A Theory of Power Wars
Online Appendix

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Proof of Lemma 2

Assume $m_t > p_t$. We need to show that $m_t + a_t > p_t + b_t$ and that $m_t - a_t > p_t - b_t$. We first show that $m_t + a_t > p_t + b_t$.

$$\begin{aligned} m_t + a_t &> p_t + b_t \\ m_t + g\left(1/2 - |m_t - 1/2|\right) &> p_t + g\left(1/2 - |p_t - 1/2|\right) \\ (m_t - p_t) &> g\left(|m_t - 1/2| - |p_t - 1/2|\right) \end{aligned}$$

There are three cases: (a) $1/2 > m_t > p_t$, (b) $m_t > p_t > 1/2$, (c) $m_t > 1/2 > p_t$.

In case (a), we have:

$$(m_t - p_t) > g\left((1/2 - m_t) - (1/2 - p_t)\right) = g\left(p_t - m_t\right)$$

which is satisfied, since $m_t > p_t$ and $g > 0$.

In case (b), we have:

$$(m_t - p_t) > g\left((m_t - 1/2) - (p_t - 1/2)\right) = g\left(m_t - p_t\right)$$

which is satisfied, since $g < 1$.

In case (c), we have:

$$(m_t - p_t) > g\left((m_t - 1/2) - (1/2 - p_t)\right) = g\left(m_t - (1 - p_t)\right)$$

which is satisfied, since $(1 - p_t) > p_t$ and $g < 1$.

We now show that $m_t - a_t > p_t - b_t$.

$$\begin{aligned} m_t - a_t &> p_t - b_t \\ m_t - g\left(1/2 - |m_t - 1/2|\right) &> p_t - g\left(1/2 - |p_t - 1/2|\right) \\ (m_t - p_t) &> g\left(|p_t - 1/2| - |m_t - 1/2|\right) \end{aligned}$$

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which is satisfied, since $g < 1$.

In case (b), we have:

$$(m_t - p_t) > g\left((p_t - 1/2) - (m_t - 1/2)\right) = g\left(p_t - m_t\right)$$

which is satisfied, since $m_t > p_t$ and $g > 0$.

In case (c), we have:

$$(m_t - p_t) > g\left((1/2 - p_t) - (m_t - 1/2)\right) = g\left((1 - m_t) - p_t\right)$$

which is satisfied, since $(1 - m_t) < m_t$ and $g < 1$. ■

Alternative Model with Bargaining

Consider the core model in the paper but — instead of assuming that peace-time consumption is proportional to relative political power — assume that, in each period, players can bargain over the allocation of that period's flow of resources before attacking or not as in the "canonical or standard model of the origins of war" discussed in Powell (2002). In this alternative model, the bargaining breakdown occurs purely because of dynamic considerations: the incentive to attack in the first period is driven by the expected gain in relative military and political power at the beginning of the second period (which, in turn, positively affects the expected share of the resources in the second-period peaceful bargaining agreement). Nonetheless, as in the models we discussed in the paper, conflict happens with positive probability and the mismatch between powers is still relevant for the chance of war.

Static Game and Value Functions

Since the game is one of complete information (i.e., players' outside options are common knowledge), then, not surprisingly, there is never conflict in the second and last period of the game. Denote with $x_2 \in [0,1]$ the allocation to player A in period 2. A prefers a peaceful agreement to war if and only if:

$$x_2 \geq m_2 - c_2 \tag{1}$$

Similarly, B prefers a peaceful agreement to war if and only if:

$$\begin{aligned} 1 - x_2 &\geq (1 - m_2) - c_2 \\ x_2 &\leq m_2 + c_2 \end{aligned} \tag{2}$$

Since $m_2 + c_2 \geq m_2 - c_2$, $m_2 + c_2 \geq 0$ and $m_2 - c_2 \leq 1$, there is always at least one feasible bargaining agreement, $x_2 \in [0,1]$, such that

$$m_2 + c_2 \geq x_2 \geq m_2 - c_2$$

In other words, there is always at least one feasible bargaining agreement, $x_2 \in [0,1]$, such that both players prefer peace to war. In fact, any

$$x_2 \in [\max\{0, m_2 - c_2\}, \min\{1, m_2 + c_2\}]$$

is a bargaining agreement which is feasible and dissuades both players from attacking. The exact point on this Pareto frontier where the second-period peaceful agreement lies depends on the players' relative bargaining power (which, in turn, can be a function of the details of the bargaining protocol employed). In this extension, we embody a player's relative bargaining power (or, in other words, the extent to which the player is able to appropriate the peace-time surplus) with his relative political power at the beginning of period 2, that is, p_2 for player A and $1 - p_2$ for player B. One way to micro-found this is to assume that the player recognized to be the agenda setter makes a take-it-or-leave-it offer to the opponent and that the chance of being recognized as the proposer is equal to a player's political power.[1]

We denote with $v_A(m_2, p_2)$ player A's expected utility from the second period when military and political power at the beginning of the second period are, respectively, m_2 and p_2. To avoid dealing with corner cases, we assume that, in the second period, both players can be made indifferent between a peaceful agreement and war with a non-negative transfer

[1]As discussed in the section of the paper on bargaining under the shadow of power, assuming that counter-offers are possible and that players alternate as proposers would lead to the same results as long as players discount an agreement which takes multiple rounds of proposals with a sufficiently low factor.

for any realization of the cost of war. This implies the following assumption:

Assumption 1 $c_2 \sim U[0, C]$ *where* $C \leq \{\min\{m_1, (1 - m_1)\} - a_1 \ \in (0, 1/2)$.

Then, in a peaceful bargaining agreement, both players will be granted at least their outside option (that is, their expected utility from war) and the peace-time surplus (that is, the total utility loss from war, $2c$) will be shared according to their relative bargaining power.

$$
\begin{aligned}
v_A(m_2, p_2) &= E[m_2 - c_2 + p_2(2c_2)] \\
&= m_2 + (2p_2 - 1)E[c_2] \\
&= m_2 + Cp_2 - \frac{C}{2} \qquad (3) \\
v_B(m_2, p_2) &= E[1 - m_2 - c_2 + (1 - p_2)(2c_2)] \\
&= 1 - m_2 - (2p_2 - 1)E[c_2] \\
&= 1 - m_2 - Cp_2 + \frac{C}{2} = 1 - v_A(m_2, p_2) \qquad (4)
\end{aligned}
$$

Note that, in this formulation, the players' relative political power — p_2 and $(1 - p_2)$ — are the players' Nash bargaining weights. These weights can also be micro-founded as the result of a non-cooperative bargaining game. For example, we would obtain the same expected values from second-period bargaining if, at the beginning of the second period, each player was selected to make a take-it-or-leave-it offer to the other player with a probability equal to his relative political power. The expected value from the second period is increasing in a player's relative military power (which determines his outside option) and in a player's relative political power (which determines his ability to appropriate the peace-time surplus).

Dynamic Game

For ease of notation, we can get rid of subscripts and denote m_1 with m and p_1 with p. Let $b \in (0, 1/2)$, $a \in (0, 1/2)$ and $\Delta(m, p)$ be, respectively, the change (in favor of the winner) of relative political power, relative military power and the expected utility from the second period in case of war in the first period.

Using the results from the previous section we have:

$$
\begin{aligned}
\Delta(m, p) &= v_A(m + a, p + b) - v_A(m, p) \\
&= m + a + C(p + b) - \frac{C}{2} - m - Cp - \frac{C}{2} \\
&= a + Cb \qquad (5)
\end{aligned}
$$

Let the transfer to A in the first-period bargaining agreement be $x_1 \in [0,1]$.

A prefers a peaceful agreement to war if and only if:

$$x_1 + v_A(m,p) \geq -c_1 + k(2m) + (1-k)\begin{pmatrix} m(p+b+v_A(m,p)+\Delta(m,p))+ \\ (1-m)(p-b+v_A(m,p)-\Delta(m,p)) \end{pmatrix} \quad (6)$$

Namely, defining K as below, we have:

$$x_1 \geq -c_1 + K \quad (7)$$

where:

$$\begin{aligned} K &= k(2m - v_A(m,p)) + (1-k)\left(p + (2m-1)(b+\Delta(m,p))\right) \\ &= k\left(m - Cp + \frac{C}{2}\right) + (1-k)\left(p + (2m-1)((1+C)b+a)\right) \end{aligned}$$

Similarly, B prefers a peaceful agreement to war if and only if:

$$1 - x_1 + v_B(m,p) \geq -c_1 + 2k(1-m) + (1-k)\begin{pmatrix} m(1-p-b+v_B-\Delta(m,p))+ \\ (1-m)(1-p+b+v_B(m,p)+\Delta(m,p)) \end{pmatrix}$$

Namely $x_1 \leq c_1 + K$. In sum, if there is a feasible bargaining agreement, $x_1 \in [0,1]$, such that

$$K + c_1 \geq x_1 \geq K - c_1$$

then, there is no war in the first period. Since $K + c_1 \geq K - c_1$, war can be avoided with a bargaining agreement if these two conditions are simultaneously satisfied:

$$K + c_1 \geq 0, \qquad K - c_1 \leq 1$$

Assumption 2 *Assume A is the militarily advantaged player, that is, $m \geq 1/2$.*

Under this assumption, we have $K + c_1 \geq 0$ for any $c_1 \in [0,1]$, and the first condition is always satisfied. Thus, there is no first-period war if and only if the second condition

$$K - c_1 \leq 1 \quad (8)$$

is satisfied for any $c_1 \in [0,1]$. This means that we have war with positive probability if and only if $K > 1 + c_1$ for some $c_1 \in [0,1]$. Thus, there is war with positive probability if and only if $K > 1$. Proposition 1 summarizes the above discussion.

Proposition 1 *When players can bargain over the allocation of resources in peace, then only if $K > 1$ there can be a first-period war and its probability is:*

$$Pr[War] = \max\{0, K - 1\} \tag{9}$$

Special Cases

We now analyze some special cases to clarify the forces at stake.

Special Case 1: Always Decisive War ($k = 1$)

Proposition 2 *When players can bargain over the allocation of resources in peace and war is always decisive ($k = 1$), there is a positive probability of first-period war if and only if the mismatch between military and political power is sufficiently large; the probability of first-period war is weakly increasing in the mismatch between military and political power (strictly increasing in the region of parameters where war happens with positive probability).*

Proof. There is war with positive probability if and only if

$$m - Cp > 1 - \frac{C}{2} \tag{10}$$

Since the LHS goes to 1 as m goes to 1 and p goes to 0 while the RHS is bounded away from 1 (since C is strictly positive), there are initial allocations of relative military and political power, (m, p), which conduce to a positive probability of war in the first period. In particular, there is a positive probability of war only if $m \geq p$ (to see this, note that the inequality can never be satisfied when $p = m$ and that the LHS is decreasing in p so the inequality is even harder to be satisfied when $p > m$). Indeed, there is a positive probability of war if and only if the mismatch between military and political power, $m - p$ is sufficiently large. When this is the case, the probability of war in the first period is

$$\Pr[\text{War}] = m - Cp + \frac{C}{2} - 1$$

which is strictly increasing in the mismatch between military and political power. ■

Special Case 2: Never Decisive War ($k = 0$)

Proposition 3 *When players can bargain over the allocation of resources in peace and war is never decisive ($k = 0$), there is a positive probability of first-period war if and only if the* *militarily advantaged player has a sufficiently large political power and the power shifts in case of war are sufficiently large.*

Proof. There is war with positive probability if and only if

$$p + (2m-1)((1+C)b + a) \quad > \quad 1 \tag{11}$$

and, in this case, the probability of war is

$$\Pr[\text{War}] \quad = \quad p + (2m-1)((1+C)b + a) - 1 \tag{12}$$

which is increasing in p, m, a and b. ■

For example under our power shift specification described in the analysis of the core model we have that if $p \leq 1/2$, we have $a = g(1-m)$ and $b = gp$, so the probability of war is

$$\Pr[\text{War}] = \max\{0, p + (2m-1)g(p(1+C) + 1 - m) - 1\} \tag{13}$$

which is strictly increasing in p, C, g; it is also strictly increasing in m if $m < \frac{(1+C)}{2}p + \frac{3}{4}$. Whereas if $p \geq 1/2$, we have $a = g(1-m)$ and $b = g(1-p)$, so the probability of war is

$$\Pr[\text{War}] = \max\{0, p + (2m-1)((1+C)g(1-p) + g(1-m)) - 1\} \tag{14}$$

which is strictly increasing in p, C, g; it is strictly increasing in m if m and p are sufficiently close to 1/2 and strictly decreasing in m if m and p are sufficiently close to 1.

Special Case 3: No Military Advantage ($m_1 = 1/2$)

Proposition 4 *When players can bargain over the allocation of resources in peace and no player is militarily advantaged ($m_1 = 1/2$), there is never a first-period war.*

Proof. There is a first-period war with positive probability if and only if

$$\begin{aligned} k\left(\frac{1}{2} - Cp + \frac{C}{2}\right) + (1-k)p \quad &> \quad 1 \\ k(1+C)\left(\frac{1}{2} - p\right) + p \quad &> \quad 1 \end{aligned}$$

which is not satisfied for any $p \in [0,1]$. ■

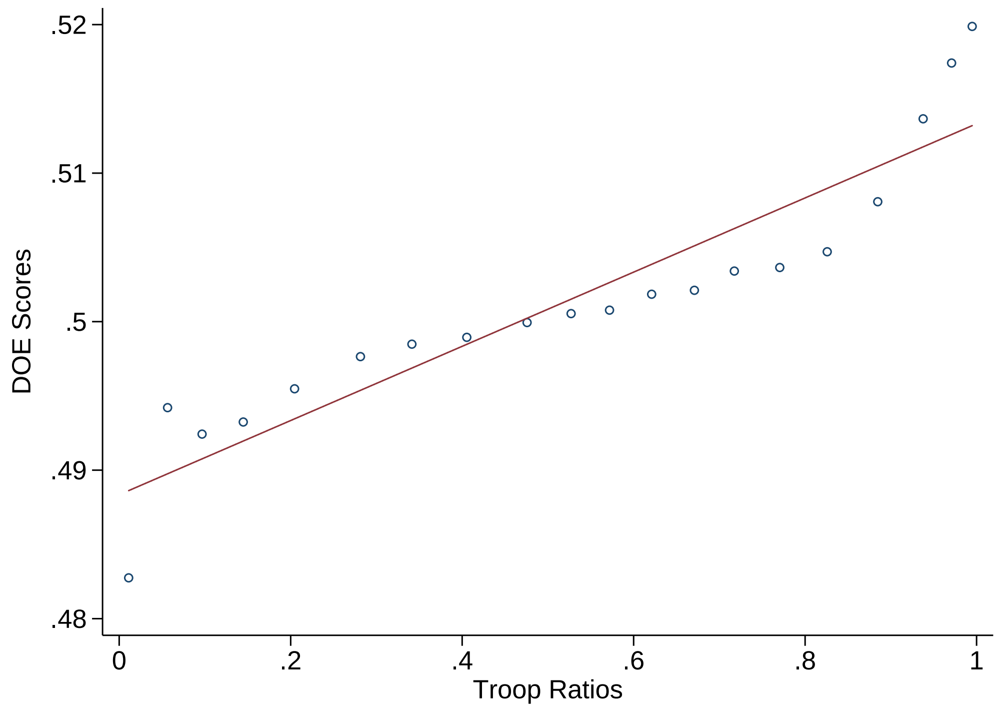

Figure 1: Binscatter DOE vs. Troop Ratio

Validity of Proxy for Relative Military Power

To test the validity of our military power proxy, it was measured against some relevant developments in the literature. A recent paper by Carroll and Kenkel (2019) applied a superlearner algorithm to militarized dispute data to create a Dispute Outcome Expectation score. This score seeks to provide probabilities on three outcomes: A wins, B wins, or stalemate. As our model does not contemplate stalemate as an ex-ante possibility, we created a ratio between the relative winning probabilities to compare to our troop ratio proxy. The correlation between the two measures, as shown by Table 2, is nearly 0.6, indicating that our measure is indeed a good proxy for military power. Moreover, the figure displays a binscatter plot of the two measures, demonstrating the positive relationship between the two.

	DOE	Mil. Prop
DOE	1	
Mil.Prop	0.5603	1

Table 2: Correlation between the two military measures