Polarized extremes and the confused centre: Campaign targeting of voters with correlation neglect

A Appendix

A.1 Appendix A: Proofs

A.1.1 Proof of Lemma 1

The symmetry around $k = \frac{n}{2}$ follows from $V(k) + V(n-k) = 1$ for all k.

Taking the first and second derivative of $V(\cdot)$ with respect to k we have:

$$\frac{\partial V(k)}{\partial k} = 2\ln(\frac{p}{1-p})\frac{p^n(1-p)^n}{[p^k(1-p)^{n-k}+(1-p)^k p^{n-k}]^2}$$

$$\frac{\partial^2 V(k)}{\partial k^2} = -4(\ln(\frac{p}{1-p}))^2\frac{p^n(1-p)^n}{[p^k(1-p)^{n-k}+(1-p)^k p^{n-k}]^3}(p^k(1-p)^{n-k} - (1-p)^k p^{n-k})$$

Hence $V(k)$ is concave (convex) if and only if $p^k(1-p)^{n-k} - (1-p)^k p^{n-k} > 0$ which is equivalent to $k > \frac{n}{2}$.

A.1.2 Proof of Proposition 1

Preliminary step: Denote by k the number of signals with realisation r. k is a sufficient statistic for the voters' expectation about the state of the world. Denote by Q_k^ω the total probability in state ω of having exactly k signals with realisation r. We restrict attention to anonymous distributions, i.e., distributions that allocate the same probability to any vector of signal realisations with the same sum $\sum_{i=1}^n s_i = k$. This restriction is without loss of generality as anonymous distributions are optimal among all, including non-anonymous ones. Denote by q_k^ω the probability assigned by the sender to some specific k signals having realisation r and the others having realisation l, in state ω. Then, $Q_k^\omega = \binom{n}{k} q_k^\omega$, and the marginal constraint can be written as:

$$p^\omega = \Pr(s_i = r|\omega) = \sum_{k=1}^n \Pr\left(\sum_j s_j = k|s_i = r, \omega\right) = \sum_{k=1}^n \binom{n-1}{k-1} q_k^\omega = \sum_{k=1}^n \frac{k}{n}\binom{n}{k} q_k^\omega = \sum_{k=1}^n \frac{k}{n} Q_k^\omega$$

The strategist problem becomes finding the set of probabilities $\{Q_k^\omega\}_{k=0}^n$ with $Q_k^\omega \geq 0$ and $\sum_{k=0}^n Q_k^\omega = 1$ to maximize $\sum_{k=0}^n Q_k^\omega V_k$ subject to the marginal constraint $\sum_{k=0}^n Q_k^\omega \frac{k}{n} = p^\omega$.

Lemma 2. *The optimal distribution never allocates positive weights to more than two points. There exists $x, y \in \{0, ..., n\}$ such that if $k \notin \{x, y\}$ $Q_k = 0$.*

Proof of Lemma 2: Fix state ω and suppose that $Q_x, Q_y, Q_z > 0$ for $x < y < z$.[1] Suppose we increase by ε the weight on y, and hence reduce the weight on x and z together by $-\varepsilon$. To satisfy the marginal constraint, we need $-x\beta + y\varepsilon - z(\varepsilon - \beta) = 0$ which implies $\beta = \frac{\varepsilon(z-y)}{z-x}$. Note that we cannot have $\beta = 0$ or $\beta = \varepsilon$. In terms of utility then, this is worth:

$$\Delta_{x,y,z} = -V(x)\left(\frac{z-y}{z-x}\right) + V(y) - V(z)\left(1 - \frac{z-y}{z-x}\right)$$

If $V(y) > V(x)\left(\frac{z-y}{z-x}\right) + V(z)\left(1 - \frac{z-y}{z-x}\right)$ (V concave over these three points), $\Delta_{x,y,z}$ is positive and hence we can increase the objective function by moving weight from x and z towards y. If $V(y) < V(x)\left(\frac{z-y}{z-x}\right) + V(z)\left(1 - \frac{z-y}{z-x}\right)$ (V convex over these three points), $\Delta_{x,y,z}$ is negative and we can increase the value of the objective function by moving weight from y towards x and y. If $V(y) = V(x)\left(\frac{z-y}{z-x}\right) + V(z)\left(1 - \frac{z-y}{z-x}\right)$, by Lemma 1, either there exists v with $y < v < z$ such that V is concave over $\{y, v, z\}$, and hence we can improve the objective function by decreasing Q_y, Q_z and increasing Q_v, or there exists $v' < x$ such that V is convex over $\{v', x, y\}$ and we can improve the objective function by decreasing Q_x and increasing $Q_{v'}, Q_y$.

Proof of Proposition 1: Define k^* such that $\frac{\partial V(k^*)}{\partial k} = \frac{V(k^*)-V(0)}{k^*}$. For simplicity we assume that k^* and np are both integers.[2] Given Lemma 2 there are at most two positive Q's. Let's denote them by Q_m, Q_k, with $m \leq k$. By the marginal constraint we know that $m \leq np \leq k$.

Suppose that $\frac{n}{2} \leq m \leq np \leq k$, then by the concavity of $V(\cdot)$ above $\frac{n}{2}$, $\Delta_{m,np,k} > 0$ and we would benefit by moving all the weight to np.

Suppose that $0 < m < \frac{n}{2} < np < k$, then $\Delta_{0,m,k} < 0$ so we would benefit by moving weight to the extremes 0 and k.

Hence, the optimal information structure is either $Q^\omega_{np} = 1$ or $Q^\omega_0 = 1 - \frac{np^\omega}{k}$, $Q^\omega_k = \frac{np^\omega}{k}$ for some $k > np^\omega$. Which one is better depends on whether $np^\omega < k^*$ or $np^\omega \geq k^*$. Suppose that $np^\omega < k^*$, then $\Delta_{0,np^\omega,k^*} < 0$ and putting weights on 0 and k^* is better than all weight at np^ω. Moreover, $\Delta_{0,k^*,k} > 0$ for all $k > k^*$ so the optimal information structure is $Q^\omega_0 = 1 - \frac{np^\omega}{k^*}$, $Q^\omega_k = \frac{np^\omega}{k^*}$. Suppose that $np^\omega \geq k^*$, then $\Delta_{0,np^\omega,k} > 0$ for all $k > np^\omega$ and putting all weight on np^ω is optimal.

Finally, the ex-ante expected belief is

$$E[V(k)] = \frac{1}{2}E[V(k) \mid \omega = 0] + \frac{1}{2}E[V(k) \mid \omega = 1] > \frac{1}{2}V(n(1-p)) + \frac{1}{2}V(np) = \frac{1}{2}.$$

[1] For simplicity of notation we ignore the reference to the state.

[2] Otherwise we would replace k^* by the highest integer such that $V(k) - V(0) \leq k(V(k) - V(k-1))$ and np by $k(p)$ and $k(p) - 1$ $k(p) - 1 < np < k(p)$.

A.1.3 Proof of Proposition 2

The proof replicates the proof of Proposition 1 replacing $V(\cdot)$ by the step function

$$a(k) = \begin{cases} 0 & if \quad V(k) < \frac{1}{2} \\ \\ 1 & if \quad V(k) > \frac{1}{2} \end{cases}$$

and where $k^* = \frac{n}{2}$.

A.1.4 Proof of Proposition 3

We provide the proof for $\omega = R$. The case $\omega = L$ is symmetric. We first show that the strategies proposed constitute an equilibrium. We start by showing that the strategies are feasible, that is, they satisfy the probability constraints (positive weights adding up to 1) and the marginal constraint. Using the notation of Proposition 1, the equilibrium strategies correspond to $Q_k = \frac{4(1-p)}{n+2} \equiv Q$ for $k \in \{0, .., \frac{n}{2} - 1\}$ and $Q_{\frac{n}{2}} = 1 - \frac{n}{2}\frac{4(1-p)}{n+2} = 1 - \frac{n}{2}Q$. It is easy to see that, $0 < Q_k < 1$ for all $0 \leq k \leq \frac{n}{2}$ and that $\sum_{k=0}^{\frac{n}{2}} Q_k = 1$, so the strategies are indeed a probability distribution. The marginal constraint becomes:

$$\sum_{k=0}^{\frac{n}{2}-1} Q\frac{k}{n/2} + Q_{\frac{n}{2}} = p$$

Replacing Q and $Q_{\frac{n}{2}}$ by their values,

$$\begin{aligned} \frac{Q}{n/2}\sum_{k=0}^{\frac{n}{2}-1} k + 1 - \frac{n}{2}Q &= \frac{Q(\frac{n}{2} - 1)}{2} + 1 - \frac{n}{2}Q \\ &= 1 - \frac{Q(\frac{n}{2} + 1)}{2} = p \end{aligned}$$

Finally, to see that this is an equilibrium note that the expected utility of S_R when she sets k_R signals with realisation r, and S_L plays according with the strategy is:

$$\begin{aligned} V(k_R) &= \Pr\left(k_L + k_R \geq \frac{n}{2}\right) \\ &= \sum_{k_L=\frac{n}{2}-k_R}^{\frac{n}{2}} Q_{k_L} \\ &= 1 - \frac{4(1-p)}{n+2}\left(\frac{n}{2} - k_R\right) \end{aligned}$$

which is linear in k_R. To see that there are no profitable deviations from this strategy, consider another information structure for S_R, $\{Q'_{k_R} = Q_{k_R} + \Delta_{k_R}\}_{k_R=0}^{\frac{n}{2}}$. For the deviation to be feasible, it needs to satisfy the following two constraints:

$$\sum \Delta_{k_R} = 0, \quad \sum \frac{k_R}{n/2}\Delta_{k_R} = 0.$$

But then the change in the expected utility for S_R given the deviation is:

$$\begin{aligned}
\sum \Delta_{k_R} V(k_R) &= \sum \Delta_{k_R} - \sum \Delta_{k_R} \frac{4(1-p)}{n+2}\left(\frac{n}{2} - k_R\right) \\
&= \sum \Delta_{k_R} - \frac{n}{2}\frac{4(1-p)}{n+2}\sum \Delta_{k_R} + \frac{4(1-p)}{n+2}\sum k_R \Delta_{k_R} \\
&= 0
\end{aligned}$$

So there is no profitable deviation for S_R. An analogous computation shows that the expected utility of S_L as a function of k_L given what S_R does and given the tie breaking rule, is also linear in k_L.

Finally, we show that this equilibrium is unique. Suppose that the strategies $\{Q_k^{S_R}\}_{k=0}^{n/2}$ and $\{Q_k^{S_L}\}_{k=0}^{n/2}$ constitute an equilibrium.

Claim 1: (i) If for S_R, $Q_k^{S_R} = 0$ for some $k < \frac{n}{2}$ then for S_L, $Q^{S_L}_{\frac{n}{2}-(k+1)} = 0$.
(ii) If for S_L, $Q_k^{S_L} = 0$ for some $k < \frac{n}{2}$ then for S_R, $Q^{S_R}_{\frac{n}{2}-k} = 0$.

Proof. Note that (ii) follows from (i) as they are mirror images. So we prove (i). Assume without loss that the state is $\omega = R$. Suppose that for some $k < \frac{n}{2}$, for $Q_k^{S_R} = 0$. Note that by the tie breaking rule, whenever the sum of signals with realisation r across both S_R and S_L are $\frac{n}{2}$ or higher S_R's utility is 1. This implies that S_L get the same level of utility from having $\frac{n}{2} - k$ signals with realisation r as from having $\frac{n}{2} - (k+1)$. As the expected utility of S_L is decreasing in the number of signals with realisation r, this implies that $\frac{n}{2} - (k+1)$ can never be part of an optimal solution for S_L; no concavication of his expected utility includes $\frac{n}{2} - (k+1)$ as a solution. Therefore, $Q^{S_L}_{\frac{n}{2}-(k+1)} = 0$. □

An implication of Claim 1 is that all equilibria are of the form whereby there exists a k^*, $p\frac{n}{2} \le k^* \le \frac{n}{2}$ such that the support of the strategy of S_R is $\{0, 1, ..., k^*\}$ and that of S_L is $\{\frac{n}{2} - k^*, \frac{n}{2} - k^* + 1, ..., \frac{n}{2}\}$.

Claim 2: Fix and equilibrium with k^*, for all $k < k^*$, $Q_k^{S_R} = \alpha$, $Q^{S_L}_{\frac{n}{2}-k} = \beta$

Proof. We provide a graphical proof. We know that the solution for S_R is to concavify his expected utility $V^{S_R}(k) = \Pr(k_L \ge \frac{n}{2} - k) = \sum_{k_L=\frac{n}{2}-k}^{n/2} Q_{k_L}^{S_L}$. The only way an optimal solution of this problem results in a support of $\{0, 1, ..., k^*\}$ is if $V^{S_R}(\cdot)$ is linear on $\{0, 1, ..., k^*\}$ which is what we need to prove. □

Consider S_L and assume that $k^* < \frac{n}{2}$. Since $V^{S_L}(\frac{n}{2} - k^*) = -Q^{S_R}_{k^*} < 0$ and $V^{S_L}(k) = -\sum_{k_R=\frac{n}{2}-k}^{k^*} Q^{S_R}_{k_R}$ for $k > \frac{n}{2} - k^*$ is linear, a convexification for S_L implies an optimal solution with strictly positive weights on $\frac{n}{2} - k^* - 1$ and $\frac{n}{2}$ only. This contradicts our premise of the equilibrium being characterised by k^*.

We conclude that the equilibrium has to have $k^* = \frac{n}{2}$ and by Claim 2 must take the form given in the proposition.■

A.2 Appendix B

We conclude by motivating our tie breaking rule leading to the result of Proposition 3. We consider a modified game in which the electorate can, at a cost, choose an informative signal before they vote.

First we show that we can replicate the exact equilibrium we have introduced in the proposition by assuming that the signal the electorate have at their disposal is fully informative and its cost, c, is not too low and not too high. We then show more generally, that for other specifications of this model we get qualitatively similar behaviour in equilibrium. In particular, we provide another example in which the signal available is not fully informative, and moderate voters end up being even more confused than they are in the equilibrium in Proposition 3.

Suppose first that the signal the electorate can acquire and is fully informative. Hence, if the median voter spends c, she gets a payoff $1 - c$. In the absence of this extra signal, the expected utility of the median voter is $\max\{V(k), 1 - V(k)\}$, which is lowest at $\frac{1}{2}$ when $k = \frac{n}{2}$, and second lowest at $\frac{p^2}{(1-p)^2+p^2}$ when $k = \frac{n}{2} + 1$ or $k = \frac{n}{2} - 1$.

Therefore, if c is such that $\frac{1}{2} < 1 - c < \frac{p^2}{(1-p)^2+p^2}$ (or equivalently, $\frac{(1-p)^2}{(1-p)^2+p^2} < c < \frac{1}{2}$) then the equilibrium of this extended model corresponds to the equilibrium in Proposition 3.

More generally, the signal does not need to be perfectly revealing and the costs might be such that the electorate acquires the signal also when the number of campaign outlets in each direction is not exactly the same. We show in an example that the qualitative result of the equilibrium in Proposition 3 is maintained.

Consider $n = 4$, and assume now that the signal the voter can acquire has accuracy β, i.e., the signals follows the correct state with probability β. Clearly, the median voter acquires this costly signal if and only if she would follow its recommendation. Let $\frac{(1-p)^4}{(1-p)^4+p^4} > \beta - c > \frac{p^2}{(1-p)^2+p^2}$. Then, the voter acquires and follows the signal unless in the political campaign she receives 4 realisations in favour of one of the options.

In this case we show that there is an equilibrium in which both strategists use the same strategy which mixes across all ks. Suppose without loss of generality that the state is R. Denote by $\{Q_0, Q_1, Q_2\}$, where

Q_k is the probability of sending k signals with realisation r, the strategy of S_L in state $\omega = R$. Given the accuracy constraint and the fact that the probabilities have to add up to 1, we can write $Q_1 = 2(1-p-Q_0)$ and $Q_2 = Q_0 + 2p - 1$, with $0 < Q_0 < 1 - p$.

We now show that we can find Q_0 such that $U_R(2) - U_R(1) = U_R(1) - U_R(0)$. In other words, S_R's utility is linear in k, implying that S_R is indifferent across all feasible strategies. In particular S_R could use the same strategy as S_L, which would imply that $U_L(k) \equiv 1 - U_R(k)$ is linear in k, and hence there is no profitable deviation for S_L either.

For any $\frac{p^2}{(1-p)^2+p^2} + c < \beta < \frac{p^4}{(1-p)^4+p^4} + c$,

$$\begin{aligned} U_R(2) &= (Q_0 + 2(1-p-Q_0))\beta + Q_0 + 2p - 1 \\ U_R(1) &= \beta \\ U_R(0) &= (1-Q_0)\beta \end{aligned}$$

Therefore, $U_R(2) - U_R(1) = U_R(1) - U_R(0)$ if and only if $Q_0 = (1-\beta)\frac{2p-1}{2\beta-1}$. Which implies that both strategists choosing $\{Q_0 = (1-\beta)\frac{2p-1}{2\beta-1}, Q_1 = \frac{2(\beta-p)}{2\beta-1}, Q_2 = \frac{\beta(2p-1)}{2\beta-1}\}$ is an equilibrium. (Note that when $\frac{p^2}{(1-p)^2+p^2} + c < \beta < \frac{p^4}{(1-p)^4+p^4} + c$, the constraint $0 < Q_0 < 1 - p$ is satisfied.)

This equilibrium construction captures the spirit of the equilibrium we had before. First, the strategy of both strategists is the same. Second, the strategies are relatively uninformative. In particular, the probability that the correct decision is taken in state R at the current equilibrium is:

$$Q_2^2 + \beta(1 - Q_0^2 - Q_2^2) = \beta + \beta(1-\beta)\frac{(2p-1)^2}{2\beta - 1}$$

This arises as when the state is R, the voter takes the correct state when both use $k = 2$, the wrong state when both use $k = 0$, and the correct state with probability β in all remaining cases.

Setting $\beta = \frac{p^2}{(1-p)^2+p^2}$ (this is the lowest β that satisfies the above constraints when $c = 0$), we get that for $p < p' \sim 0.7$ this probability is higher than $1 - \frac{4(1-p)^2}{3}$, which was the probability of correct decision in the original equilibrium.