Online Appendix
Social Conflict and the Predatory State

Contents

A.1 Additional Notation

Throughout the appendix, let $\Phi(c) = \sum_i \phi(c_i)$. We have $\log \omega_i(c) = \log \phi(c_i) - \log \Phi(c)$ and thus

$$\begin{aligned}\frac{\partial \log \omega_i(c)}{\partial c_i} &= \frac{\phi'(c_i)}{\phi(c_i)} - \frac{\phi'(c_i)}{\Phi(c)} \\ &= \frac{\phi'(c_i)}{\phi(c_i)}\left(1 - \frac{\phi(c_i)}{\Phi(c)}\right) \\ &= \hat{\phi}'(c_i)(1 - \omega_i(c)),\end{aligned}$$

where $\hat{\phi} = \log \phi$. Because ϕ is strictly increasing and log-concave, $\hat{\phi}' > 0$ and $\hat{\phi}'' \leq 0$.

A.2 Equilibrium Existence and Uniqueness

For the existence and uniqueness results, I consider a more general version of the model presented in the text. I allow groups to be asymmetric in their size and productivities, which entails generalizing each faction i's budget constraint (1) to

$$\frac{p_i}{\pi_i^p} + \frac{r_i}{\pi_i^r} + \frac{c_i}{\pi_i^c} = L_i, \tag{A.1}$$

where $\pi_i^p, \pi_i^r, \pi_i^c, L_i > 0$. I assume a labor-financed government throughout the existence and uniqueness results, as this is the more difficult case; all claims here also apply to a capital-financed state in which $f(p) = X$. In addition, the results here do not depend on Assumption 1.

Let $\Gamma(t)$ denote the subgame that follows the government's selection of t, in which the factions simultaneously decide how to allocate their labor. Let $\sigma_i = (p_i, r_i, c_i)$ be a strategy for faction i in the subgame, and let

$$\Sigma_i = \left\{ (p_i, r_i, c_i) \;\middle|\; \frac{p_i}{\pi_i^p} + \frac{r_i}{\pi_i^r} + \frac{c_i}{\pi_i^c} = L_i \right\}$$

denote the strategy space. Let $\sigma = (\sigma_1, \ldots, \sigma_N)$ and $\Sigma = \bigtimes_{i=1}^N \Sigma_i$.

I begin by proving that a Nash equilibrium exists in each subgame. The task is complicated by the potential discontinuity of the factions' payoffs, namely at $c = 0$ when $\phi(0) = 0$. I rely on Reny's (1999) conditions for the existence of pure strategy equilibria in a discontinuous game. The key condition is *better-reply security*—informally, that at least one player can assure a strict benefit by deviating from any non-equilibrium strategy profile, even if the other players make slight deviations.

Lemma A.1. $\Gamma(t)$ *is better-reply secure.*

Proof. Let $U^t : \Sigma \to \mathbb{R}_+^N$ be the vector payoff function for the factions in $\Gamma(t)$, so that $U^t(\sigma) = (u_1(t, \sigma), \ldots, u_N(t, \sigma))$. Take any convergent sequence in the graph of U^t, call it $(\sigma^k, U^t(\sigma^k)) \to (\sigma^*, U^*)$, such that σ^* is not an equilibrium of $\Gamma(t)$. Because production and the effective tax rate are continuous in (p, r), we have

$$U_i^* = w_i^* \times \bar{\tau}(t, r^*) \times f(p^*)$$

for each i, where $w_i^* \geq 0$ and $\sum_{i=1}^N w_i^* = 1$. I must show there is a player i who can secure a payoff $\bar{U}_i > U_i^*$ at σ^*; i.e., there exists $\bar{\sigma}_i \in \Sigma_i$ such that $u_i(t, \bar{\sigma}_i, \sigma'_{-i}) \geq \bar{U}_i$ for all σ'_{-i} in a neighborhood of σ^*_{-i} (Reny 1999, 1032).

If $N = 1$ or $\Phi(c^*) > 0$, then U^t is continuous in a neighborhood of σ^*, so the conclusion is immediate. If $\bar{\tau}(t, r^*) \times f(p^*) = 0$, then each $U_i^* = 0$ and each faction can assure a strictly greater payoff by deviating to a strategy with positive production, resistance, and conflict.

For the remaining cases, suppose $N > 1$, $\bar{\tau}(t, r^*) \times f(p^*) > 0$, and $\Phi(c^*) = 0$, the latter of which implies $c^* = 0$ and $\phi(0) = 0$. Since $N > 1$, there is a faction i such that $w_i^* < 1$. Take any $\epsilon \in (0, (1 - w_i^*)/2)$ and any $\delta_1 > 0$ such that

$$\bar{\tau}(t, r') \times f(p') \geq (w_i^* + 2\epsilon) \times \bar{\tau}(t, r^*) \times f(p^*)$$

for all σ' in a δ_1-neighborhood of σ^*. Since $w_i^* + 2\epsilon < 1$ and $\bar{\tau}(t, r) \times f(p)$ is continuous in (p, r), such a δ_1 exists. Then let $\bar{\sigma}_i = (\bar{p}_i, \bar{r}_i, \bar{c}_i)$ be any strategy in a δ_1-neighborhood of σ_i^* such that $\bar{c}_i > 0$. Because $c_{-i}^* = 0$ and ϕ is continuous, there exists $\delta_2 > 0$ such that

$$\omega_i(\bar{c}_i, c'_{-i}) = \frac{\phi(\bar{c}_i)}{\phi(\bar{c}_i) + \sum_{j \in \mathcal{N} \setminus \{i\}} \phi(c'_j)} \geq \frac{w_i^* + \epsilon}{w_i^* + 2\epsilon}$$

for all σ'_{-i} in a δ_2-neighborhood of σ_{-i}^*. Therefore, for all σ'_{-i} in a $\min\{\delta_1, \delta_2\}$-neighborhood of σ_{-i}^*, we have

$$u_i(t, \bar{\sigma}_i, \sigma'_{-i}) \geq (w_i^* + \epsilon) \times \bar{\tau}(t, r^*) \times f(p^*) > U_i^*,$$

establishing the claim. □

The other main condition for equilibrium existence is that each faction's utility function be quasiconcave in its own actions. I prove this by showing that the logarithm of a faction's utility function is concave in its actions.

Lemma A.2. $\Gamma(t)$ *is log-concave.*

Proof. Take any (p, r, c) such that $u_i(t, p, r, c) > 0$, and let $P = \sum_j p_j$ and $R = \sum_j r_j$. First, assume $\sum_{j \neq i} \phi(c_j) > 0$, so that u_i is continuously differentiable in (p_i, r_i, c_i). We have

$$\begin{aligned}
\frac{\partial \log u_i(t, p, r, c)}{\partial p_i} &= \frac{1}{P}, \\
\frac{\partial \log u_i(t, p, r, c)}{\partial r_i} &= \frac{-tg'(R)}{1 - tg(R)}, \\
\frac{\partial \log u_i(t, p, r, c)}{\partial c_i} &= \hat{\phi}'(c_i)(1 - \omega_i(c)),
\end{aligned}$$

and therefore

$$\begin{aligned}
\frac{\partial^2 \log u_i(t, p, r, c)}{\partial p_i^2} &= \frac{-1}{P^2} < 0, \\
\frac{\partial^2 \log u_i(t, p, r, c)}{\partial r_i^2} &= \frac{-tg''(R)(1 - tg(R)) - (tg'(R))^2}{(1 - tg(R))^2} \leq 0,
\end{aligned}$$

$$\frac{\partial^2 \log u_i(t,p,r,c)}{\partial c_i^2} = \hat{\phi}''(c_i)(1-\omega_i(c)) - \hat{\phi}'(c_i)\frac{\partial \omega_i(c)}{\partial c_i} \leq 0,$$
$$\frac{\partial^2 \log u_i(t,p,r,c)}{\partial p_i \partial r_i} = \frac{\partial^2 \log u_i(t,p,r,c)}{\partial p_i \partial c_i} = \frac{\partial^2 \log u_i(t,p,r,c)}{\partial r_i \partial c_i} = 0,$$

so $\log u_i$ is concave in (p_i, r_i, c_i). By the same token, $\bar{\tau}(t,r) \times f(p)$ is log-concave in (p,r).

Now assume $\sum_{j \neq i} \phi(c_j) = 0$. Take any (p_i', r_i', c_i') such that $u_i(t, p', r', c') > 0$, where $(p', r', c') = ((p_i', p_{-i}), (r_i', r_{-i}), (c_i', c_{-i}))$. Take any $\alpha \in [0,1]$, and let $(p^\alpha, r^\alpha, c^\alpha) = \alpha(p,r,c) + (1-\alpha)(p', r', c')$. If $c_i = c_i' = 0$, then $\omega_i(c^\alpha) = \omega_i(c) = \omega_i(c') = 1/N$ and thus

$$\begin{aligned}
\log u_i(t, p^\alpha, r^\alpha, c^\alpha) &= \log \frac{1}{N} + \log \bar{\tau}(t, r^\alpha) + \log f(p^\alpha) \\
&\geq \log \frac{1}{N} + \alpha \left(\log \bar{\tau}(t,r) + \log f(p)\right) + (1-\alpha)\left(\log \bar{\tau}(t,r') + \log f(p')\right) \\
&= \alpha \log u_i(t,p,r,c) + (1-\alpha) \log u_i(t, p', r', c'),
\end{aligned}$$

where the inequality follows from the log-concavity of $\bar{\tau}(t,r) \times f(p)$ in (p,r). If $c_i > 0$ and $c_i' = 0$, then $\omega_i(c^\alpha) = \omega_i(c) = 1$, $\omega_i(c') = 1/N$, and thus

$$\begin{aligned}
\log u_i(t, p^\alpha, r^\alpha, c^\alpha) &= \log \bar{\tau}(t, r^\alpha) + \log f(p^\alpha) \\
&\geq \alpha \left(\log \bar{\tau}(t,r) + \log f(p)\right) + (1-\alpha)\left(\log \frac{1}{N} + \log \bar{\tau}(t,r') + \log f(p')\right) \\
&= \alpha \log u_i(t,p,r,c) + (1-\alpha) \log u_i(t, p', r', c').
\end{aligned}$$

The same argument holds in case $c_i = 0$ and $c_i' > 0$. It is easy to see that the same conclusion holds if $c_i > 0$ and $c_i' > 0$, in which case $\omega_i(c^\alpha) = \omega_i(c) = \omega_i(c') = 1$. Therefore, $\log u_i$ is concave in (p_i, r_i, c_i). □

Equilibrium existence follows immediately from the two preceding lemmas.

Proposition A.1. $\Gamma(t)$ *has a pure strategy equilibrium.*

Proof. The strategy space Σ is compact, each payoff function u_i is bounded on Σ, and $\Gamma(t)$ is better-reply secure (Lemma A.1) and quasiconcave (Lemma A.2). Therefore, a pure strategy equilibrium exists (Reny 1999, Theorem 3.1). □

I now turn to the question of uniqueness. I show that although $\Gamma(t)$ may have multiple equilibria, these equilibria are identical in terms of three essential characteristics: total production, $\sum_i p_i$; total resistance, $\sum_i r_i$; and the vector of individual expenditures on internal conflict, c.

To prove essential uniqueness, I must characterize the equilibrium more fully than I have up to this point. The following result rules out equilibria in which (1) a faction's share in the internal competition is zero or (2) a faction could raise its share to one by an infinitesimal change in strategy.

Lemma A.3. *If $N > 1$, then each $\phi(c_i) > 0$ in any equilibrium of $\Gamma(t)$.*

Proof. Assume $N > 1$, and let (p, r, c) be a strategy profile of $\Gamma(t)$ in which $c_i = 0$ for some $i \in \mathcal{N}$. The claim holds trivially if $\phi(0) > 0$, so assume $\phi(0) = 0$. If $\Phi(c) > 0$ or $\bar{\tau}(t, r) \times f(p) = 0$, then $u_i(t, p, r, c) = 0$. But i could ensure a strictly positive payoff with any strategy that allocated nonzero labor to production, resistance, and conflict, so (p, r, c) is not an equilibrium. Conversely, suppose $\Phi(c) = 0$, which implies $c_j = 0$ for all $j \in \mathcal{N}$, and $\bar{\tau}(t, r) \times f(p) > 0$. Then $u_i(t, p, r, c) = (\bar{\tau}(t, r) \times f(p))/N$. But i could obtain a payoff arbitrarily close to $\bar{\tau}(t, r) \times f(p)$ by diverting an infinitesimal amount of labor away from production or resistance and into internal conflict, so (p, r, c) is not an equilibrium. □

This result is important because it implies utility functions are continuously differentiable in the neighborhood of any equilibrium. Equilibria can therefore be characterized in terms of first-order conditions.

Lemma A.4. *(p', r', c') is an equilibrium of $\Gamma(t)$ if and only if, for each $i \in \mathcal{N}$,*

$$p_i' \left(\pi_i^p \frac{\partial \log f(p')}{\partial p_i} - \mu_i \right) = 0, \tag{A.2}$$

$$r_i' \left(\pi_i^r \frac{\partial \log \bar{\tau}(t, r')}{\partial r_i} - \mu_i \right) = 0, \tag{A.3}$$

$$c_i' \left(\pi_i^c \frac{\partial \log \omega_i(c')}{\partial c_i} - \mu_i \right) = 0, \tag{A.4}$$

$$\frac{p_i'}{\pi_i^p} + \frac{r_i'}{\pi_i^r} + \frac{c_i'}{\pi_i^c} - L_i = 0, \tag{A.5}$$

where

$$\mu_i = \max \left\{ \pi_i^p \frac{\partial \log f(p')}{\partial p_i}, \pi_i^r \frac{\partial \log \bar{\tau}(t, r')}{\partial r_i}, \pi_i^c \frac{\partial \log \omega_i(c')}{\partial c_i} \right\}.$$

Proof. In equilibrium, each faction's strategy must solve the constrained maximization problem

$$\begin{aligned} \max_{p_i, r_i, c_i} \quad & \log u_i(t, p, r, c) \\ \text{s.t.} \quad & \frac{p_i}{\pi_i^p} + \frac{r_i}{\pi_i^r} + \frac{c_i}{\pi_i^c} - L_i = 0, \end{aligned}$$

$$p_i \geq 0, r_i \geq 0, c_i \geq 0.$$

It follows from Lemma A.3 that each u_i is C^1 in (p_i, r_i, c_i) in a neighborhood of any equilibrium. This allows use of the Karush–Kuhn–Tucker conditions to characterize solutions of the above problem. The "only if" direction holds because (A.2)–(A.5) are the first-order conditions for the problem and the linearity constraint qualification holds. The "if" direction holds because $\log u_i$ is concave in (p_i, r_i, c_i), per Lemma A.2. □

A weak welfare optimality result follows almost immediately from this equilibrium characterization. If (p', r', c') is an equilibrium of $\Gamma(t)$, then there is no other equilibrium (p'', r'', c'') such that $c'' = c'$ and $\bar{\tau}(t, r'') \times f(p'') > \bar{\tau}(t, r') \times f(p')$. In other words, taking as fixed the factions' allocations toward internal conflict, there is no inefficient misallocation of labor between production and resistance.

Corollary A.1. *If (p', r', c') is an equilibrium of $\Gamma(t)$, then (p', r') solves*

$$\begin{aligned} \max_{p,r} \quad & \log \bar{\tau}(t, r) + \log f(p) \\ s.t. \quad & \frac{p_i}{\pi_i^p} + \frac{r_i}{\pi_i^r} = L_i - \frac{c_i'}{\pi_i^c}, \qquad i = 1, \ldots, N, \\ & p_i \geq 0, r_i \geq 0, \qquad i = 1, \ldots, N. \end{aligned}$$

Proof. This is a C^1 concave maximization problem with linear constraints, so the Karush–Kuhn–Tucker first-order conditions are necessary and sufficient for a solution. The result then follows from Lemma A.4. □

I next prove that if post-tax output is weakly greater in one equilibrium of $\Gamma(t)$ than another, then each of the two individual components (production and the factions' total share) is weakly greater. The proof relies on the fact that if $c_i' \leq c_i''$ and $\omega_i(c') \leq \omega_i(c'')$, then

$$\frac{\partial \log \omega_i(c')}{\partial c_i} = \hat{\phi}'(c_i')(1 - \omega_i(c')) \geq \hat{\phi}'(c_i'')(1 - \omega_i(c'')) = \frac{\partial \log \omega_i(c'')}{\partial c_i}.$$

If in addition $\omega_i(c') < \omega_i(c'')$, the inequality is strict.

Lemma A.5. *If (p', r', c') and (p'', r'', c'') are equilibria of $\Gamma(t)$ such that $\bar{\tau}(t, r') \times f(p') \geq \bar{\tau}(t, r'') \times f(p'')$, then $\bar{\tau}(t, r') \geq \bar{\tau}(t, r'')$ and $f(p') \geq f(p'')$.*

Proof. Suppose the claim of the lemma does not hold, so there exist equilibria such that $\bar{\tau}(t, r') \times f(p') \geq \bar{\tau}(t, r'') \times f(p'')$ but $\bar{\tau}(t, r') < \bar{\tau}(t, r'')$. Together, these inequalities imply

$f(p') > f(p'')$. (The proof in case $\bar{\tau}(t, r') > \bar{\tau}(t, r'')$ and $f(p') < f(p'')$ is analogous.)

I will first establish that $p'_i > 0$ implies $r''_i = 0$. Per Lemma A.4 and the log-concavity of f and $\bar{\tau}$, $p'_i > 0$ implies

$$\pi_i^p \frac{\partial \log f(p'')}{\partial p_i} > \pi_i^p \frac{\partial \log f(p')}{\partial p_i} \geq \pi_i^r \frac{\partial \log \bar{\tau}(t, r')}{\partial r_i} > \pi_i^r \frac{\partial \log \bar{\tau}(t, r'')}{\partial r_i}.$$

Therefore, again by Lemma A.4, $r''_i = 0$.

Next, I establish that $\Phi(c'') > \Phi(c')$. Since $f(p') > f(p'')$, there is a faction $i \in \mathcal{N}$ such that $p'_i > p''_i$. As this implies $r''_i = 0$, the budget constraint gives $c''_i > c'_i$. If $\Phi(c'') \leq \Phi(c')$, then $\omega_i(c'') > \omega_i(c')$ and thus by Lemma A.4

$$\pi_i^c \frac{\partial \log \omega_i(c')}{\partial c_i} > \pi_i^c \frac{\partial \log \omega_i(c'')}{\partial c_i} \geq \pi_i^p \frac{\partial \log f(p'')}{\partial p_i} > \pi_i^p \frac{\partial \log f(p')}{\partial p_i}.$$

But this implies $p'_i = 0$, a contradiction. Therefore, $\Phi(c'') > \Phi(c')$.

Using these intermediate results, I can now establish the main claim by contradiction. Since $\bar{\tau}(t, r'') > \bar{\tau}(t, r')$, there is a faction $j \in \mathcal{N}$ such that $r''_j > r'_j$. This implies $p'_j = 0$, so the budget constraint gives $c''_j < c'_j$. Since $\Phi(c'') > \Phi(c')$, this in turn gives $\omega_j(c'') < \omega_j(c')$ and thus

$$\pi_j^c \frac{\partial \log \omega_j(c'')}{\partial c_j} > \pi_j^c \frac{\partial \log \omega_j(c')}{\partial c_j} \geq \pi_j^r \frac{\partial \log \bar{\tau}(t, r')}{\partial r_j} > \pi_j^r \frac{\partial \log \bar{\tau}(t, r'')}{\partial r_j}.$$

But this implies $r''_j = 0$, a contradiction. □

I can now state and prove the essential uniqueness of the equilibrium of each labor allocation subgame.

Proposition A.2. *If (p', r', c') and (p'', r'', c'') are equilibria of $\Gamma(t)$, then $f(p') = f(p'')$, $\bar{\tau}(t, r') = \bar{\tau}(t, r'')$, and $c' = c''$.*

Proof. First I prove that $\bar{\tau}(t, r') \times f(p') = \bar{\tau}(t, r'') \times f(p'')$. Suppose not, so that, without loss of generality, $\bar{\tau}(t, r') \times f(p') > \bar{\tau}(t, r'') \times f(p'')$. Then Lemma A.5 implies $\bar{\tau}(t, r') \geq \bar{\tau}(t, r'')$ and $f(p') \geq f(p'')$, at least one strictly so, and thus

$$\max \left\{ \pi_i^p \frac{\partial \log f(p'')}{\partial p_i}, \pi_i^r \frac{\partial \log \bar{\tau}(t, r'')}{\partial r_i} \right\} \geq \max \left\{ \pi_i^p \frac{\partial \log f(p')}{\partial p_i}, \pi_i^r \frac{\partial \log \bar{\tau}(t, r')}{\partial r_i} \right\}$$

for all $i \in \mathcal{N}$, strictly so for some $j \in \mathcal{N}$. Since $\bar{\tau}(t, r'') \times f(p'') < \bar{\tau}(t, r') \times f(p')$, it follows from Corollary A.1 that the set $\mathcal{N}^+ = \{i \in \mathcal{N} \mid c''_i > c'_i\}$ is nonempty. For any $i \in \mathcal{N}^+$ such that $\omega_i(c'') > \omega_i(c')$,

$$\begin{aligned}\pi_i^c \frac{\partial \log \omega_i(c')}{\partial c_i} &> \pi_i^c \frac{\partial \log \omega_i(c'')}{\partial c_i} \\ &\geq \max\left\{\pi_i^p \frac{\partial \log f(p'')}{\partial p_i}, \pi_i^r \frac{\partial \log \bar{\tau}(t, r'')}{\partial r_i}\right\} \\ &\geq \max\left\{\pi_i^p \frac{\partial \log f(p')}{\partial p_i}, \pi_i^r \frac{\partial \log \bar{\tau}(t, r')}{\partial r_i}\right\}.\end{aligned}$$

But this implies $p'_i = r'_i = 0$, contradicting $c''_i > c'_i$. So $\omega_i(c'') \leq \omega_i(c')$ for all $i \in \mathcal{N}^+$. Since $\mathcal{N}^+$ is nonempty and the conflict shares are increasing in c_i and sum to one, this can hold only if $\mathcal{N}^+ = \mathcal{N}$ and $\omega_i(c'') = \omega_i(c')$ for all $i \in \mathcal{N}$. This implies

$$\begin{aligned}\pi_j^c \frac{\partial \log \omega_j(c')}{\partial c_j} &\geq \pi_j^c \frac{\partial \log \omega_j(c'')}{\partial c_j} \\ &\geq \max\left\{\pi_j^p \frac{\partial \log f(p'')}{\partial p_j}, \pi_j^r \frac{\partial \log \bar{\tau}(t, r'')}{\partial r_j}\right\} \\ &> \max\left\{\pi_j^p \frac{\partial \log f(p')}{\partial p_j}, \pi_j^r \frac{\partial \log \bar{\tau}(t, r')}{\partial r_j}\right\},\end{aligned}$$

which in turn implies $p'_j = r'_j = 0$, contradicting $c''_j > c'_j$. I conclude that $\bar{\tau}(t, r') \times f(p') = \bar{\tau}(t, r'') \times f(p'')$ and thus, by Lemma A.5, $\bar{\tau}(t, r') = \bar{\tau}(t, r'')$ and $f(p') = f(p'')$.

Next, I prove that $c' = c''$. Suppose not, so $c' \neq c''$. Without loss of generality, suppose $\Phi(c') \geq \Phi(c'')$. Since $\bar{\tau}(t, r') \times f(p') = \bar{\tau}(t, r'') \times f(p'')$ yet $c' \neq c''$, by Corollary A.1 there exists $i \in \mathcal{N}$ such that $c'_i > c''_i$ and $j \in \mathcal{N}$ such that $c'_j < c''_j$. It follows from $\Phi(c') \geq \Phi(c'')$ that $\omega_j(c') < \omega_j(c'')$ and therefore

$$\begin{aligned}\pi_j^c \frac{\partial \log \omega_j(c')}{\partial c_j} &> \pi_j^c \frac{\partial \log \omega_j(c'')}{\partial c_j} \\ &\geq \max\left\{\pi_j^p \frac{\partial \log f(p'')}{\partial p_j}, \pi_j^r \frac{\partial \log \bar{\tau}(t, r'')}{\partial r_j}\right\} \\ &= \max\left\{\pi_j^p \frac{\partial \log f(p')}{\partial p_j}, \pi_j^r \frac{\partial \log \bar{\tau}(t, r')}{\partial r_j}\right\}.\end{aligned}$$

But this implies $p'_j = r'_j = 0$, contradicting $c''_j > c'_j$. □

Proposition A.2 allows me to write the equilibrium values of total production, total resistance, and individual conflict allocations as functions of the tax rate. For each $t \in [0, 1]$, let $P^*(t) = P$ if and only if there is an equilibrium (p, r, c) of $\Gamma(t)$ such that $\sum_i p_i = P$. Let the functions $R^*(t)$ and $c^*(t)$, the latter of which is vector-valued, be defined analogously.

The only remaining step to prove the existence of an equilibrium in the full game is to show that an optimal tax rate exists. An important consequence of Proposition A.2 is that the optimal tax rate (if one exists) does not depend on the equilibrium that is selected in each labor allocation subgame, since the government's payoff depends only on total production and resistance. The main step toward proving the existence of an optimal tax rate is to show that total production and resistance are continuous in t.

Lemma A.6. *P^*, R^*, and c^* are continuous.*

Proof. Define the equilibrium correspondence $E : [0,1] \rightrightarrows \Sigma$ by

$$E(t) = \{(p, r, c) \mid (p, r, c) \text{ is an equilibrium of } \Gamma(t)\}.$$

Standard arguments (e.g., Fudenberg and Tirole 1991, 30–32) imply that E has a closed graph.[17] This in turn implies E is upper hemicontinuous, as its codomain, Σ, is compact. Let $F : \Sigma \to \mathbb{R}_+^{N+2}$ be the function defined by $F(p, r, c) = (\sum_i p_i, \sum_i r_i, c)$. Since F is continuous as a function, it is upper hemicontinuous as a correspondence. Then we can write the functions in the lemma as the composition of F and E:

$$(P^*(t), R^*(t), c^*(t)) = \{F(p, r, c) \mid (p, r, c) \in E(t)\} = (F \circ E)(t).$$

As the composition of upper hemicontinuous correspondences, (P^*, R^*, c^*) is upper hemicontinuous (Aliprantis and Border 2006, Theorem 17.23). Then, as an upper hemicontinuous correspondence that is single-valued (per Proposition A.2), (P^*, R^*, c^*) is continuous as a function. □

Continuity of total production and resistance in the tax rate imply that the government's payoff is continuous in the tax rate, so an equilibrium exists.

Proposition A.3. *There is a pure strategy equilibrium.*

Proof. For each labor allocation subgame $\Gamma(t)$, let $\sigma^*(t)$ be a pure strategy equilibrium of $\Gamma(t)$. Proposition A.1 guarantees the existence of these equilibria. By Proposition A.2, the government's payoff from any $t \in [0,1]$ is

$$u_G(t, \sigma^*(t)) = t \times g(R^*(t)) \times P^*(t).$$

[17] The only complication in applying the usual argument is that the model is discontinuous at $c = 0$ in case $\phi(0) = 0$. However, by the same arguments as in the proof of Lemma A.1, if $\phi(0) = 0$ there cannot be a sequence $(t^k, (p^k, r^k, c^k))$ in the graph of E such that $c^k \to 0$.

This expression is continuous in t, per Lemma A.6, and therefore attains its maximum on the compact interval $[0,1]$. A maximizer t^* exists, and the pure strategy profile $(t^*, (\sigma^*(t))_{t\in[0,1]})$ is an equilibrium. $\square$

A.3 Symmetric Game Properties

In the remainder of the appendix, I consider the special symmetric case of the model discussed in the text, in which each $\pi_i^p = \pi^p$, $\pi_i^r = \pi^r$, $\pi_i^c = \pi^c$, and $L_i = L/N$. Let $\Gamma(t)$ denote the labor allocation subgame with tax rate t in the model with a labor-financed government, and let $\Gamma_X(t)$ denote the same with a capital-financed government.

An important initial result for the symmetric case is that in every equilibrium of every labor allocation subgame, every faction spends the same amount on internal conflict.

Lemma A.7. *If the game is symmetric and (p, r, c) is an equilibrium of $\Gamma(t)$ or $\Gamma_X(t)$, then $c_i = c_j$ for all $i, j \in \mathcal{N}$.*

Proof. Because the game is symmetric, there exists an equilibrium (p', r', c') in which $(p'_i, r'_i, c'_i) = (p_j, r_j, c_j)$, $(p'_j, r'_j, c'_j) = (p_i, r_i, c_i)$, and $(p'_k, r'_k, c'_k) = (p_k, r_k, c_k)$ for all $k \in \mathcal{N} \setminus \{i, j\}$. Proposition A.2 then implies $c_i = c_j$. $\square$

This means we can characterize an equilibrium of the labor allocation subgame in terms of just three variables: total production, total resistance, and the (common across factions) individual allocation to internal conflict. Using the characterization result of Lemma A.4, we can identify an equilibrium as a solution to some subset of the following system of equations derived from the first-order conditions. Each Q^{xy} represents marginal equality of the returns to x and y, while Q^b is the budget constraint. I write these as functions of t as well as the exogenous parameters $\pi = (\pi^p, \pi^r, \pi^c, L, N)$ to allow for comparative statics via implicit differentiation:

$$Q^{pr}(P, R, C; t, \pi) = \pi^p(1 - tg(R)) + \pi^r tPg'(R) = 0, \tag{A.6}$$

$$Q^{pc}(P, R, C; t, \pi) = \frac{\pi^p}{P} - \frac{N-1}{N}\pi^c \hat{\phi}'(C) = 0, \tag{A.7}$$

$$Q^{rc}(P, R, C; t, \pi) = \frac{\pi^r tg'(R)}{1 - tg(R)} + \frac{N-1}{N}\pi^c \hat{\phi}'(C) = 0, \tag{A.8}$$

$$Q^{b}(P, R, C; t, \pi) = L - \frac{P}{\pi^p} - \frac{R}{\pi^r} - \frac{NC}{\pi^c} = 0. \tag{A.9}$$

The condition (A.8) is redundant when the government is labor-financed and (A.6) and (A.7) both hold, but it is useful for characterizing equilibrium with a capital-financed government.

A.4 Proofs of Named Results

A.4.1 Proof of Proposition 1

The quantities defined in Proposition 1 are as follows. $(\tilde{R}_X(t), \tilde{c}_X(t))$ is the solution to the system

$$Q^{rc}(0, \tilde{R}_X(t), \tilde{c}_X(t); t, \pi) = \frac{\pi^r t g'(\tilde{R}_X(t))}{1 - tg(\tilde{R}_X(t))} + \frac{N-1}{N}\pi^c \hat{\phi}'(\tilde{c}_X(t)) = 0, \tag{A.10}$$

$$Q^b(0, \tilde{R}_X(t), \tilde{c}_X(t); t, \pi) = L - \frac{\tilde{R}_X(t)}{\pi^r} - \frac{N\tilde{c}_X(t)}{\pi^c} = 0. \tag{A.11}$$

The cutpoint tax rates are

$$\hat{t}_0^X = \frac{\eta \pi^c \hat{\phi}'(\pi^c L/N)}{\eta \pi^c \hat{\phi}'(\pi^c L/N) - \pi^r g'(0)}, \tag{A.12}$$

$$\hat{t}_1^X = \frac{\eta \pi^c \hat{\phi}'(0)}{\eta \pi^c g(\pi^r L) \hat{\phi}'(0) - \pi^r g'(\pi^r L)}, \tag{A.13}$$

where $\eta = (N-1)/N$. Observe that

$$\pi^r \frac{\partial \log \bar{\tau}(\hat{t}_0^X, 0)}{\partial r_i} = \eta \pi^c \hat{\phi}'(\pi^c L/N) = \pi^c \frac{\partial \log \omega_i((\pi^c L/N)\mathbf{1}_N)}{\partial c_i},$$

$$\pi^r \frac{\partial \log \bar{\tau}(\hat{t}_1^X, (\pi^r L/N)\mathbf{1}_N)}{\partial r_i} = \eta \pi^c \hat{\phi}'(0) = \pi^c \frac{\partial \log \omega_i(0)}{\partial c_i}$$

for each $i \in \mathcal{N}$, where $\mathbf{1}_N$ is an N-vector of 1s.

Proposition 1. *If the government is capital-financed, every labor allocation subgame has a unique equilibrium. There exists a tax rate $\hat{t}_0^X \in (0,1)$ such that each $r_i = 0$ in equilibrium if and only if $t \leq \hat{t}_0^X$. There exists $\hat{t}_1^X > \hat{t}_0^X$ such that each $c_i = 0$ in equilibrium if and only if $t \geq \hat{t}_1^X$. For $t \in (\hat{t}_0^X, \hat{t}_1^X)$, in equilibrium each $r_i = \tilde{R}_X(t)/N > 0$ (strictly increasing in t) and each $c_i = \tilde{c}_X(t) > 0$ (strictly decreasing).*

Proof. $\Gamma_X(t)$ has an equilibrium (Proposition A.1), and there exists $c_X^*(t)$ such that each $c_i = c_X^*(t)$ in all of its equilibria (Proposition A.2 and Lemma A.7). The budget constraint then implies each $r_i = \pi^r(L/N - c_X^*(t)/\pi^c)$ in every equilibrium of $\Gamma_X(t)$, so the equilibrium is unique.

Let (r, c) be the equilibrium of $\Gamma_X(t)$. To simplify expressions in what follows, let $C = c_i = c_X^*(t)$ and $R = \sum_i r_i = \pi^r(L - NC/\pi^c)$. If $t \leq \hat{t}_0^X$ and $R > 0$, then the first-order conditions give

$$\pi^r \frac{\partial \log \bar{\tau}(t,r)}{\partial r_i} < \pi^r \frac{\partial \log \bar{\tau}(\hat{t}_0^X, 0)}{\partial r_i} = \pi^c \frac{\partial \log \omega_i((\pi^c L/N)\mathbf{1}_N)}{\partial c_i} \leq \pi^c \frac{\partial \log \omega_i(c)}{\partial c_i}$$

for each $i \in \mathcal{N}$. But this implies each $r_i = 0$, contradicting $R > 0$. Therefore, if $t \leq \hat{t}_0^X$, then $R = 0$. Similarly, if $t > \hat{t}_0^X$ and $R = 0$, then each $c_i = \pi^c L/N$ and thus

$$\pi^c \frac{\partial \log \omega_i(c)}{\partial c_i} = \pi^r \frac{\partial \log \bar{\tau}(\hat{t}_0^X, 0)}{\partial r_i} < \pi^r \frac{\partial \log \bar{\tau}(t,r)}{\partial r_i}.$$

But this implies each $c_i = 0$, a contradiction. Therefore, if $t > \hat{t}_0^X$, then $R > 0$. The proof that $C > 0$ if and only if $t < \hat{t}_1^X$ is analogous.

For $t \in (\hat{t}_0^X, \hat{t}_1^X)$, the first-order conditions imply that R and C solve $Q^{rc}(0, R, C; t, \pi) = Q^b(0, R, C; t, \pi) = 0$; therefore, $R = \tilde{R}_X(t)$ and $C = \tilde{c}_X(t)$. To reduce clutter in what follows, I omit the evaluation point $(0, \tilde{R}_X(t), \tilde{c}_X(t); t, \pi)$ from all partial derivative expressions. The Jacobian of the system defining $(\tilde{R}_X(t), \tilde{c}_X(t))$ is

$$\begin{aligned} \mathbf{J}_X &= \begin{bmatrix} \partial Q^{rc}/\partial R & \partial Q^{rc}/\partial C \\ \partial Q^b/\partial R & \partial Q^b/\partial C \end{bmatrix} \\ &= \begin{bmatrix} \pi^r t \dfrac{g''(\tilde{R}_X(t)) - tg(\tilde{R}_X(t))^2 \hat{g}''(\tilde{R}_X(t))}{(1 - tg(\tilde{R}_X(t)))^2} & \eta \pi^c \hat{\phi}''(\tilde{c}_X(t)) \\ -1/\pi^r & -N/\pi^c \end{bmatrix} \end{aligned}$$

where $\eta = (N-1)/N$ and $\hat{g} = \log g$. Its determinant is

$$|\mathbf{J}_X| = \frac{\pi^c}{\pi^r} \left(\eta \hat{\phi}''(\tilde{c}_X(t)) - N \pi^r t \frac{g''(\tilde{R}_X(t)) - tg(\tilde{R}_X(t))^2 \hat{g}''(\tilde{R}_X(t))}{(1 - tg(\tilde{R}_X(t)))^2} \right) < 0.$$

By the implicit function theorem and Cramer's rule,

$$\begin{aligned} \frac{d\tilde{R}_X(t)}{dt} &= \frac{\begin{vmatrix} -\partial Q^{rc}/\partial t & \partial Q^{rc}/\partial C \\ -\partial Q^b/\partial t & \partial Q^b/\partial C \end{vmatrix}}{|\mathbf{J}_X|} \\ &= \frac{\begin{vmatrix} -\pi^r g'(\tilde{R}_X(t))/(1 - tg(\tilde{R}_X(t)))^2 & \eta \pi^c \hat{\phi}''(\tilde{c}_X(t)) \\ 0 & -N/\pi^c \end{vmatrix}}{|\mathbf{J}_X|} \\ &= \frac{N \pi^r g'(\tilde{R}_X(t))}{\pi^c (1 - tg(\tilde{R}_X(t)))^2 |\mathbf{J}_X|} \end{aligned}$$

$$> 0,$$

as claimed. The budget constraint then implies $d\tilde{c}_X(t)/dt < 0$, as claimed. □

A.4.2 Proof of Propositions 2 and 3

Before proving the results, I separately derive the comparative statics of $\hat{t}_0^X$ and $\tilde{R}_X(t)$ in N and π^c.

Lemma A.8. *The lower cutpoint $\hat{t}_0^X$ is strictly increasing in the number of factions, N. It is locally decreasing in the effectiveness of conflict, π^c, if and only if*

$$\hat{\phi}'\left(\frac{\pi^c L}{N}\right) + \frac{\pi^c L}{N}\hat{\phi}''\left(\frac{\pi^c L}{N}\right) \geq 0.$$

Proof. Recall that

$$\hat{t}_0^X = \frac{((N-1)/N)\pi^c\hat{\phi}'(\pi^c L/N)}{((N-1)/N)\pi^c\hat{\phi}'(\pi^c L/N) - \pi^r g'(0)}.$$

Observe that $g'(0) < 0$, $(N-1)/N$ is strictly increasing in N, and $\hat{\phi}'(\pi^c L/N)$ is weakly increasing in N. Therefore, $\hat{t}_0^X$ is strictly increasing in N. Notice that

$$\frac{\partial}{\partial \pi^c}\left[\pi^c\hat{\phi}'\left(\frac{\pi^c L}{N}\right)\right] = \hat{\phi}'\left(\frac{\pi^c L}{N}\right) + \frac{\pi^c L}{N}\hat{\phi}''\left(\frac{\pi^c L}{N}\right),$$

so $\hat{t}_0^X$ is locally increasing in π^c if and only if the above expression is positive. □

Lemma A.9. *For fixed $t \in (\hat{t}_0^X, \hat{t}_1^X)$, total resistance, $\tilde{R}_X(t)$, is strictly decreasing in the number of factions, N. It is locally decreasing in the effectiveness of conflict, π^c, if and only if*

$$\hat{\phi}'(\tilde{c}_X(t)) + \tilde{c}_X(t)\hat{\phi}''(\tilde{c}_X(t)) \geq 0.$$

Proof. I will treat N as if it were continuous in order to obtain comparative statics by implicit differentiation. Throughout the proof I write $\tilde{R}_X(t)$ and $\tilde{c}_X(t)$ as functions of (N, π^c).

I first consider comparative statics in N. To reduce clutter in what follows, I omit the evaluation point $(0, \tilde{R}_X(t; N, \pi^c), \tilde{c}_X(t; N, \pi^c); t, \pi)$ from all partial derivative expressions. By the implicit function theorem and Cramer's rule,

$$\frac{\partial \tilde{R}_X(t; N, \pi^c)}{\partial N} = \frac{\begin{vmatrix} -\partial Q^{rc}/\partial N & \partial Q^{rc}/\partial C \\ -\partial Q^b/\partial N & \partial Q^b/\partial C \end{vmatrix}}{|\mathbf{J}_X(t; N, \pi^c)|}$$

$$= \frac{\begin{vmatrix} -\pi^c \hat{\phi}'(\tilde{c}_X(t;N,\pi^c))/N^2 & ((N-1)/N)\pi^c \hat{\phi}''(\tilde{c}_X(t;N,\pi^c)) \\ \tilde{c}_X(t;N,\pi^c)/\pi^c & -N/\pi^c \end{vmatrix}}{|\mathbf{J}_X(t;N,\pi^c)|}$$
$$= \frac{\hat{\phi}'(\tilde{c}_X(t;N,\pi^c)) - (N-1)\tilde{c}_X(t;N,\pi^c)\hat{\phi}''(\tilde{c}_X(t;N,\pi^c))}{N|\mathbf{J}_X(t;N,\pi^c)|}$$
$$< 0,$$

as claimed, where $|\mathbf{J}_X(t;N,\pi^c)| < 0$ is defined as in the proof of Proposition 1.

I now consider comparative statics in π^c. Again by the implicit function theorem and Cramer's rule,

$$\frac{\partial \tilde{R}_X(t;N,\pi^c)}{\partial \pi^c} = \frac{\begin{vmatrix} -\partial Q^{rc}/\partial \pi^c & \partial Q^{rc}/\partial C \\ -\partial Q^b/\partial \pi^c & \partial Q^b/\partial C \end{vmatrix}}{|\mathbf{J}_X(t;N,\pi^c)|}$$
$$= \frac{\begin{vmatrix} -((N-1)/N)\hat{\phi}'(\tilde{c}_X(t;N,\pi^c)) & ((N-1)/N)\pi^c \hat{\phi}''(\tilde{c}_X(t;N,\pi^c)) \\ -N\tilde{c}_X(t;N,\pi^c)/(\pi^c)^2 & -N/\pi^c \end{vmatrix}}{|\mathbf{J}_X(t;N,\pi^c)|}$$
$$= \frac{(N-1)\left(\hat{\phi}'(\tilde{c}_X(t;N,\pi^c)) + \tilde{c}_X(t;N,\pi^c)\hat{\phi}''(\tilde{c}_X(t;N,\pi^c))\right)}{\pi^c|\mathbf{J}_X(t;N,\pi^c)|}.$$

Therefore, $\partial \tilde{R}_X(t;N,\pi^c)/\partial \pi^c \leq 0$ if and only if

$$\hat{\phi}'(\tilde{c}_X(t;N,\pi^c)) + \tilde{c}_X(t;N,\pi^c)\hat{\phi}''(\tilde{c}_X(t;N,\pi^c)) \geq 0,$$

as claimed. □

The propositions, which I prove together, follow mainly from these lemmas.

Proposition 2. *A capital-financed government's equilibrium payoff is increasing in the number of factions, N.*

Proposition 3. *If there is a unique equilibrium tax rate t^*, the government's equilibrium payoff is locally increasing in competitive effectiveness, π^c, if and only if the incentive effect outweighs the labor-saving effect (i.e., Equation 8 holds) at the corresponding equilibrium level of internal competition.*

Proof. Throughout the proof I write various equilibrium quantities, including the cutpoints $\hat{t}_0^X$ and $\hat{t}_1^X$, as functions of (N, π^c). Let the government's equilibrium payoff as a function of these parameters be

$$u_G^*(N, \pi^c) = \max_{t \in [0,1]} t \times g(R^*(t; N, \pi^c)) \times X.$$

I begin with the comparative statics on N. First, suppose $t = \hat{t}_0^X(N', \pi^c)$ is an equilibrium for all N' in a neighborhood of N.[18] Then $u_G^*(N', \pi^c) = \hat{t}_0^X(N', \pi^c) \times X$ in a neighborhood of N, which by Lemma A.8 is strictly increasing in N'. Next, suppose there is an equilibrium with $t \in (\hat{t}_0^X(N', \pi^c), \hat{t}_1^X(N', \pi^c))$ for all N' in a neighborhood of N. Then, by the envelope theorem,

$$\frac{\partial u_G^*(N, \pi^c)}{\partial N} = g'(\tilde{R}_X(t; N, \pi^c)) \frac{\partial \tilde{R}_X(t; N, \pi^c)}{\partial N} \times X > 0,$$

where the inequality follows from Lemma A.9. Finally, suppose $\hat{t}_1^X(N', \pi^c) < 1$ and $t = 1$ is an equilibrium for all N' in a neighborhood of N. Then $u_G^*(N, \pi^c) = g(\pi^r L) \times X$ is locally constant in N, and thus weakly increasing.

I now consider the comparative statics on π^c. First, suppose $t = \hat{t}_0^X(N, \pi^{c\prime})$ is an equilibrium for all $\pi^{c\prime}$ in a neighborhood of π^c. Then $u_G^*(N, \pi^{c\prime}) = \hat{t}_0^X(N, \pi^{c\prime}) \times X$ in a neighborhood of $\pi^{c\prime}$, which by Lemma A.8 is locally increasing at π^c if and only if

$$\hat{\phi}'\left(\frac{\pi^c L}{N}\right) + \frac{\pi^c L}{N} \hat{\phi}''\left(\frac{\pi^c L}{N}\right) \geq 0.$$

Next, suppose there is a unique equilibrium with $t \in (\hat{t}_0^X(N, \pi^{c\prime}), \hat{t}_1^X(N, \pi^{c\prime}))$ for all $\pi^{c\prime}$ in a neighborhood of π^c. Then, by the envelope theorem,

$$\frac{\partial u_G^*(N, \pi^c)}{\partial \pi^c} = g'(\tilde{R}_X(t; N, \pi^c)) \frac{\partial \tilde{R}_X(t; N, \pi^c)}{\partial \pi^c} \times X.$$

This is positive if and only if $\hat{\phi}'(\tilde{c}_X(t)) + \tilde{c}_X(t)\hat{\phi}''(\tilde{c}_X(t)) \geq 0$, per Lemma A.9. Finally, suppose $\hat{t}_1^X(N, \pi^{c\prime}) < 1$ and $t = 1$ is an equilibrium for all $\pi^{c\prime}$ in a neighborhood of π^c. Then $u_G^*(N, \pi^c) = g(\pi^r L) \times X$ is locally constant in π^c, and thus weakly increasing. □

I also prove the claim in footnote 13.

Lemma A.10. *Let $\theta, \lambda > 0$. If $\phi(C) = \theta \exp(\lambda C)$, then the incentive effect outweighs the labor-saving effect for all $C \geq 0$. If $\phi(C) = \theta C^\lambda$, then the incentive and labor-saving effects are exactly offsetting for all $C > 0$.*

Proof. First consider the difference contest success function, $\phi(C) = \theta \exp(\lambda C)$. Then

[18] There cannot be an equilibrium tax rate $t < \hat{t}_0^X$ or (if $\hat{t}_1^X < 1$) $t \in [\hat{t}_1^X, 1)$, as R^* is constant on $[0, \hat{t}_0^X]$ and on $[\hat{t}_1^X, 1]$.

$\hat{\phi}(C) = \log\theta + \lambda C$, $\hat{\phi}'(C) = \lambda$, and $\hat{\phi}''(C) = 0$ for all $C \geq 0$. Therefore,

$$\hat{\phi}'(C) + C\hat{\phi}''(C) = \lambda > 0.$$

Now consider the ratio contest success function, $\phi(C) = \theta C^{\lambda}$. Then $\hat{\phi}(C) = \log\theta + \lambda \log C$, $\hat{\phi}'(C) = \lambda/C$, and $\hat{\phi}''(C) = -\lambda/C^2$. Therefore,

$$\hat{\phi}'(C) + C\hat{\phi}''(C) = \frac{\lambda}{C} + C\left(\frac{-\lambda}{C^2}\right) = 0. \qquad \square$$

A.4.3 Proof of Proposition 4

The quantities defined in Proposition 4 are as follows. $(\bar{P}_0, \bar{c}_0)$ is the solution to the system

$$Q^{pc}(\bar{P}_0, 0, \bar{c}_0; t, \pi) = \frac{\pi^p}{\bar{P}_0} - \frac{N-1}{N}\pi^c\hat{\phi}'(\bar{c}_0) = 0, \tag{A.14}$$

$$Q^b(\bar{P}_0, 0, \bar{c}_0; t, \pi) = L - \frac{\bar{P}_0}{\pi^p} - \frac{N\bar{c}_0}{\pi^c} = 0. \tag{A.15}$$

$(\tilde{P}_1(t), \tilde{R}_1(t), \tilde{c}_1(t))$ is the solution to the system

$$Q^{pr}(\tilde{P}_1(t), \tilde{R}_1(t), \tilde{c}_1(t); t, \pi) = \pi^p(1 - tg(\tilde{R}_1(t)) + \pi^r t\tilde{P}_1(t)g'(\tilde{R}_1(t)) = 0, \tag{A.16}$$

$$Q^{pc}(\tilde{P}_1(t), \tilde{R}_1(t), \tilde{c}_1(t); t, \pi) = \frac{\pi^p}{\tilde{P}_1(t)} - \frac{N-1}{N}\pi^c\hat{\phi}'(\tilde{c}_1(t)) = 0, \tag{A.17}$$

$$Q^b(\tilde{P}_1(t), \tilde{R}_1(t), \tilde{c}_1(t); t, \pi) = L - \frac{\tilde{P}_1(t)}{\pi^p} - \frac{\tilde{R}_1(t)}{\pi^r} - \frac{N\tilde{c}_1(t)}{\pi^c} = 0. \tag{A.18}$$

$(\tilde{P}_2(t), \tilde{R}_2(t))$ is the solution to the system

$$Q^{pr}(\tilde{P}_2(t), \tilde{R}_2(t), 0; t, \pi) = \pi^p(1 - tg(\tilde{R}_2(t)) + \pi^r t\tilde{P}_2(t)g'(\tilde{R}_2(t)) = 0, \tag{A.19}$$

$$Q^b(\tilde{P}_2(t), \tilde{R}_2(t), 0; t, \pi) = L - \frac{\tilde{P}_2(t)}{\pi^p} - \frac{\tilde{R}_2(t)}{\pi^r} = 0. \tag{A.20}$$

The first cutpoint tax rate is

$$\hat{t}_0 = \frac{\pi^p}{\pi^p - \pi^r\bar{P}_0 g'(0)}. \tag{A.21}$$

Lemma A.11 below shows that $\bar{P}_0 > 0$ and therefore, since $g'(0) < 0$, that $\hat{t}_0 < 1$. The second cutpoint tax rate is

$$\hat{t}_1 = \frac{\pi^p}{\pi^p g(\bar{R}_1) - \pi^r\bar{P}_1 g'(\bar{R}_1)}, \tag{A.22}$$

where

$$\bar{P}_1 = \frac{N}{N-1}\frac{\pi^p}{\pi^c}\frac{1}{\hat{\phi}'(0)}, \tag{A.23}$$

$$\bar{R}_1 = \pi^r\left(L - \frac{\bar{P}_1}{\pi^p}\right). \tag{A.24}$$

The next three lemmas give conditions on the tax rate under which there is positive production, resistance, and internal conflict in the equilibrium of the labor allocation subgame. Jointly, these lemmas constitute the bulk of the proof of Proposition 4. The proofs rely on the following equalities:

$$\begin{aligned}
\pi^r \frac{\partial \log \bar{\tau}(\hat{t}_0, 0)}{\partial r_i} &= -\pi^r \frac{\hat{t}_0 g'(0)}{1-\hat{t}_0} &&= \frac{\pi^p}{\bar{P}_0}, \\
\pi^r \frac{\partial \log \bar{\tau}(\hat{t}_1, (\bar{R}_1/N)\mathbf{1}_N)}{\partial r_i} &= -\pi^r \frac{\hat{t}_1 g'(\bar{R}_1)}{1-\hat{t}_1 g(\bar{R}_1)} &&= \frac{\pi^p}{\bar{P}_1}
\end{aligned}$$

for all $i \in \mathcal{N}$.

Lemma A.11. *If the game is symmetric, Assumption 1 holds, and (p, r, c) is an equilibrium of $\Gamma(t)$, then $0 < \sum_i p_i \leq \bar{P}_0 < \pi^p L$.*

Proof. Assumption 1 implies

$$Q^{pc}(\pi^p L, 0, 0; 0, \pi) = \frac{1}{L} - \frac{N-1}{N}\pi^c \hat{\phi}'(0) < 0.$$

Since Q^{pc} is decreasing in P and weakly increasing in C, this gives $\bar{P}_0 < \pi^p L$.

Let $P = \sum_i p_i$, and suppose $P > \bar{P}_0$. The budget constraint and Lemma A.7 then give $c_i = C < \bar{c}_0$ for each $i \in \mathcal{N}$. But then we have

$$\pi^c \frac{\partial \log \omega_i(c)}{\partial c_i} \geq \frac{N-1}{N}\pi^c \hat{\phi}'(\bar{c}_0) = \frac{\pi^p}{\bar{P}_0} > \pi^p \frac{\partial \log f(p)}{\partial p_i}$$

for each $i \in \mathcal{N}$. By Lemma A.4, this implies each $p_i = 0$, a contradiction. Therefore, $P \leq \bar{P}_0$.

Finally, since $P = 0$ implies each $u_i(t, p, r, c) = 0$, but any faction can assure itself a positive payoff with any $(p_i, r_i, c_i) \gg 0$, in equilibrium $P > 0$. □

Lemma A.12. *If the game is symmetric, Assumption 1 holds, and (p, r, c) is an equilibrium of $\Gamma(t)$, then $\sum_i r_i > 0$ if and only if $t > \hat{t}_0$.*

Proof. Let $P = \sum_i p_i$ and $R = \sum_i r_i$. To prove the "if" direction, suppose $t > \hat{t}_0$ and $R = 0$. Since $P \leq \bar{P}_0$, this implies each $c_i = C > 0$; the first-order conditions of Lemma A.4 then give $P = \bar{P}_0$ and $C = \bar{c}_0$. It follows that

$$\pi^r \frac{\partial \log \bar{\tau}(t, r)}{\partial r_i} > \pi^r \frac{\partial \log \bar{\tau}(\hat{t}_0, r)}{\partial r_i} = \frac{\pi^p}{\bar{P}_0} = \pi^p \frac{\partial \log f(p)}{\partial p_i}.$$

This implies each $p_i = 0$, a contradiction.

To prove the "only if" direction, suppose $t \leq \hat{t}_0$ and $R > 0$. For each $i \in \mathcal{N}$,

$$\pi^r \frac{\partial \log \bar{\tau}(t, r)}{\partial r_i} < \pi^r \frac{\partial \log \bar{\tau}(\hat{t}_0, 0)}{\partial r_i} = \frac{\pi^p}{\bar{P}_0} \leq \pi^p \frac{\partial \log f(p)}{\partial p_i}.$$

This implies each $r_i = 0$, a contradiction. □

Lemma A.13. *If the game is symmetric and Assumption 1 holds, then $\hat{t}_1 > \hat{t}_0$. If, in addition, (p, r, c) is an equilibrium of $\Gamma(t)$, then each $c_i > 0$ if and only if $t < \hat{t}_1$.*

Proof. To prove that $\hat{t}_1 > \hat{t}_0$, note that

$$\bar{P}_1 = \frac{N}{N-1} \frac{\pi^p}{\pi^c} \frac{1}{\hat{\phi}'(0)} \leq \frac{N}{N-1} \frac{\pi^p}{\pi^c} \frac{1}{\hat{\phi}'(\bar{c}_0)} = \bar{P}_0$$

by log-concavity of ϕ. This implies $\bar{R}_1 > 0$, so $g(\bar{R}_1) < g(0) = 1$ and $g'(0) \leq g'(\bar{R}_1) < 0$. Therefore,

$$\pi^p - \pi^r \bar{P}_0 g'(0) > \pi^p g(\bar{R}_1) - \pi^r \bar{P}_1 g'(\bar{R}_1) > 0,$$

which implies $\hat{t}_1 > \hat{t}_0$.

Let $P = \sum_i p_i$ and $R = \sum_i r_i$. To prove the "if" direction of the second statement, suppose $t \geq \hat{t}_1$ and some $c_i > 0$. By Lemma A.7, $c_j = c_i = C > 0$ for each $j \in \mathcal{N}$. Since $P > 0$ by Lemma A.11, the first-order conditions give

$$P = \frac{N}{N-1} \frac{\pi^p}{\pi^c} \frac{1}{\hat{\phi}'(C)} \geq \bar{P}_1.$$

The budget constraint then gives $R < \bar{R}_1$ and thus

$$\pi^r \frac{\partial \log \bar{\tau}(t, r)}{\partial r_i} > \pi^r \frac{\partial \log \bar{\tau}(\hat{t}_1, (\bar{R}_1/N)\mathbf{1}_N)}{\partial r_i} = \frac{\pi^p}{\bar{P}_1} \geq \pi^p \frac{\partial \log f(p)}{\partial p_i}.$$

But this implies each $p_i = 0$, a contradiction.

To prove the "only if" direction, suppose $t < \hat{t}_1$ and each $c_i = 0$. The first-order conditions then give $P \leq \bar{P}_1$, so $R \geq \bar{R}_1 > 0$ by the budget constraint. This in turn gives

$$\pi^p \frac{\partial \log f(p)}{\partial p_i} = \frac{\pi^p}{P} \geq \frac{\pi^p}{\bar{P}_1} \geq \pi^r \frac{\partial \log \bar{\tau}(\hat{t}_1, r)}{\partial r_i} > \pi^r \frac{\partial \log \bar{\tau}(t, r)}{\partial r_i}$$

for each $i \in \mathcal{N}$. But this implies each $r_i = 0$, a contradiction. □

The last thing we need to prove the propositions is how the labor allocations change with the tax rate when $t > \hat{t}_0$.

Lemma A.14. *Let the game be symmetric and Assumption 1 hold. For all* $t \in (\hat{t}_0, \hat{t}_1)$,

$$\frac{d\tilde{P}_1(t)}{dt} = \frac{-(N-1)\pi^p \pi^c \hat{\phi}''(\tilde{c}_1(t))}{N\pi^r t \Delta_1(t)} \leq 0, \tag{A.25}$$

$$\frac{d\tilde{R}_1(t)}{dt} = \frac{-\pi^p \left(N\pi^p/\pi^c \tilde{P}_1(t)^2 - (N-1)\pi^c \hat{\phi}''(\tilde{c}_1(t))/N\pi^p\right)}{t\Delta_1(t)} > 0, \tag{A.26}$$

$$\frac{d\tilde{c}_1(t)}{dt} = \frac{(\pi^p)^2}{\pi^r t \tilde{P}_1(t)^2 \Delta_1(t)} < 0, \tag{A.27}$$

where

$$\begin{aligned}\Delta_1(t) = &\left(\pi^p t g'(\tilde{R}_1(t)) - \pi^r t \tilde{P}_1(t) g''(\tilde{R}_1(t))\right)\left(\frac{N\pi^p}{\pi^c \tilde{P}_1(t)^2} - \frac{N-1}{N}\frac{\pi^c}{\pi^p}\hat{\phi}''(\tilde{c}_1(t))\right) \\ &- \frac{N-1}{N}\pi^c t g'(\tilde{R}_1(t))\hat{\phi}''(\tilde{c}_1(t)) \\ &< 0.\end{aligned} \tag{A.28}$$

For all $t > \hat{t}_1$,

$$\frac{d\tilde{P}_2(t)}{dt} = \frac{-\pi^p}{\pi^r t \Delta_2(t)} < 0, \tag{A.29}$$

$$\frac{d\tilde{R}_2(t)}{dt} = \frac{1}{t\Delta_2(t)} > 0, \tag{A.30}$$

where

$$\Delta_2(t) = \frac{\pi^r}{\pi^p} t \tilde{P}_2(t) g''(\tilde{R}_2(t)) - 2t g'(\tilde{R}_2(t)) > 0. \tag{A.31}$$

Proof. Throughout the proof, let $\eta = (N-1)/N$.

First consider $t \in (\hat{t}_0, \hat{t}_1)$. To reduce clutter in what follows, I omit the evaluation point $(\tilde{P}_1(t), \tilde{R}_1(t), \tilde{c}_1(t); t, \pi)$ from all partial derivative expressions. The Jacobian of the system of equations that defines $(\tilde{P}_1(t), \tilde{R}_1(t), \tilde{c}_1(t))$ is

$$\mathbf{J}_1(t) = \begin{bmatrix} \partial Q^{pr}/\partial P & \partial Q^{pr}/\partial R & \partial Q^{pr}/\partial C \\ \partial Q^{pc}/\partial P & \partial Q^{pc}/\partial R & \partial Q^{pc}/\partial C \\ \partial Q^{b}/\partial P & \partial Q^{b}/\partial R & \partial Q^{b}/\partial C \end{bmatrix}$$
$$= \begin{bmatrix} \pi^r t g'(\tilde{R}_1(t)) & \pi^r t \tilde{P}_1(t) g''(\tilde{R}_1(t)) - \pi^p t g'(\tilde{R}_1(t)) & 0 \\ -\pi^p/\tilde{P}_1(t)^2 & 0 & -\eta\pi^c \hat{\phi}''(\tilde{c}_1(t)) \\ -1/\pi^p & -1/\pi^r & -N/\pi^c \end{bmatrix}.$$

It is easy to verify that $|\mathbf{J}_1(t)| = \Delta_1(t) < 0$. Notice that

$$\begin{aligned} \frac{\partial Q^{pr}}{\partial t} &= \pi^r \tilde{P}_1(t) g'(\tilde{R}_1(t)) - \pi^p g(\tilde{R}_1(t)) \\ &= \pi^r \left(-\frac{\pi^p(1 - tg(\tilde{R}_1(t)))}{\pi^r t g'(\tilde{R}_1(t))} \right) g'(\tilde{R}_1(t)) - \pi^p g(\tilde{R}_1(t)) \\ &= -\frac{\pi^p}{t}. \end{aligned}$$

Then, by the implicit function theorem and Cramer's rule,

$$\begin{aligned} \frac{d\tilde{P}_1(t)}{dt} &= \frac{\begin{vmatrix} -\partial Q^{pr}/\partial t & \partial Q^{pr}/\partial R & \partial Q^{pr}/\partial C \\ -\partial Q^{pc}/\partial t & \partial Q^{pc}/\partial R & \partial Q^{pc}/\partial C \\ -\partial Q^{b}/\partial t & \partial Q^{b}/\partial R & \partial Q^{b}/\partial C \end{vmatrix}}{|\mathbf{J}_1(t)|} \\ &= \frac{-\eta\pi^p\pi^c\hat{\phi}''(\tilde{c}_1(t))}{\pi^r t \Delta_1(t)} \\ &\leq 0, \\ \frac{d\tilde{R}_1(t)}{dt} &= \frac{\begin{vmatrix} \partial Q^{pr}/\partial P & -\partial Q^{pr}/\partial t & \partial Q^{pr}/\partial C \\ \partial Q^{pc}/\partial P & -\partial Q^{pc}/\partial t & \partial Q^{pc}/\partial C \\ \partial Q^{b}/\partial P & -\partial Q^{b}/\partial t & \partial Q^{b}/\partial C \end{vmatrix}}{|\mathbf{J}_1(t)|} \\ &= \frac{-\pi^p \left(N\pi^p/\pi^c \tilde{P}_1(t)^2 - \eta\pi^c\hat{\phi}''(\tilde{c}_1(t))/\pi^p \right)}{t\Delta_1(t)} \\ &> 0, \end{aligned}$$

$$\frac{d\tilde{c}_1(t)}{dt} = \frac{\begin{vmatrix} \partial Q^{pr}/\partial P & \partial Q^{pr}/\partial R & -\partial Q^{pr}/\partial t \\ \partial Q^{pc}/\partial P & \partial Q^{pc}/\partial R & -\partial Q^{pc}/\partial t \\ \partial Q^{b}/\partial P & \partial Q^{b}/\partial R & -\partial Q^{b}/\partial t \end{vmatrix}}{|\mathbf{J}_1(t)|}$$
$$= \frac{(\pi^p)^2}{\pi^r t \tilde{P}_1(t)^2 \Delta_1(t)}$$
$$< 0,$$

as claimed.

Now consider $t > \hat{t}_1$. Again to reduce clutter in what follows, I omit the evaluation point $(\tilde{P}_2(t), \tilde{R}_2(t), 0; t, \pi)$ from all partial derivative expressions. The Jacobian of the system of equations that defines $(\tilde{P}_2(t), \tilde{R}_2(t))$ is

$$\mathbf{J}_2(t) = \begin{bmatrix} \partial Q^{pr}/\partial P & \partial Q^{pr}/\partial R \\ \partial Q^{b}/\partial P & \partial Q^{b}/\partial R \end{bmatrix}$$
$$= \begin{bmatrix} \pi^r t g'(\tilde{R}_2(t)) & \pi^r t \tilde{P}_2(t) g''(\tilde{R}_2(t)) - \pi^p t g'(\tilde{R}_2(t)) \\ -1/\pi^p & -1/\pi^r \end{bmatrix}.$$

It is easy to verify that $|\mathbf{J}_2(t)| = \Delta_2(t) > 0$. As before, $\partial Q^{pr}/\partial t = -\pi^p/t$. So by the implicit function theorem and Cramer's rule,

$$\frac{d\tilde{P}_2(t)}{dt} = \frac{\begin{vmatrix} -\partial Q^{pr}/\partial t & \partial Q^{pr}/\partial R \\ -\partial Q^{b}/\partial t & \partial Q^{b}/\partial R \end{vmatrix}}{|\mathbf{J}_2(t)|}$$
$$= \frac{-\pi^p}{\pi^r t \Delta_2(t)}$$
$$< 0,$$
$$\frac{d\tilde{R}_2(t)}{dt} = \frac{\begin{vmatrix} \partial Q^{pr}/\partial P & -\partial Q^{pr}/\partial t \\ \partial Q^{b}/\partial P & -\partial Q^{b}/\partial t \end{vmatrix}}{|\mathbf{J}_2(t)|}$$
$$= \frac{1}{t \Delta_2(t)}$$
$$> 0,$$

as claimed. □

The proof of Proposition 4 follows almost immediately from these lemmas.

Proposition 4. *Assume the government is labor-financed. There exist tax rates $\hat{t}_0 \in (0,1)$ and $\hat{t}_1 > \hat{t}_0$ such that in every equilibrium of the labor allocation subgame with tax rate t:*

- *If $t \leq \hat{t}_0$, then each $p_i = \bar{P}_0/N > 0$, each $r_i = 0$, and each $c_i = \bar{c}_0 > 0$.*
- *If $t \in (\hat{t}_0, \hat{t}_1)$, then $\sum_i p_i = \tilde{P}_1(t) > 0$ (weakly decreasing in t), $\sum_i r_i = \tilde{R}_1(t) > 0$ (strictly increasing), and each $c_i = \tilde{c}_1(t) > 0$ (strictly decreasing).*
- *If $t \geq \hat{t}_1$, then $\sum_i p_i = \tilde{P}_2(t) > 0$ (strictly decreasing in t), $\sum_i r_i = \tilde{R}_2(t) > 0$ (strictly increasing), and each $c_i = 0$.*

Proof. For fixed t, every equilibrium of $\Gamma(t)$ has the same total production, total resistance, and individual conflict allocations, per Proposition A.2. Consider any $t \in [0,1]$ and let (p, r, c) be an equilibrium of $\Gamma(t)$.

If $t \leq \hat{t}_0$, then $\sum_i p_i = P > 0$, $\sum_i r_i = 0$, and each $c_i = C > 0$ by Lemmas A.11–A.13. The first-order conditions (Lemma A.4) imply that P and C solve $Q^{pc}(P, 0, C; t, \pi) = Q^b(P, 0, C; t, \pi) = 0$; therefore, $P = \bar{P}_0$ and $C = \bar{c}_0$. Since each $r_i = 0$, each $p_i = \pi^p(L/N - \bar{c}_0/\pi^c) = \bar{P}_0/N$, so the equilibrium is unique.

Similarly, if $t \in (\hat{t}_0, \hat{t}_1)$, then $\sum_i p_i = P > 0$, $\sum_i r_i = R > 0$, and each $c_i = C > 0$ by Lemmas A.11–A.13. The first-order conditions then imply that these solve the system (A.6)–(A.9); therefore, $P = \tilde{P}_1(t)$, $R = \tilde{R}_1(t)$, and $C = \tilde{c}_1(t)$. The comparative statics on $\tilde{P}_1$, $\tilde{R}_1$, and $\tilde{c}_1$ follow from Lemma A.14.

Finally, if $t \geq \hat{t}_1$, then $\sum_i p_i = P > 0$, $\sum_i r_i = R > 0$, and each $c_i = 0$ by Lemmas A.11–A.13. The first-order conditions then imply that P and R solve $Q^{pr}(P, R, 0; t, \pi) = Q^b(P, R, 0; t, \pi) = 0$; therefore, $P = \tilde{P}_2(t)$ and $R = \tilde{R}_2(t)$. The comparative statics on $\tilde{P}_2$ and $\tilde{R}_2$ follow from Lemma A.14. □

A.4.4 Proof of Proposition 5

Proposition 5. *If the government is labor-financed, there is an equilibrium in which the government chooses the greatest tax rate that engenders no resistance, $t = \hat{t}_0$. If g or ϕ is strictly log-concave, this is the unique equilibrium tax rate.*

Proof. As in the proof of Lemma A.14, let $\eta = (N-1)/N$.

For each $t \in [0,1]$, fix an equilibrium $(p(t), r(t), c(t))$ of $\Gamma(t)$. By Propositions 4 and A.2, the government's induced utility function is

$$u_G^*(t) = u_G(t, p(t), r(t), c(t)) = \begin{cases} t\bar{P}_0 & t \le \hat{t}_0, \\ tg(\tilde{R}_1(t))\tilde{P}_1(t) & \hat{t}_0 < t < \hat{t}_1, \\ tg(\tilde{R}_2(t))\tilde{P}_2(t) & t \ge \hat{t}_1. \end{cases}$$

It is immediate from the above expression that $u_G^*(t) < u_G^*(\hat{t}_0)$ for all $t < \hat{t}_0$.

Now consider $t \in (\hat{t}_0, \hat{t}_1)$. By Lemma A.14,

$$\begin{aligned} \frac{du_G^*(t)}{dt} &= g(\tilde{R}_1(t))\tilde{P}_1(t) + tg'(\tilde{R}_1(t))\frac{d\tilde{R}_1(t)}{dt}\tilde{P}_1(t) + tg(\tilde{R}_1(t))\frac{d\tilde{P}_1(t)}{dt} \\ &= g(\tilde{R}_1(t))\tilde{P}_1(t) - \frac{\pi^p g'(\tilde{R}_1(t))\tilde{P}_1(t)\left(N\pi^p/\pi^c\tilde{P}_1(t)^2 - \eta\pi^c\hat{\phi}''(\tilde{c}_1(t))/\pi^p\right)}{\Delta_1(t)} \\ &\quad - \frac{\eta\pi^p\pi^c g(\tilde{R}_1(t))\hat{\phi}''(\tilde{c}_1(t))}{\pi^r\Delta_1(t)}, \end{aligned}$$

where $\Delta_1(t)$ is defined by (A.28). To reduce clutter in what follows, let $\tilde{P} = \tilde{P}_1(t)$, $\tilde{R} = \tilde{R}_1(t)$, and $\tilde{c} = \tilde{c}_1(t)$. Since $\Delta_1(t) < 0$, the sign of the above expression is the same as that of

$$\begin{aligned} & g'(\tilde{R})\tilde{P}\left(\frac{N(\pi^p)^2}{\pi^c\tilde{P}^2} - \eta\pi^c\hat{\phi}''(\tilde{c})\right) + \frac{\eta\pi^p\pi^c g(\tilde{R})\hat{\phi}''(\tilde{c})}{\pi^r} - g(\tilde{R})\tilde{P}\Delta_1(t) \\ &= \tilde{P}g'(\tilde{R})\left(\frac{N(\pi^p)^2}{\pi^c\tilde{P}^2} - \eta\pi^c\hat{\phi}''(\tilde{c})\right) + \frac{\eta\pi^p\pi^c g(\tilde{R})\hat{\phi}''(\tilde{c})}{\pi^r} \\ &\quad - \tilde{P}g(\tilde{R})\left(\pi^p tg'(\tilde{R}) - \pi^r t\tilde{P}g''(\tilde{R})\right)\left(\frac{N\pi^p}{\pi^c\tilde{P}^2} - \eta\frac{\pi^c}{\pi^p}\hat{\phi}''(\tilde{c})\right) \\ &\quad + \eta\pi^c t\tilde{P}g(\tilde{R})g'(\tilde{R})\hat{\phi}''(\tilde{c}) \\ &= \tilde{P}\left(g'(\tilde{R}) - \frac{g(\tilde{R})\left(\pi^p tg'(\tilde{R}) - \pi^r t\tilde{P}g''(\tilde{R})\right)}{\pi^p}\right)\left(\frac{N(\pi^p)^2}{\pi^c\tilde{P}^2} - \eta\pi^c\hat{\phi}''(\tilde{c})\right) \\ &\quad + \eta\pi^c g(\tilde{R})\hat{\phi}''(\tilde{c})\left(\frac{\pi^p}{\pi^r} + t\tilde{P}g'(\tilde{R})\right) \\ &= \frac{\tilde{P}}{g'(\tilde{R})}(1 - tg(\tilde{R}))\left(g'(\tilde{R})^2 - g(\tilde{R})g''(\tilde{R})\right)\left(\frac{N(\pi^p)^2}{\pi^c\tilde{P}^2} - \eta\pi^c\hat{\phi}''(\tilde{c})\right) \\ &\quad + \eta\pi^c g(\tilde{R})\hat{\phi}''(\tilde{c})\left(\frac{\pi^p}{\pi^r}tg(\tilde{R})\right). \end{aligned}$$

The first term is weakly negative, strictly so if g is strictly log-concave. The second term is weakly negative, strictly so if ϕ is strictly log-concave. Therefore, $du_G^*(t)/dt \le 0$ for all

$t \in (\hat{t}_0, \hat{t}_1)$, strictly so if g or ϕ is strictly log-concave. This implies $u^*_G(\hat{t}_0) \geq u^*_G(t)$ for all $t \in (\hat{t}_0, \hat{t}_1]$, strictly so if g or ϕ is strictly log-concave.

Finally, consider $t > \hat{t}_1$. Again by Lemma A.14,

$$\begin{aligned}\frac{du^*_G(t)}{dt} &= g(\tilde{R}_2(t))\tilde{P}_2(t) + tg'(\tilde{R}_2(t))\frac{d\tilde{R}_2(t)}{dt}\tilde{P}_2(t) + tg(\tilde{R}_2(t))\frac{d\tilde{P}_2(t)}{dt} \\ &= g(\tilde{R}_2(t))\tilde{P}_2(t) + \frac{g'(\tilde{R}_2(t))\tilde{P}_2(t)}{\Delta_2(t)} - \frac{\pi^p g(\tilde{R}_2(t))}{\pi^r \Delta_2(t)},\end{aligned}$$

where $\Delta_2(t)$ is defined by (A.31). To reduce clutter in what follows, let $\tilde{P} = \tilde{P}_2(t)$ and $\tilde{R} = \tilde{R}_2(t)$. Since $\Delta_2(t) > 0$, the sign of the above expression is the same as that of

$$\begin{aligned}&\tilde{P}g(\tilde{R})\Delta_2(t) + \tilde{P}g'(\tilde{R}) - \frac{\pi^p g(\tilde{R})}{\pi^r} \\ &\quad = \tilde{P}g(\tilde{R})\left(\frac{\pi^r t\tilde{P}g''(\tilde{R})}{\pi^p} - 2tg'(\tilde{R})\right) + \tilde{P}g'(\tilde{R}) - \frac{\pi^p g(\tilde{R})}{\pi^r} \\ &\quad = \frac{\pi^p}{\pi^r tg'(\tilde{R})^2}\left((1 - tg(\tilde{R}))^2\left(g(\tilde{R})g''(\tilde{R}) - g'(\tilde{R})^2\right) - \left(tg(\tilde{R}g'(\tilde{R}))\right)^2\right) \\ &\quad < \frac{\pi^p(1 - tg(\tilde{R}))^2\left(g(\tilde{R})g''(\tilde{R}) - g'(\tilde{R})^2\right)}{\pi^r tg'(\tilde{R})^2} \\ &\quad \leq 0.\end{aligned}$$

Therefore, $u^*_G(\hat{t}_0) \geq u^*_G(\hat{t}_1) > u^*_G(t)$ for all $t > \hat{t}_1$.

Combining these findings, $u^*_G(\hat{t}_0) \geq u^*_G(t)$ for all $t \in [0, 1] \setminus \{\hat{t}_0\}$, strictly so if g or ϕ is strictly log-concave. Therefore, there is an equilibrium in which $t = \hat{t}_0$, and every equilibrium has this tax rate if g or ϕ is strictly log-concave. □

A.4.5 Proof of Proposition 6

Proposition 6. *Equilibrium production with a labor-financed government, $\bar{P}_0$, is strictly decreasing in the number of factions, N. It is locally decreasing in conflict effectiveness, π^c, if and only if the incentive effect outweighs the labor-saving effect at $\bar{c}_0$.*

Proof. I will treat N as if it were continuous in order to obtain comparative statics by implicit differentiation. Throughout the proof I write $\bar{P}_0$ and $\bar{c}_0$ as functions of (N, π^c).

Recall that $(\bar{P}_0(N, \pi^c), \bar{c}_0(N, \pi^c))$ is defined as the solution to (A.14) and (A.15). To reduce clutter in what follows, I omit the evaluation point $(\bar{P}_0(N, \pi^c), 0, \bar{c}_0(N, \pi^c); t, \pi)$ from all partial derivative expressions. The Jacobian of the system is

$$\mathbf{J}_0 = \begin{bmatrix} \partial Q^{pc}/\partial P & \partial Q^{pc}/\partial C \\ \partial Q^b/\partial P & \partial Q^b/\partial C \end{bmatrix} = \begin{bmatrix} -\pi^p/\bar{P}_0(N,\pi^c)^2 & -(N-1)\pi^c\hat{\phi}''(\bar{c}_0(N,\pi^c))/N \\ -1/\pi^p & -N/\pi^c \end{bmatrix},$$

with determinant

$$|\mathbf{J}_0| = \frac{N\pi^p}{\pi^c\bar{P}_0(N,\pi^c)^2} - \frac{N-1}{N}\frac{\pi^c}{\pi^p}\hat{\phi}''(\bar{c}_0(N,\pi^c)) > 0.$$

By the implicit function theorem and Cramer's rule,

$$\begin{aligned}
\frac{\partial \bar{P}_0(N,\pi^c)}{\partial N} &= \frac{\begin{vmatrix} -\partial Q^{pc}/\partial N & \partial Q^{pc}/\partial C \\ -\partial Q^b/\partial N & \partial Q^b/\partial C \end{vmatrix}}{|\mathbf{J}_0|} \\
&= \frac{\begin{vmatrix} \pi^c\hat{\phi}'(\bar{c}_0(N,\pi^c))/N^2 & -(N-1)\pi^c\hat{\phi}''(\bar{c}_0(N,\pi^c))/N \\ \bar{c}_0(N,\pi^c)/\pi^c & -N/\pi^c \end{vmatrix}}{|\mathbf{J}_0|} \\
&= \frac{1}{|\mathbf{J}_0|}\left(\frac{N-1}{N}\bar{c}_0(N,\pi^c)\hat{\phi}''(\bar{c}_0(N,\pi^c)) - \frac{\hat{\phi}'(\bar{c}_0(N,\pi^c))}{N}\right) \\
&< 0,
\end{aligned}$$

as claimed. Similarly,

$$\begin{aligned}
\frac{\partial \bar{P}_0(N,\pi^c)}{\partial \pi^c} &= \frac{\begin{vmatrix} -\partial Q^{pc}/\partial \pi^c & \partial Q^{pc}/\partial C \\ -\partial Q^b/\partial \pi^c & \partial Q^b/\partial C \end{vmatrix}}{|\mathbf{J}_0|} \\
&= \frac{\begin{vmatrix} (N-1)\hat{\phi}'(\bar{c}_0(N,\pi^c))/N & -(N-1)\pi^c\hat{\phi}''(\bar{c}_0(N,\pi^c))/N \\ -N\bar{c}_0(N,\pi^c)/(\pi^c)^2 & -N/\pi^c \end{vmatrix}}{|\mathbf{J}_0|} \\
&= -\frac{N-1}{\pi^c|\mathbf{J}_0|}\left(\hat{\phi}'(\bar{c}_0(N,\pi^c)) + \bar{c}_0(N,\pi^c)\hat{\phi}''(\bar{c}_0(N,\pi^c))\right),
\end{aligned}$$

which is negative if and only if the incentive effect outweighs the labor-saving effect at $\bar{c}_0(N,\pi^c)$. □

A.4.6 Proof of Proposition 7

Proposition 7. *The equilibrium tax rate of a labor-financed government,*

$$\hat{t}_0 = \frac{\pi^p}{\pi^p - \pi^r \bar{P}_0 g'(0)},$$

is strictly increasing in fractionalization, N. It strictly increases with a marginal increase in competitive effectiveness, π^c, if and only if the incentive effect outweighs the labor-saving effect at $\bar{c}_0$.

Proof. Immediate from Proposition 6, as $\hat{t}_0$ is strictly decreasing in $\bar{P}_0$, and N and π^c enter the expression for $\hat{t}_0$ only via $\bar{P}_0$. □

A.4.7 Proof of Proposition 8

Proposition 8. *A labor-financed government's equilibrium payoff is strictly decreasing in the number of factions, N. It strictly decreases with a marginal increase in conflict effectiveness, π^c, if and only if the incentive effect outweighs the labor-saving effect at $\bar{c}_0$.*

Proof. By Proposition 5, the government's equilibrium payoff is

$$\hat{t}_0 \bar{P}_0 = \frac{\pi^p \bar{P}_0}{\pi^p - \pi^r \bar{P}_0 g'(0)} = \frac{\pi^p}{(\pi^p / \bar{P}_0) - \pi^r g'(0)}.$$

This expression is strictly increasing in $\bar{P}_0$. Since N and π^c only enter through the equilibrium value of $\bar{P}_0$, the claim follows from Proposition 6. □

A.5 Extensions

A.5.1 Endogenous Inequality

In the model with *asymmetric taxation*, the government is labor-financed and taxes each faction's production separately. To keep the analysis simple, I assume throughout the extension that $N = 2$. The government chooses a pair of tax rates, t_1 and t_2, where each $t_i \in [0, 1]$ as before. The factions then respond as before, by allocating their labor among production, resistance, and internal conflict, (p_i, r_i, c_i), subject to the budget constraint, Equation 1. I consider tax schemes such that $t_1 \geq t_2$; as the factions remain identical *ex ante*, this restriction is without loss of generality. The utility functions for the government and the factions are now

$$u_G(t,p,r,c) = \tau(t_1,r)p_1 + \tau(t_2,r)p_2,$$
$$u_i(t,p,r,c) = \omega_i(c)\left[\bar{\tau}(t_1,r)p_1 + \bar{\tau}(t_2,r)p_2\right].$$

If the government chooses the same tax rate for both groups, $t_1 = t_2$, then each player's utility is the same as in the original model with that rate. Throughout the analysis of this extension, I impose an additional technical condition on the function that translates c_i into effective strength in the internal conflict: I assume ϕ'/ϕ is convex.[19]

Similar to above, let $\Gamma(t_1, t_2)$ denote the labor allocation subgame following the government's choice of the given tax rates. Notice that in the model with asymmetric taxation we have

$$\frac{\partial u_i(t,p,r,c)}{\partial p_i} = \frac{\phi(c_i)}{\phi(c_i)+\phi(c_j)}(1 - t_i g(r_i + r_j)),$$
$$\frac{\partial u_i(t,p,r,c)}{\partial r_i} = \frac{\phi(c_i)}{\phi(c_i)+\phi(c_j)}(-g'(r_i + r_j))(t_i p_i + t_j p_j),$$
$$\frac{\partial u_i(t,p,r,c)}{\partial c_i} = \frac{\phi'(c_i)\phi(c_j)}{(\phi(c_i)+\phi(c_j))^2}\left[(1 - t_i g(r_i + r_j))p_i + (1 - t_j g(r_i + r_j))p_j\right].$$

To analyze the extension, I first consider how the factions would respond to the choice of unequal tax rates. Naturally, as taxation reduces the marginal benefit of production, the faction that is taxed more produces less in equilibrium. The more highly taxed faction then shifts some of the labor it would have spent on economic production into resistance and internal conflict. This has the counterintuitive implication that the equilibrium payoff for the more-taxed faction is no less than that of the less-taxed faction. By reducing a group's incentive to produce, the government increases its incentive to appropriate from the other group, resulting in it taking home a disproportionate share of the total post-tax output. This result is reminiscent of the "paradox of power" characterized by Hirshleifer (1991), wherein seemingly weaker groups expend disproportionate effort on appropriation. The following proposition summarizes the equilibrium responses to unequal taxation.

Proposition A.4. *In the game with asymmetric taxation, if the government chooses $t_1 > t_2$, then $p_1 \leq p_2$, $r_1 \geq r_2$, and $c_1 \geq c_2$ in any equilibrium of the subsequent labor allocation subgame.*

Proof. First I will prove $c_1 \geq c_2$. To this end, suppose $c_i > c_j$; I will show this implies $t_i > t_j$

[19] The baseline assumptions imply that ϕ'/ϕ is positive and decreasing, so convexity is a natural restriction. The difference and ratio functional forms described above in footnote 13 both satisfy this condition.

and thus $i = 1$. Log-concavity of ϕ implies $\phi'(c_i)\phi(c_j) \leq \phi'(c_j)\phi(c_i)$, so we have

$$\pi^c \frac{\partial u_i(t,p,r,c)}{\partial c_i} \leq \pi^c \frac{\partial u_j(t,p,r,c)}{\partial c_j}.$$

The first-order conditions of equilibrium then imply

$$\max\left\{\pi^p \frac{\partial u_i(t,p,r,c)}{\partial p_i}, \pi^r \frac{\partial u_i(t,p,r,c)}{\partial r_i}\right\} \leq \max\left\{\pi^p \frac{\partial u_j(t,p,r,c)}{\partial p_j}, \pi^r \frac{\partial u_j(t,p,r,c)}{\partial r_j}\right\}.$$

As $\phi(c_i) > \phi(c_j)$, this can hold only if $1 - t_i g(r_i + r_j) < 1 - t_j g(r_i + r_j)$; i.e., $t_i > t_j$.

I now prove $r_1 \geq r_2$. First suppose $c_1 > 0$. The first-order conditions of equilibrium, combined with the fact that $c_1 \geq c_2$, imply

$$\pi^c \frac{\partial u_2(t,p,r,c)}{\partial c_2} \geq \pi^c \frac{\partial u_1(t,p,r,c)}{\partial c_1} \geq \pi^r \frac{\partial u_1(t,p,r,c)}{\partial r_1} > \pi^r \frac{\partial u_2(t,p,r,c)}{\partial r_2}.$$

It then follows from the first-order conditions that $r_2 = 0$, which implies $r_1 \geq r_2$. On the other hand, suppose $c_1 = 0$, in which case $c_2 = 0$ per above. It is then immediate from the budget constraint that $r_1 \geq r_2$ if $p_1 = 0$, so suppose $p_1 > 0$. The first-order conditions and $t_1 > t_2$ imply

$$\pi^p \frac{\partial u_2(t,p,r,c)}{\partial p_2} > \pi^p \frac{\partial u_1(t,p,r,c)}{\partial p_1} \geq \pi^r \frac{\partial u_1(t,p,r,c)}{\partial r_1} = \pi^r \frac{\partial u_2(t,p,r,c)}{\partial r_2}.$$

It then follows from the first-order conditions that $r_2 = 0$, which implies $r_1 \geq r_2$.

Finally, under the budget constraint, $c_1 \geq c_2$ and $r_1 \geq r_2$ imply $p_1 \leq p_2$. □

The factions' responses show why asymmetric taxation is ultimately unprofitable for a labor-financed government. There is obviously no profit to be made from the more highly taxed faction, as it reduces its production in response to the greater taxation. But as the more-taxed faction increases its appropriative efforts, the less-taxed faction also loses some of its incentive to engage in productive activity. The decrease in the less-taxed faction's incentive to resist does not make up the difference, as the marginal benefit of resistance remains a function of the government's overall payoff, just as in the baseline model. Ultimately, then, the government is no better off having the ability to set unequal tax rates across groups.

Proposition A.5. *Asymmetric taxation does not raise the equilibrium payoff of a labor-financed government.*

Proof. Assume $t_1 > t_2$, and let (p, r, c) be an equilibrium of $\Gamma(t_1, t_2)$. Let $\hat{t}_0$, $\bar{P}_0$, and $\bar{c}_0$ be defined as in Proposition 4. In addition, let $P = p_1 + p_2$, $R = r_1 + r_2$, and $C = c_1 + c_2$.

My first task is to prove $P \le \bar{P}_0$. As any equilibrium entails $P > 0$, it follows from Proposition A.4 that $p_2 > 0$. If $p_1 = 0$, in which case $P = p_2$, the first-order conditions for equilibrium imply

$$\pi^p \phi(c_2) \ge \frac{\pi^c \phi'(c_2)\phi(c_1)}{\phi(c_1) + \phi(c_2)} P.$$

Rearranging terms and applying the fact that $\phi(c_1) \ge \phi(c_2)$ (per Proposition A.4) gives

$$P \le \frac{\pi^p (\phi(c_1) + \phi(c_2))\phi(c_2)}{\pi^c \phi(c_1)\phi'(c_2)} \le \frac{2\pi^p \phi(c_2)}{\pi^c \phi'(c_2)}.$$

Under this inequality, $P > \bar{P}_0$ would imply $c_2 > \bar{c}_0$, violating the budget constraint. Therefore, $P \le \bar{P}_0$. Next, suppose $p_1 > 0$, so the first-order conditions for equilibrium imply

$$\begin{aligned} \pi^p(1 - t_1 g(R)) &\ge \frac{\pi^c \phi'(c_1)\phi(c_2)}{\phi(c_1)(\phi(c_1) + \pi(c_2))} \left[P - (t \cdot p) g(R)\right], \\ \pi^p(1 - t_2 g(R)) &\ge \frac{\pi^c \phi'(c_2)\phi(c_1)}{\phi(c_2)(\phi(c_1) + \pi(c_2))} \left[P - (t \cdot p) g(R)\right]. \end{aligned} \tag{A.32}$$

As $\log \phi$ is concave and its derivative is convex, $c_1 \ge c_2$ (per Proposition A.4) implies $\phi'(c_1)/\phi(c_1) \le \phi'(c_2)/\phi(c_2)$ and

$$\frac{1}{2}\left(\frac{\phi'(c_1)}{\phi(c_1)} + \frac{\phi'(c_2)}{\phi(c_2)}\right) \ge \frac{\phi'(C/2)}{\phi(C/2)}.$$

In addition, $p_1 \le p_2$ and $t_1 > t_2$ imply

$$\left(\frac{t_1 + t_2}{2}\right) P \ge t_1 p_1 + t_2 p_2.$$

Summing the conditions in (A.32) and applying these inequalities gives

$$\begin{aligned} &2\pi^p \left(1 - \frac{t_1 + t_2}{2} g(R)\right) \\ &\quad \ge \pi^c \left[P - (t \cdot p) g(R)\right] \left[\left(\frac{\phi(c_2)}{\phi(c_1) + \phi(c_2)}\right) \frac{\phi'(c_1)}{\phi(c_1)} + \left(\frac{\phi(c_1)}{\phi(c_1) + \phi(c_2)}\right) \frac{\phi'(c_2)}{\phi(c_2)}\right] \\ &\quad \ge \pi^c \left[P - (t \cdot p) g(R)\right] \left[\frac{1}{2}\left(\frac{\phi'(c_1)}{\phi(c_1)} + \frac{\phi'(c_2)}{\phi(c_2)}\right)\right] \\ &\quad \ge \pi^c \left[P - (t \cdot p) g(R)\right] \left(\frac{\phi'(C/2)}{\phi(C/2)}\right) \\ &\quad \ge \pi^c P \left(1 - \frac{t_1 + t_2}{2} g(R)\right) \left(\frac{\phi'(C/2)}{\phi(C/2)}\right) \end{aligned}$$

Simplifying and rearranging terms gives

$$P \le \frac{2\pi^p \phi'(C/2)}{\pi^c \phi(C/2)}.$$

As in the previous case, the budget constraint then implies $P \le \bar{P}_0$.

If $t_i g(R) \le \hat{t}_0$ for each $i = 1, 2$ such that $p_i > 0$, then $P \le \bar{P}_0$ implies $u_G(t, c, p, r) \le \hat{t}_0 \bar{P}_0$, as claimed. So suppose there is a group i such that $p_i > 0$ and $t_i g(R) > \hat{t}_0$. The first-order conditions for equilibrium imply

$$\pi^p(1 - t_i g(R)) \ge \pi^r(-g'(R))(t_i p_i + t_j p_j).$$

This inequality, combined with log-concavity of g and the assumption that $t_i g(R) > \hat{t}_0$, gives

$$\begin{aligned} u_G(t, p, c, r) &= (t_i p_i + t_j p_j) g(R) \\ &\le -\left(\frac{g(R)}{g'(R)}\right) \frac{\pi^p(1 - t_i g(R))}{\pi^r} \\ &< -\left(\frac{1}{g'(0)}\right) \frac{\pi^p(1 - \hat{t}_0)}{\pi^r} \\ &= \left(\frac{\pi^p}{\pi^p - \pi^r \bar{P}_0 g'(0)}\right) \bar{P}_0 \\ &= \hat{t}_0 \bar{P}_0, \end{aligned}$$

as claimed. □

This brief extension demonstrates that the earlier results for labor-financed extraction do not depend on the assumption of equal tax rates across factions. All else equal, a labor-financed government benefits from social order and has no incentive to create inequality where none exists before. What this extension does not answer is how asymmetric tax rates would interact with *ex ante* asymmetries in productivity or size among factions, a topic that is beyond the scope of the present analysis.

A.5.2 Conquest

In the *conquest model*, a set of N factions compete with each other and with an outsider, denoted O, for the chance to be the government in the future. The incremental value of being the government is $v(N) > 0$, which may increase with N (when capital is the main source of revenue) or decrease (when labor is the main source of revenue). Each faction has L/N units of labor, which it may divide between two activities: $s_i \ge 0$, to prevent the outsider from taking over; and $d_i \geq 0$, to influence its own chance of becoming the government if the outsider fails. Each faction's budget constraint is[20]

$$s_i + d_i = \frac{L}{N}. \tag{A.33}$$

The success of the attempted takeover depends on how much the factions spend to combat the outsider. I assume the outsider's military strength is a fixed value, $\bar{s}_O > 0$, so the outsider is not a strategic player here. The assumption that the outsider's strength is exogenous is of course a simplification, but it is plausible in situations where the outsider marshals its forces before fully understanding the internal political situation—such as in Cortés's incursion into the Mexican mainland, and other early maritime colonial ventures. The probability that the outsider becomes the government is

$$\frac{\bar{s}_O}{\bar{s}_O + \chi(\sum_{i=1}^N s_i)}, \tag{A.34}$$

where $\chi : [0, L] \to \mathbb{R}_+$ represents the translation of society's labor into its strength against the outsider. In case the outsider fails, the probability that faction i becomes the government is

$$\frac{\psi(d_i)}{\sum_{j=1}^N \psi(d_j)}, \tag{A.35}$$

where $\psi : [0, L/N] \to \mathbb{R}_+$ represents the translation of an individual faction's labor into its proportional chance of success against other factions. As with the function ϕ in the original model, I assume χ and ψ are strictly increasing and log-concave.

The factions simultaneously choose how to allocate their labor, subject to the budget constraint (A.33). A faction's utility function is

$$u_i(s, d) = \frac{\psi(d_i)}{\sum_{j=1}^N \psi(d_j)} \times \frac{\chi(\sum_{j=1}^N s_j)}{\bar{s}_O + \chi(\sum_{j=1}^N s_j)} \times v(N), \tag{A.36}$$

where $s = (s_1, \ldots, s_N)$ and $d = (d_1, \ldots, d_N)$.

I begin by characterizing the equilibrium of the conquest game. Throughout the proofs, let $\hat{\chi} = \log \chi$ and $\hat{\psi} = \log \psi$. I will characterize equilibria in terms of the criterion function

$$Q^{ds}(S; N) = \frac{N-1}{N}(\psi(S) + \bar{s}_O)\hat{\chi}'\left(\frac{L-S}{N}\right) - \hat{\psi}'(S)\bar{s}_O, \tag{A.37}$$

[20] The assumption of unit productivity for each activity is without loss of generality. The model here with functional forms $\chi(S) = \tilde{\chi}(\pi^s S)$ and $\psi(D) = \tilde{\psi}(\pi^d D)$ is isomorphic to a model with the common budget constraint $s_i/\pi^s + d_i/\pi^d = L/N$ and functional forms $\tilde{\chi}$ and $\tilde{\psi}$.

which is strictly increasing in both S and N.

Lemma A.15. *The conquest game has a unique equilibrium in which each*

$$s_i = \begin{cases} 0 & Q^{ds}(0; N) \geq 0, \\ \tilde{S}(N)/N & Q^{ds}(0; N) < 0, Q^{ds}(L; N) > 0, \\ L/N & Q^{ds}(L; N) \leq 0, \end{cases}$$

and each $d_i = L/N - s_i$*, where* $\tilde{S}(N)$ *is the unique solution to* $Q^{ds}(\tilde{S}(N); N) = 0$*.*

Proof. Like the original game, the conquest game is log-concave, so a pure-strategy equilibrium exists can be characterized by first-order conditions. In addition, the proof of Lemma A.7 carries over to the conquest game, so in equilibrium each $d_i = d_j$ for $i, j \in \mathcal{N}$. The claim then follows from the first-order conditions for maximization of each faction's utility. □

The strategic tradeoff for the factions here is analogous to the tradeoff between resistance and internal conflict in the baseline model. Critically, the relative marginal benefit of fighting the outsider declines as the number of factions increases. When the number of factions is large, any individual faction's chance of becoming the government if the outsider loses is small, which in turn reduces its incentive to contribute to the collective effort against the outsider. Consequently, as the following result states, the outsider is more likely to win the more divided the society is.

Proposition A.6. *In the conquest model, the probability that the outsider wins is increasing in the number of factions,* N*.*

Proof. I will prove that the equilibrium value of $\sum_i s_i$ decreases with the number of factions. Let (d, s) and (d', s') be the equilibria at N and N' respectively, where $N' > N$, and let $S = \sum_i s_i$ and $S' = \sum_i s_i'$. If $S = 0$, then $Q^{ds}(0; N) \geq 0$ and thus $Q^{ds}(0; N') \geq 0$, so $S' = 0$ as well. If $S \in (0, L)$, then $S = \tilde{S}(N)$, which implies $Q^{ds}(L; N) > 0$ and thus $Q^{ds}(L; N') > 0$. This in turn implies either $S' = \tilde{S}(N') < \tilde{S}(N)$ or $S' = 0 < \tilde{S}(N)$. Finally, if $S = L$, then it is trivial that $S' \leq S$. □

A.5.3 Informal Sector Production

In the baseline model with a capital-financed government, the factions' only choices are resistance and internal competition. Here I extend the model to allow factions to allocate labor to the informal sector, which has a fixed marginal value and cannot be expropriated by the government or by other factions. Let o_i denote informal sector production, and let $\kappa > 0$ denote the per-unit consumption value. Each faction's utility function in the extended model is

$$u_i(t, o, r, c) = \omega_i(c) \times \bar{\tau}(t, r) \times X + \kappa o_i, \tag{A.38}$$

and they are subject to the budget constraint

$$o_i + \frac{r_i}{\pi^r} + \frac{c_i}{\pi^c} = \frac{L}{N}. \tag{A.39}$$

(There is no loss of generality in normalizing the labor cost of informal production to 1.) Because the government cannot expropriate informal production, its utility function remains as in Equation 6.

The existence and uniqueness arguments from the baseline model do not immediately carry over to the extended model, as quasiconcavity is not necessarily preserved under addition. If u_i is strictly quasiconcave in a faction's own choices,[21] then the arguments from Proposition A.1 and Proposition A.2 can be adapted to prove that the labor allocation subgame in the extended model has a unique equilibrium characterized by first-order conditions, in which $c_i = C$ and $o_i = O$ for all factions i. From there, the argument of Lemma A.6 can be adapted to show that total equilibrium resistance is continuous in t, which in turn implies existence of an optimal tax rate per the argument of Proposition A.3. For the remainder of the analysis of this extension, I will proceed under the assumption that u_i is strictly quasiconcave in a faction's own choices, so that the existence and continuity of equilibria in the labor allocation subgame can be presumed. Additionally, I will assume $\phi(0) > 0$ to ensure the utility function is everywhere continuously differentiable.

My goal is to show that the main substantive result of the baseline analysis with a capital-financed government—namely, that the government's equilibrium payoff increases with the extent of social fractionalization (Proposition 2)—holds up in the environment with informal production. I first consider a global result, showing that the government extracts the full resource endowment in equilibrium if fractionalization is great enough.

Proposition A.7. *In the model with a capital-financed government and informal production, if $N \geq \frac{-\pi^r g'(0) X}{\kappa}$, then there is an equilibrium where $t^* = 1$ and each $(o_i^*(1), r_i^*(1), c_i^*(1)) = (\frac{L}{N}, 0, 0)$.*

Proof. I begin by proving that each $o_i^*(1) = \frac{L}{N}$ (and thus $r_i^*(1) = c_i^*(1) = 0$ by the budget

[21] A sufficient condition is that $\omega_i(c) \cdot \bar{\tau}(t, r)$ is strictly concave in (c_i, r_i).

constraint) under the hypothesis of the proposition. It will suffice to show that the first-order conditions for a best response are satisfied for each faction at the proposed allocation. The relevant partial derivatives are:

$$\frac{\partial u_i(t,o,r,c)}{\partial o_i} = \kappa,$$
$$\frac{\partial u_i(t,o,r,c)}{\partial r_i} = \frac{-g'(0)}{N} \cdot X,$$
$$\frac{\partial u_i(t,o,r,c)}{\partial c_i} = 0.$$

If $N \geq \frac{-\pi^r g'(0)X}{\kappa}$, then we have

$$\frac{\partial u_i(t,o,r,c)}{\partial o_i} \geq \max\left\{\pi^r \frac{\partial u_i(t,o,r,c)}{\partial r_i}, \pi^c \frac{\partial u_i(t,o,r,c)}{\partial c_i}\right\},$$

confirming that the proposed allocation is an equilibrium.

To conclude, I must prove that $t^* = 1$ is optimal for 1. Because $\sum_i r_i^*(1) = 0$, the government's expected utility from $t^* = 1$ is $u_G(1, o^*(1), r^*(1), c^*(1)) = X$. Regardless of the factions' responses, no other tax rate can yield a strictly greater payoff for the government, so the proposed tax rate is optimal. □

I next prove a local result, showing that the government's utility decreases with fractionalization when there is positive informal production in equilibrium. As in the proof of Proposition 2 above, I treat N as though it were continuous so as to obtain comparative statics via implicit differentiation.

Proposition A.8. *In the model with a capital-financed government and informal production, if there is an equilibrium where $t = t^*$ and each $o_i^*(t^*) > 0$, then the government's equilibrium utility is locally non-decreasing in N.*

Proof. As in the proof of Proposition 2 above, it will suffice to show that $R^*(t^*)$ locally decreases with N. The claim holds trivially if $R^*(t^*) = 0$, so consider the case where $R^*(t^*) > 0$. In this case, equilibrium resistance is defined by the condition

$$\pi^r \frac{\partial u_i(t^*,o,r,c)}{\partial r_i} = \frac{\pi^r(-t^* g'(R^*(t^*)))}{N} = \kappa = \frac{\partial u_i(t^*,o,r,c)}{\partial o_i}.$$

Convexity of g implies $-g'$ decreases with $R^*(t^*)$. Therefore, as N increases, $R^*(t^*)$ must decrease in order to maintain the condition. □

A.5.4 Incomplete Information

In the baseline model with a labor-financed government, the equilibrium tax rate is $\hat{t}_0$, the highest level at which there is no resistance (Proposition 5). The proof that such a government's equilibrium payoff decreases with fractionalization (Proposition 8) depends on this property. In this section, to probe the robustness of the result relating fractionalization to government revenues, I consider a simple extension of the baseline labor-financed environment in which the optimal tax rate may engender positive resistance. The main result still holds true in this environment: the government's equilibrium payoff decreases with the extent of fractionalization.

To introduce the possibility of positive resistance in equilibrium, I assume there is incomplete information about the labor cost of resistance at the time that the government chooses the tax rate. After the government chooses t, Nature draws π^r from a distribution over $\{\pi^r_\ell, \pi^r_h\}$, where $\pi^r_\ell < \pi^r_h$, and reveals its value to all players. The factions then play the labor subgame as usual. The prior distribution of π^r and the values of all other parameters are common knowledge from the outset of the game. Let $\xi \equiv \Pr(\pi^r = \pi^r_h)$, and to avoid trivialities assume $\xi \in (0, 1)$.

The incomplete-information setup creates the possibility of nonzero resistance occurring with positive probability along the equilibrium path. Let $\hat{t}_{0\ell}$ denote the greatest tax rate that engenders no resistance if $\pi^r = \pi^r_\ell$ (Equation A.21), and let $\hat{t}_{0h}$ be defined analogously. We naturally have $\hat{t}_{0h} \leq \hat{t}_{0\ell}$, so the government can guarantee no resistance by choosing $t = \hat{t}_{0h}$. A choice of $t \in (\hat{t}_{0h}, \hat{t}_{0\ell}]$ will raise the government's payoff in case $\pi^r = \pi^r_\ell$ but will lower it in case $\pi^r = \pi^r_h$. Following the same risk-reward tradeoff logic as in bargaining models with incomplete information (Fearon 1995), it is optimal to choose $t > \hat{t}_{0h}$ if ξ is sufficiently small.

For tractability, I impose specific functional forms to analyze the extended model. Let $\phi(c_i) = c_i$, so that ω_i is a ratio-form contest success function. Let $g(R) = 1 - R$, so that the relationship between total resistance and the effective tax rate is linear. Finally, to reduce clutter in the analysis, I normalize $L = 1$. Consequently, I also assume $\pi^r_h \leq 1$, so that $g(R) \geq 0$ for all feasible $R \in [0, \pi^r L]$.

With these functional forms, the labor allocation subgame has a closed-form solution for each value of π^r. Lemma A.3 implies each $c_i > 0$ in equilibrium, so each $\hat{t}_1 > 1$; the equilibrium following any $t > \hat{t}_0$ will thus be characterized by (A.16)–(A.18). It is straightforward to verify that the equilibrium conditions imply the following:

$$\bar{P}_{0\ell} = \bar{P}_{0h} = \frac{\pi^p}{N},$$

$$\hat{t}_{0\ell} = \frac{N}{N + \pi^r_\ell},$$

$$\hat{t}_{0h} = \frac{N}{N + \pi_h^r},$$

$$\tilde{P}_{1h}(t) = \frac{\pi^p \left(\frac{1}{t} - 1 + \pi_h^r\right)}{\pi_h^r (N+1)},$$

$$\tilde{R}_{1h}(t) = \frac{N\left(1 - \frac{1}{t}\right) + \pi_h^r}{N+1}.$$

For any $t \geq \hat{t}_{0h}$, the effective tax rate in case $\pi^r = \pi_h^r$ is

$$t\left(1 - \tilde{R}_{1h}(t)\right) = t\left[\frac{[N+1] - [N(1-\frac{1}{t}) + \pi_h^r]}{N+1}\right] = \frac{N + (1-\pi_h^r)t}{N+1}.$$

The government's utility in this case is therefore

$$\begin{aligned} t\left(1 - \tilde{R}_{1h}(t)\right)\tilde{P}_{1h}(t) &= \frac{N + (1-\pi_h^r)t}{N+1} \times \frac{\pi^p \left(\frac{1}{t} - 1 + \pi_r^h\right)}{\pi_h^r (N+1)} \\ &= \frac{\pi^p N}{\pi_h^r (N+1)^2} \underbrace{\left[\frac{1}{t} - 1 + \pi_h^r\right]}_{>0} + \frac{\pi^p (1-\pi_h^r)[1 - (1-\pi_h^r)t]}{\pi_h^r (N+1)^2}. \end{aligned} \tag{A.40}$$

I now prove that the government's expected utility strictly decreases with N in the incomplete information game, even when the equilibrium entails positive probability of resistance along the equilibrium path. Arguments from the baseline analysis imply that no $t < \hat{t}_{0h}$ or $t > \hat{t}_{0\ell}$ may be optimal, so we may restrict attention to $t \in [\hat{t}_{0h}, \hat{t}_{0\ell}]$. In this range, the government's expected utility from any tax rate is

$$\begin{aligned} &\mathbb{E}[u_G(t, p^*(t), r^*(t), c^*(t))] \\ &\quad = \xi\left[t\left(1 - \tilde{R}_{1h}(t)\right)\tilde{P}_{1h}(t)\right] + (1-\xi)\left[t\bar{P}_{0\ell}\right] \\ &\quad = \frac{\xi\pi^p N}{\pi_h^r (N+1)^2}\left[\frac{1}{t} - 1 + \pi_h^r\right] + \frac{\xi\pi^p (1-\pi_h^r)[1-(1-\pi_h^r)t]}{\pi_h^r (N+1)^2} + \frac{(1-\xi)\pi^p t}{N}. \end{aligned}$$

This expression is strictly convex in t, so it is maximized at one of the boundary points, $t \in \{\hat{t}_{0h}, \hat{t}_{0\ell}\}$. At the lower boundary, $t = \hat{t}_{0h}$, the tax demand engenders no resistance regardless of the factions' type. This yields a government payoff of

$$\xi\hat{t}_{0h}\bar{P}_{0h} + (1-\xi)\hat{t}_{0h}\bar{P}_{0\ell} = \frac{\pi^p}{N + \pi_h^r},$$

which is strictly decreasing in N. At the upper boundary, $t = \hat{t}_{0\ell}$, the government's payoff is

$$\xi \hat{t}_{0\ell} \left(1 - \tilde{R}_{1h}(\hat{t}_{0\ell})\right) \tilde{P}_{1h}(\hat{t}_{0\ell}) + (1-\xi)\hat{t}_{0\ell}\bar{P}_{0\ell}.$$

It is evident from Equation A.40 that $t(1 - \tilde{R}_{1h}(t))\tilde{P}_{1h}(t)$ is strictly decreasing in both N and t. Since $\hat{t}_{0\ell}$ is increasing in N, this means the first term of the expression here is strictly decreasing in N. Meanwhile, the baseline model result Proposition 8 implies that the second term is strictly decreasing in N, as it is simply a scalar multiple of the government's utility from the complete-information game where $\pi^r = \pi^r_\ell$. Altogether, we have that the government's equilibrium expected utility,

$$\begin{aligned} &\max_{t \in [0,1]} \mathbb{E}[u_G(t, p^*(t), r^*(t), c^*(t))] \\ &\qquad = \max \left\{ \mathbb{E}[u_G(\hat{t}_{0h}, p^*(\hat{t}_{0h}), r^*(\hat{t}_{0h}), c^*(\hat{t}_{0h}))], \mathbb{E}[u_G(\hat{t}_{0\ell}, p^*(\hat{t}_{0\ell}), r^*(\hat{t}_{0\ell}), c^*(\hat{t}_{0\ell}))] \right\}, \end{aligned}$$

is strictly decreasing in N, just as in the baseline model.

A.5.5 Combined Capital and Labor Financing

In the baseline model, each player's utility is a fraction of either exogenous resources X or endogenous production $\sum_i p_i$. I now extend the model to allow these to be combined, altering the production function to $f(p) = X + \sum_i p_i$ (where $X > 0$). In the extended model, the government's utility is U-shaped as a function of social fractionalization. Marginal increases in fractionalization reduce government revenues at low levels (when equilibrium behavior resembles the baseline model with a labor-financed government), but increase revenue once N is high enough (when behavior is more like in the capital-financed baseline). Additionally, as the value of the exogenous resource X increases, the portion of the parameter space where fractionalization increases revenues grows.

As in the extension with incomplete information (subsubsection A.5.4), I impose particular functional forms to allow for a closed-form solution. Specifically, I again let $\phi(c_i) = c_i$ and $g(R) = 1 - R$, and I assume $L = 1$ and $\pi^r \leq 1$ to ensure that $g(R) \geq 0$.

Equilibrium with low fractionalization. If $N < 1 + \frac{\pi^p}{X}$, then the equilibrium closely resembles that of the baseline model with a labor-financed government. (All of the following statements can be verified by checking them against the first-order conditions for optimal labor allocation.) Behavior in the labor allocation subgame is determined by two cutpoints on the tax rate. If $t \leq \hat{t}_0 \equiv \frac{N\pi^p}{N\pi^p + \pi^r(\pi^p + X)}$, then we have

$$P^*(t) = \frac{\pi^p - (N-1)X}{N};$$
$$R^*(t) = 0.$$

If $\hat{t}_0 < t < \hat{t}_1 \equiv \frac{\pi^p}{\pi^p(1-\pi^r) + N\pi^r X}$, then we have

$$P^*(t) = \frac{\pi^p(\pi^r + \frac{1}{t} - 1) - N\pi^r X}{(N+1)\pi^r};$$
$$R^*(t) = \frac{\pi^r(1 + \frac{X}{\pi^p}) - N(\frac{1}{t} - 1)}{N+1}.$$

Finally, if $t \geq \hat{t}_1$, then we have

$$P^*(t) = 0;$$
$$R^*(t) = \frac{\pi^r - (N-1)\left(\frac{1}{t} - 1\right)}{N}.$$

Note that $\hat{t}_1 \geq 1$ if and only if $N \leq \frac{\pi^p}{X}$.

I now solve for the government's optimal tax rate. Clearly no $t < \hat{t}_0$ may be optimal. For $t \in [\hat{t}_0, \hat{t}_1]$, government revenues given equilibrium responses by the factions are

$$\begin{aligned}
&u_G(t, p^*(t), r^*(t), c^*(t)) \\
&\quad = t \times (1 - R^*(t)) \times (X + P^*(t)) \\
&\quad = \frac{\pi^p}{(N+1)^2}\left[t\left(1 - \pi^r - \frac{\pi^r X}{\pi^p}\right) + N\right]\left[\frac{1}{t} + \pi^r\left(1 + \frac{X}{\pi^p}\right) - 1\right] \\
&\quad = \frac{\pi^p}{(N+1)^2}\left[\left(1 - \pi^r - \frac{\pi^r X}{\pi^p}\right) + \frac{N}{t} - t\left(1 - \pi^r - \frac{\pi^r X}{\pi^p}\right) - N\right]
\end{aligned}$$

Because $\pi^r \leq 1$ and $\frac{X}{\pi^p} < \frac{1}{N-1}$, we have

$$\begin{aligned}
\frac{du_G(t, p^*(t), r^*(t), c^*(t))}{dt} &= \frac{\pi^p}{(N+1)^2}\left[-\frac{N}{t^2} - 1 + \pi^r + \frac{\pi^r X}{\pi^p}\right] \\
&\leq \frac{\pi^p}{(N+1)^2}\left[\frac{X}{\pi^p} - \frac{N}{t^2}\right] \\
&< \frac{\pi^p}{(N+1)^2}\left[\frac{1}{N-1} - N\right] \\
&< 0.
\end{aligned}$$

Therefore, no $t \in (\hat{t}_0, \hat{t}_1]$ may be optimal for the government. Meanwhile, for $t > \hat{t}_1$ the government's payoff is

$$\begin{aligned} u_G(t, p^*(t), r^*(t), c^*(t)) &= t \times (1 - R^*(t)) \times X \\ &= \frac{(1-\pi^r)t + N - 1}{N} \times X, \end{aligned}$$

which is weakly increasing in t (strictly if $\pi^r < 1$). Therefore, no $t \in [\hat{t}_1, 1)$ may be optimal for the government.

If $N \leq \frac{\pi^p}{X}$, so that $\hat{t}_1 \geq 1$, then the argument above implies that $t^* = \hat{t}_0$. Otherwise, if $\frac{\pi^p}{X} < N < 1 + \frac{\pi^p}{X}$, then the optimal tax rate depends on the government's preference between $t = \hat{t}_0$ and $t = 1$. We have

$$\begin{aligned} u_G(\hat{t}_0, p^*(\hat{t}_0), r^*(\hat{t}_0), c^*(\hat{t}_0)) = \hat{t}_0 \times (X + P^*(\hat{t}_0)) &= \frac{\pi^p(\pi^p + X)}{N\pi^p + \pi^r(\pi^p + X)}; \\ u_G(1, p^*(1), r^*(1), c^*(1)) = (1 - R^*(1)) \cdot X \quad &= \frac{(N - \pi^r)X}{N}. \end{aligned}$$

It is evident from these expressions that the government's utility from $t = \hat{t}_0$ is decreasing in N, while its utility from $t = 1$ is increasing in N. This implies that there exists a cutpoint $N^* \in [\frac{\pi^p}{X}, 1 + \frac{\pi^p}{X}]$ such that $t^* = \hat{t}_0$ if $N \in [0, N^*]$ and $t^* = 1$ if $N \in (N^*, 1 + \frac{\pi^p}{X})$. Additionally, the government's equilibrium utility as a function of N is decreasing on $[2, N^*)$ and increasing on $(N^*, 1 + \frac{\pi^p}{X})$.

Equilibrium with high fractionalization. If $N \geq 1 + \frac{\pi^p}{X}$, then the equilibrium resembles that of the baseline model with a capital-financed government. If $t \leq \hat{t}_X \equiv \frac{N-1}{N-1+\pi^r}$, then $P^*(t) = R^*(t) = 0$. Otherwise, if $t > \hat{t}_X$, then equilibrium labor allocation is the same as in the case above with $t > \hat{t}_1$:

$$\begin{aligned} P^*(t) &= 0; \\ R^*(t) &= \frac{\pi^r - (N-1)\left(\frac{1}{t} - 1\right)}{N}. \end{aligned}$$

It then follows from the same arguments as above that the government's optimal tax rate is $t^* = 1$. Combined with the results from above, we now have that the government's equilbrium utility as a function of N is decreasing on $[2, N^*)$ and increasing on (N^*, ∞).

Exogenous resources and the cutpoint. The last step of the argument is to show that the government benefits from fractionalization under a wider set of parameters as the resource

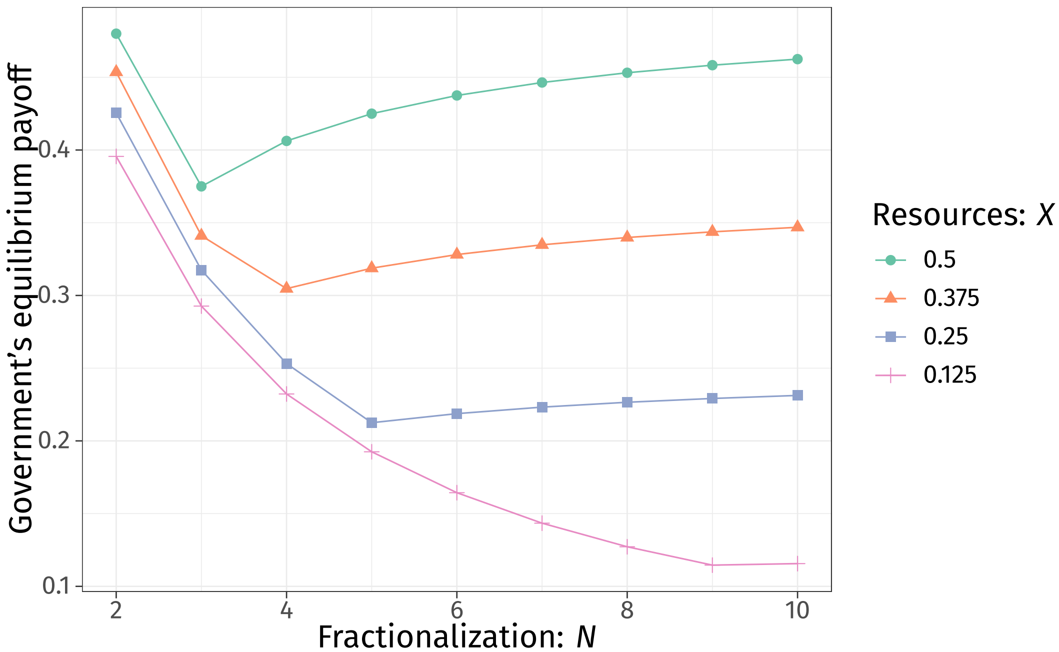

Figure A.1. Government's equilibrium payoff as a function of N and X in the extension with combined resources and production.

endowment increases; i.e., that the cutpoint N^* is decreasing in X. This is immediate in case $N^* = \frac{\pi^p}{X}$ or $N^* = 1 + \frac{\pi^p}{X}$. The last case to consider is when $N^* \in (\frac{\pi^p}{X}, 1 + \frac{\pi^p}{X})$, in which case N^* is defined as the root of

$$u_G(1, p^*(1), r^*(1), c^*(1)) - u_G(\hat{t}_0, p^*(\hat{t}_0), r^*(\hat{t}_0), c^*(\hat{t}_0)) = \frac{(N - \pi^r)X}{N} - \frac{\pi^p(\pi^p + X)}{N\pi^p + \pi^r(\pi^p + X)}.$$

Since this expression is increasing in N, to prove that N^* decreases with X it will suffice to prove that this expression is also increasing in X. We have

$$\begin{aligned} \frac{d}{dX}\left[\frac{(N - \pi^r)X}{N} - \frac{\pi^p(\pi^p + X)}{N\pi^p + \pi^r(\pi^p + X)}\right] &= 1 - \frac{\pi^r}{N} - \frac{N(\pi^p)^2}{[N\pi^p + \pi^r(\pi^p + X)]^2} \\ &> \frac{N - \pi^r - 1}{N} \\ &\geq 0, \end{aligned}$$

where the final inequality holds because $\pi^r \leq 1$ and $N \geq 2$. Therefore, N^* decreases with X. Figure A.1 illustrates how the government's utility is U-shaped in N, as well as how the range where N increases revenues expands as X increases.