;=\; \begin{cases} f(\theta) & \text{if } \theta\in[-1,z]\cup[-z,1]\setminus\Theta^n \\ 0 & \text{else,}\end{cases}$$

we then have $\Pr(\text{error})=\int g_n(\theta)d\theta$. For each $\theta\in[-1,z)\cup(-z,1]$, we have $\beta_i(\theta)>0$ for each $i=1,-1$, and thus for high enough n, we have $\theta\in\Theta^n$. Letting $g_0\equiv 0$ be the zero function, it follows that for almost every θ, we have $g_n(\theta)\rightarrow g_0(\theta)$. The family $\{g_n\}$ is bounded in absolute value by $|f|$, and thus Lebesgue's dominated convergence theorem again implies that

$$\Pr(\text{error}) \;=\; \int g_n(\theta)d\theta \;\rightarrow\; \int g_0(\theta)d\theta \;=\; 0,$$

so that the probability of error goes to zero.