# Online Appendix: Formal Proofs

We prove the formal statements in a slightly different order than they are presented in the text. We use a single proof for propositions 1 and 5, because proving the former proposition requires to proof the latter. We begin by proving several intermediary results.

## Intermediary results

**Lemma 1.** *A distribution of internal support* $\boldsymbol{p}$ *is feasible if and only if*

$$\sum_{a=0,1} \int_0^1 \left( \frac{1}{1+a} - (1+\kappa)(1-\theta) \right) p_a(\theta) dF(\theta) \geq 1 - (1+\kappa)\left(1 - \mathbb{E}[\theta]\right). \tag{1}$$

***Proof.*** We first prove the sufficiency by constructing an information structure from a given distribution of support that satisfies (1). Then, we prove the necessity by showing that any distribution of support induced by a subgame equilibrium under some information structure must satisfy (1). To prove sufficiency, assume that $\boldsymbol{p}$ satisfies (1). Then, suppose $(M, \sigma)$ is such that $M = \{0, 1\}$ and for each $\theta$,

$$\begin{aligned} \sigma(0,0|\theta) &= p_0(\theta) \\ \sigma(1,0|\theta) = \sigma(0,1|\theta) &= \frac{1}{2} p_1(\theta) \\ \sigma(1,1|\theta) &= p_2(\theta). \end{aligned}$$

Let $(q_1^*, q_2^*)$ be a strategy profile such that $q_i^*(m_i) = m_i$ for each $m_i = 0, 1$ and $i = 1, 2$. We now verify that $(q_1^*, q_2^*)$ is an equilibrium under $(M, \sigma)$. Consider the best response of an ally $i$ to the strategy of his peer $q_{-i}^*$. If $m_i = 0$, then the expected payoff from $a_i = 0$ is

$$\begin{aligned} &\int_0^1 \sigma(0,0|\theta) dF(\theta) + \int_0^1 \left(\theta - (1-\theta)\kappa\right) \sigma(0,1|\theta) dF(\theta) \\ &= \int_0^1 p_0(\theta) dF(\theta) + \int_0^1 \left(\theta - (1-\theta)\kappa\right) \frac{1}{2} p_1(\theta) dF(\theta). \end{aligned}$$

The latter expression can be rearranged to get

$$\begin{aligned} &\sum_{a=0,1} \int_0^1 \left( \frac{1}{1+a} - (1+\kappa)(1-\theta) \right) p_a(\theta) dF(\theta) + (1+\kappa) \int_0^1 (1-\theta) \left( p_0(\theta) + \frac{1}{2} p_1(\theta) \right) dF(\theta) \\ &\geq \sum_{a=0,1} \int_0^1 \left( \frac{1}{1+a} - (1+\kappa)(1-\theta) \right) p_a(\theta) dF(\theta) \\ &\geq 1 - (1+\kappa)\left(1 - \mathbb{E}[\theta]\right) > 0, \end{aligned}$$

and so $a_i = 0$ is optimal following $m_i = 0$. Conversely, $a_i = 1$ is optimal after $m_i = 1$, since, otherwise, by choosing $a_i = 0$, the ally $i$ would expect to obtain

$$\begin{aligned}
&\int_0^1 (\theta - (1-\theta)\kappa)\,\sigma(1,1|\theta)dF(\theta) + \int_0^1 \sigma(1,0|\theta)dF(\theta) \\
=& \int_0^1 (\theta - (1-\theta)\kappa)\,p_2(\theta)dF(\theta) + \int_0^1 \frac{1}{2}p_1(\theta)dF(\theta) \\
=& \int_0^1 (\theta - (1-\theta)\kappa)\,(1 - p_0(\theta) - p_1(\theta))\,dF(\theta) + \int_0^1 \frac{1}{2}p_1(\theta)dF(\theta) \\
=& 1 - (1+\kappa)\,(1 - \mathbb{E}[\theta]) - \sum_{a=0,1}\int_0^1 \left(\frac{1}{1+a} - (1+\kappa)(1-\theta)\right) p_a(\theta)dF(\theta) \le 0.
\end{aligned}$$

So $q_i^*$ is a best response of $i$ given $q_{-i}^*$ and so $(q_1^*, q_2^*)$ constitutes a Bayesian Nash equilibrium under $(M, \sigma)$. By construction, $(q_1^*, q_2^*)$ induces $\boldsymbol{p}$ under $(M, \sigma)$, and so $\boldsymbol{p}$ is feasible.

To prove the necessity, suppose that $\boldsymbol{p}$ is feasible. If so, then by definition there exists an information structure $(M, \sigma)$ and an equilibrium $(q_1^*, q_2^*)$ under this information structure that induces $\boldsymbol{p}$. Because $q_1^*$ is a best response against $q_2^*$, if ally 1 observes any $m_1$ such that $q_1^*(m_1) > 0$, it must be true that he prefers to oppose the ruler and so

$$\int_0^1 \int_{M_2} ((\theta - (1-\theta)\kappa)\,q_2^*(m_2) + 1 - q_2^*(m_2))\,\sigma(m_1, dm_2|\theta)dF(\theta) \le 0.$$

Integrating both sides with respect to $m_1$ implies that

$$\begin{aligned}
&\int_{M_1} q_1^*(m_1) \int_0^1 \int_{M_2} ((\theta - (1-\theta)\kappa)\,q_2^*(m_2) + 1 - q_2^*(m_2))\,\sigma(dm_1, dm_2|\theta)dF(\theta) \\
=& \int_0^1 (1 - (1+\kappa)(1-\theta)) \left(\int_M q_1^*(m_1)q_2^*(m_2)\sigma(dm_1, dm_2|\theta)\right) \mathrm{d}F(\theta) \\
&+ \int_0^1 \left(\int_M (q_1^*(m_1) - q_1^*(m_1)q_2^*(m_2))\,\sigma(dm_1, dm_2|\theta)\right) dF(\theta) \le 0.
\end{aligned}$$

Equivalent steps yield

$$\begin{aligned}
&\int_0^1 (1 - (1+\kappa)(1-\theta)) \left(\int_M q_1^*(m_1)q_2^*(m_2)\sigma(dm_1, dm_2|\theta)\right) dF(\theta) \\
&+ \int_0^1 \left(\int_M (q_2^*(m_2) - q_1^*(m_1)q_2^*(m_2))\,\sigma(dm_1, dm_2|\theta)\right) dF(\theta) \le 0.
\end{aligned}$$

Adding up the above two inequalities and dividing by 2 we get

$$\begin{aligned}
&\int_0^1 (1 - (1+\kappa)(1-\theta)) \left(\int_M q_1^*(m_1)q_2^*(m_2)\sigma(dm_1, dm_2|\theta)\right) dF(\theta) \\
&+ \frac{1}{2}\int_0^1 \left(\int_M (q_1^*(m_1) + q_2^*(m_2) - 2q_1^*(m_1)q_2^*(m_2))\,\sigma(dm_1, dm_2|\theta)\right) dF(\theta) \\
=& \int_0^1 (1 - (1+\kappa)(1-\theta))\,p_2(\theta)dF(\theta) + \frac{1}{2}\int_0^1 p_1(\theta)\mathrm{d}F(\theta) \\
=& \int_0^1 (1 - (1+\kappa)(1-\theta))\,(1 - p_0(\theta) - p_1(\theta))\,dF(\theta) + \frac{1}{2}\int_0^1 p_1(\theta)\mathrm{d}F(\theta) \\
=& 1 - (1+\kappa)\,(1 - \mathbb{E}[\theta]) - \sum_{a=0,1}\int_0^1 \left(\frac{1}{1+a} - (1+\kappa)(1-\theta)\right) p_a(\theta)dF(\theta) \le 0,
\end{aligned}$$

as stated in the lemma. ■

**Lemma 2.** *The feasibility constraint (1) is binding for any optimal distribution of internal support.*

***Proof.*** Given Lemma 1, we can write the ruler's optimization problem as

$$\begin{aligned} &\max_{\boldsymbol{p}} \ V(\boldsymbol{p}) = 1 + \int_0^1 \left((\lambda(1-\theta)-1)\, p_1(\theta) - p_0(\theta)\right) dF(\theta) \\ &s.t. \ \sum_{a=0,1} \int_0^1 \left(\frac{1}{1+a} - (1+\kappa)(1-\theta)\right) p_a(\theta) dF(\theta) \geq 1 - (1+\kappa)\left(1 - \mathbb{E}[\theta]\right). \end{aligned} \tag{2}$$

The Lagrangian of problem (2) is

$$\ell(p_0, p_1, \theta, \mu) := (\lambda(1-\theta)-1)\, p_1 - p_0 + \mu \left( \sum_{a=0,1} \left(\frac{1}{1+a} - (1+\kappa)(1-\theta)\right) p_a - c \right),$$

where $\mu \geq 0$ is the multiplier of the feasibility constraint (1) and $c := 1 - (1+\kappa)\left(1 - \mathbb{E}[\theta]\right)$ is a constant. Let

$$\begin{aligned} \ell_0(\theta, \mu) &:= \frac{\partial}{\partial p_0} \ell(p_0, p_1, \theta, \mu) = -1 + \mu \left(1 - (1+\kappa)(1-\theta)\right) \\ \ell_1(\theta, \mu) &:= \frac{\partial}{\partial p_1} \ell(p_0, p_1, \theta, \mu) = \lambda(1-\theta) - 1 + \mu \left(\frac{1}{2} - (1+\kappa)(1-\theta)\right). \end{aligned}$$

Consider any optimal distribution of internal support $\boldsymbol{p}^*$. Clearly, for $a = 0, 1$ and $\theta \in [0, 1]$,

$$p_a^*(\theta) = \begin{cases} 1, & \ell_a(\theta, \mu) > \max\left\{0, \ell_{1-a}(\theta, \mu)\right\} \\ 0, & \ell_a(\theta, \mu) < \max\left\{0, \ell_{1-a}(\theta, \mu)\right\} \end{cases}. \tag{3}$$

Assume $\mu = 0$. Then, $\ell_0(\theta, 0) = -1 < 0$ for all $\theta$, so that $p_0^*(\theta) = 0$ for all $\theta$. Moreover, $\ell_1(\theta, 0) = \lambda(1-\theta) - 1 > 0$ if and only if $\theta < 1 - \frac{1}{\lambda}$, so that $p_1^*(\theta) = \mathbb{1}\left(\theta \leq 1 - \frac{1}{\lambda}\right)$. But then,

$$\begin{aligned} &\sum_{a=0,1} \int_0^1 \left(\frac{1}{1+a} - (1+\kappa)(1-\theta)\right) p_a^*(\theta) dF(\theta) \\ &= \int_0^{1-\frac{1}{\lambda}} \left(\frac{1}{2} - (1+\kappa)(1-\theta)\right) dF(\theta) \\ &< \int_0^{1-\frac{1}{\lambda}} \left(1 - (1+\kappa)(1-\theta)\right) dF(\theta) \\ &\leq 1 - (1+\kappa)\left(1 - \mathbb{E}[\theta]\right), \end{aligned}$$

where the last inequality is because $\int_0^t \left(1 - (1+\kappa)(1-\theta)\right) dF(\theta)$ is strictly convex in $t$, equals to $1 - (1+\kappa)\left(1 - \mathbb{E}[\theta]\right)$ at $t = 1$, and equals to $0 < 1 - (1+\kappa)\left(1 - \mathbb{E}[\theta]\right)$ at $t = 0$. This is a contradiction to the feasibility constraint (1). Hence, it must be true that $\mu > 0$ and, as a result, (1) must be binding for $\boldsymbol{p}^*$. ■

**Lemma 3.** *Any optimal distribution of support $\boldsymbol{p}^*$ takes one in three possibilities:*

1. $p_0^*(\theta) = \mathbb{1}(\theta > x)$ *and* $p_1^*(\theta) = 0$ *for all* $\theta$*, where* $x \in \left(\frac{\kappa}{1+\kappa}, 1\right)$ *is the unique positive root of equation*

$$\int_x^1 (1 - (1+\kappa)(1-\theta))\, dF(\theta) = 1 - (1+\kappa)(1 - \mathbb{E}[\theta]); \tag{4}$$

2. $p_0^*(\theta) = \mathbb{1}(\theta > t)$ *for some* $t \in \left(\frac{\kappa}{1+\kappa}, x\right)$ *and* $p_1^*(\theta) = \mathbb{1}(\theta \le \alpha(t))$*, where* $\alpha(t) \in (0, t)$ *is strictly decreasing in* $t$ *and* $\alpha(t) < \frac{\kappa}{1+\kappa}$*;*

3. $p_0^*(\theta) = \mathbb{1}(\theta > t)$ *for some* $t \in \left(\frac{\kappa}{1+\kappa}, x\right)$ *and* $p_1^*(\theta) = \mathbb{1}(\beta(t) < \theta \le t)$*, where* $\beta(t) \in (0, t)$ *is strictly increasing in* $t$ *and* $\beta(t) < 1 - \frac{1}{2(1+\kappa)}$*.*

***Proof.*** We first show that (4) has a unique positive root. The left hand side of (4) is strictly quasi-concave in $x$, strictly increasing in $x < \frac{\kappa}{1+\kappa}$, strictly decreasing in $x > \frac{\kappa}{1+\kappa}$. At $x = \frac{\kappa}{1+\kappa}$,

$$\int_{\frac{\kappa}{1+\kappa}}^1 (1 - (1+\kappa)(1-\theta))\, dF(\theta) > \int_0^1 (1 - (1+\kappa)(1-\theta))\, dF(\theta) = 1 - (1+\kappa)(1 - \mathbb{E}[\theta]).$$

At $x = 1$,

$$\int_1^1 (1 - (1+\kappa)(1-\theta))\, dF(\theta) = 0 < 1 - (1+\kappa)(1 - \mathbb{E}[\theta]).$$

Hence, other than 0, (4) admits another root $x \in \left(\frac{\kappa}{1+\kappa}, 1\right)$.

Let $\boldsymbol{p}^*$ be optimal. We first characterize $p_0^*$. To do so, we show that $p_0^*(\theta) = \mathbb{1}(\theta > t)$ for some $t > \frac{\kappa}{1+\kappa}$. Then, we show that $t \le x$.

First, because $\mu > 0$, $\ell_0(\theta, \mu) = -\mu(1+\kappa)(1-\theta) + \mu - 1$ is strictly increasing in $\theta$ and $\ell_0(\theta, \mu) > 0$ if and only if $\theta > 1 - \frac{\mu - 1}{\mu(1+\kappa)} > \frac{\kappa}{1+\kappa}$. Moreover, note that

$$\ell_1(\theta, \mu) - \ell_0(\theta, \mu) = \lambda(1-\theta) - \frac{1}{2}\mu > 0$$

if and only if $\theta < 1 - \frac{\mu}{2\lambda}$. Therefore, $p_0^*(\theta) = 0$ for all $\theta < 1 - \frac{\mu}{2\lambda}$ and $p_1^*(\theta) = 0$ for all $\theta > 1 - \frac{\mu}{2\lambda}$. Let $t := 1 - \min\left\{\frac{\mu}{2\lambda}, \frac{\mu-1}{\mu(1+\kappa)}\right\}$ if $\mu > 1$ and $t := 1$ if $\mu \le 1$. Note that $t > \frac{\kappa}{1+\kappa}$. Then, $\ell_0(\theta, \mu) > \max\{0, \ell_1(\theta, \mu)\}$ if and only if $\theta > t$, which implies that $p_0^*(\theta) = \mathbb{1}(\theta > t)$.

Second, assume that $p_1^*(\theta) > 0$ for some $\theta > 1 - \frac{1}{2(1+\kappa)}$. Because $p_1^*(\theta) = 0$ for all $\theta > 1 - \frac{\mu}{2\lambda}$, it must be true that $\frac{1}{2(1+\kappa)} > \frac{\mu}{2\lambda}$ and so $\lambda - \mu(1+\kappa) > 0$. This implies that $\ell_1(\theta, \mu) = (\lambda - \mu(1+\kappa))(1-\theta) + \frac{1}{2}\mu - 1$ is strictly decreasing in $\theta$ and $\ell_1(\theta, \mu) > 0$ if and only if $\theta < 1 - \frac{1 - \frac{1}{2}\mu}{\lambda - \mu(1+\kappa)}$. Let $t' := 1 - \max\left\{\frac{\mu}{2\lambda}, \frac{1 - \frac{1}{2}\mu}{\lambda - \mu(1+\kappa)}\right\}$, then $\ell_1(\theta, \mu) > \max\{0, \ell_0(\theta, \mu)\}$ if and only if $\theta < t'$ and so $p_1^*(\theta) = \mathbb{1}\{\theta \le t'\}$. Then, as $p_1^*(\theta) > 0$ for some $\theta > 1 - \frac{1}{2(1+\kappa)}$, it must be true that $t' > 1 - \frac{1}{2(1+\kappa)}$. Moreover, by definition, $t' \leq 1 - \frac{\mu}{2\lambda} \leq t$. But then,

$$\begin{aligned}
&\sum_{a=0,1} \int_0^1 \left( \frac{1}{1+a} - (1+\kappa)(1-\theta) \right) p_a^*(\theta) dF(\theta) \\
&= \int_t^1 \left(1 - (1+\kappa)(1-\theta)\right) dF(\theta) + \int_0^{t'} \left( \frac{1}{2} - (1+\kappa)(1-\theta) \right) dF(\theta) \\
&< \int_t^1 \left(1 - (1+\kappa)(1-\theta)\right) dF(\theta) + \int_0^{t'} \left(1 - (1+\kappa)(1-\theta)\right) dF(\theta) \\
&\leq \int_t^1 \left(1 - (1+\kappa)(1-\theta)\right) dF(\theta) + \int_0^{t} \left(1 - (1+\kappa)(1-\theta)\right) dF(\theta) \\
&= 1 - (1+\kappa)\left(1 - \mathbb{E}[\theta]\right),
\end{aligned}$$

where the third inequality is due to $t' > 1 - \frac{1}{2(1+\kappa)} > 1 - \frac{1}{1+\kappa} = \frac{\kappa}{1+\kappa}$ and the fact that $\int_0^{t'} \left(1 - (1+\kappa)(1-\theta)\right) dF(\theta)$ is strictly increasing in $t' \geq \frac{\kappa}{1+\kappa}$. This contradicts the feasibility constraint (1). Hence, it must be true that $p_1^*(\theta) = 0$ for all $\theta > 1 - \frac{1}{2(1+\kappa)}$.

But because $p_1^*(\theta) = 0$ for all $\theta > 1 - \frac{1}{2(1+\kappa)}$,

$$\int_0^1 \left( \frac{1}{2} - (1+\kappa)(1-\theta) \right) p_1^*(\theta) dF(\theta) \leq 0,$$

so that

$$\begin{aligned}
&\int_t^1 \left(1 - (1+\kappa)(1-\theta)\right) dF(\theta) \\
&= \int_0^1 \left(1 - (1+\kappa)(1-\theta)\right) p_0^*(\theta) dF(\theta) \\
&\geq \int_0^1 \left(1 - (1+\kappa)(1-\theta)\right) p_0^*(\theta) dF(\theta) + \int_0^1 \left( \frac{1}{2} - (1+\kappa)(1-\theta) \right) p_1^*(\theta) dF(\theta) \\
&= \sum_{a=0,1} \int_0^1 \left( \frac{1}{1+a} - (1+\kappa)(1-\theta) \right) p_a^*(\theta) dF(\theta) \\
&\geq 1 - (1+\kappa)\left(1 - \mathbb{E}[\theta]\right),
\end{aligned}$$

where the last inequality is due to the feasibility constraint (1). By the definition of $x$ and the fact that $\int_t^1 \left(1 - (1+\kappa)(1-\theta)\right) dF(\theta)$ is strictly decreasing in $t > \frac{\kappa}{1+\kappa}$. It must be true that $t \leq x$.

Now we characterize $p_1^*$. Suppose $t = x$. Because $p_1^*(\theta) = 0$ for all $\theta > 1 - \frac{1}{2(1+\kappa)}$. Assume $p_1^*(\theta) > 0$ in a set of positive measure, then

$$\begin{aligned}
&\sum_{a=0,1} \int_0^1 \left( \frac{1}{1+a} - (1+\kappa)(1-\theta) \right) p_a^*(\theta) dF(\theta) \\
&= \int_x^1 \left(1 - (1+\kappa)(1-\theta)\right) dF(\theta) + \int_0^{1-\frac{1}{2(1+\kappa)}} \left( \frac{1}{2} - (1+\kappa)(1-\theta) \right) dF(\theta) \\
&= 1 - (1+\kappa)\left(1 - \mathbb{E}[\theta]\right) + \int_0^{1-\frac{1}{2(1+\kappa)}} \left( \frac{1}{2} - (1+\kappa)(1-\theta) \right) dF(\theta) \\
&< 1 - (1+\kappa)\left(1 - \mathbb{E}[\theta]\right),
\end{aligned}$$

where the last inequality is due to $\frac{1}{2}-(1+\kappa)(1-\theta)<0$ for all $\theta<1-\frac{1}{2(1+\kappa)}$. This contradicts the feasibility constraint (1). Therefore, it must be true that $p_1^*(\theta)=0$ almost everywhere.

Suppose $t<x$. The same steps above shows that $p_1^*(\theta)>0$ must hold in a set of positive measure to bind the feasibility constraint (1). Because $\ell_1(\theta,\mu)$ is linear in $\theta$ and because $p_0^*(\theta)=\mathbb{1}(\theta>t)$, $p_1^*$ takes one in two possible forms: either $p_1^*(\theta)=\mathbb{1}(\theta\le t')$ for some $t'<t$ or $p_1^*(\theta)=\mathbb{1}(t'<\theta\le t)$ for some $t'<t$.

Consider the first possibility, $p_1^*(\theta)=\mathbb{1}(\theta\le t')$ for some $t'<t$. Because the feasibility constraint (1) must be binding, it must be true that

$$\begin{aligned}(5)\qquad &\int_t^1 (1-(1+\kappa)(1-\theta))\,dF(\theta)+\int_0^{t'}\left(\frac{1}{2}-(1+\kappa)(1-\theta)\right)dF(\theta)\\ &=1-(1+\kappa)\left(1-\mathbb{E}[\theta]\right).\end{aligned}$$

Because the left hand side of the above equation is strictly quasi-convex in $t'$,

$$\int_t^1 (1-(1+\kappa)(1-\theta))\,dF(\theta)>1-(1+\kappa)\left(1-E[\theta]\right)$$

at $t'=0$ as $t<x$, and

$$\begin{aligned}&\int_t^1 (1-(1+\kappa)(1-\theta))\,dF(\theta)+\int_0^{t}\left(\frac{1}{2}-(1+\kappa)(1-\theta)\right)dF(\theta)\\ &<\int_t^1 (1-(1+\kappa)(1-\theta))\,dF(\theta)+\int_0^{t}(1-(1+\kappa)(1-\theta))\,dF(\theta)\\ &=1-(1+\kappa)\left(1-\mathbb{E}[\theta]\right)\end{aligned}$$

at $t'=t$, there exists a unique $\alpha(t)\in(0,t)$ that solves (5). It follows that $t'=\alpha(t)$. Note that the left hand side of (5) is strictly increasing in $t\in\left[\frac{\kappa}{1+\kappa},t\right]$, which implies that it is strictly bounded below $1-(1+\kappa)\left(1-\mathbb{E}[\theta]\right)$ for all $t'\in\left[\frac{\kappa}{1+\kappa},t\right]$. As a result, $\alpha(t)<\frac{\kappa}{1+\kappa}$.

By the implicit function theorem,

$$\alpha'(t)=\frac{1-(1+\kappa)(1-t)}{\frac{1}{2}-(1+\kappa)\left(1-\alpha(t)\right)}\frac{f(t)}{f\left(\alpha(t)\right)}<0,$$

where the second inequality is due to $t>\frac{\kappa}{1+\kappa}$, so that $1-(1+\kappa)(1-t)>0$, and $\alpha(t)<\frac{\kappa}{1+\kappa}<1-\frac{1}{2(1+\kappa)}$, so that $\frac{1}{2}-(1+\kappa)\left(1-\alpha(t)\right)<0$.

Now consider the second possibility that $p_1^*(\theta)=\mathbb{1}(t'<\theta\le t)$ for some $t'<t$. Because the feasibility constraint (1) must be binding, it must be true that

$$\begin{aligned}(6)\qquad &\int_t^1 (1-(1+\kappa)(1-\theta))\,dF(\theta)+\int_{t'}^{t}\left(\frac{1}{2}-(1+\kappa)(1-\theta)\right)dF(\theta)\\ &=1-(1+\kappa)\left(1-\mathbb{E}[\theta]\right).\end{aligned}$$

The left hand side of the above equation is strictly quasi-concave in $t'$,

$$\begin{aligned}&\int_t^1 (1-(1+\kappa)(1-\theta))\,dF(\theta)+\int_0^{t}\left(\frac{1}{2}-(1+\kappa)(1-\theta)\right)dF(\theta)\\ &<\int_t^1 (1-(1+\kappa)(1-\theta))\,dF(\theta)+\int_0^{t}(1-(1+\kappa)(1-\theta))\,dF(\theta)\\ &=1-(1+\kappa)\left(1-\mathbb{E}[\theta]\right)\end{aligned}$$

at $t' = 0$, and

$$\int_t^1 (1 - (1+\kappa)(1-\theta))\, dF(\theta) > 1 - (1+\kappa)(1 - \mathbb{E}[\theta])$$

at $t' = t$ due to $t < x$, there exists a unique $\beta(t) \in (0, t)$ that solves (6). It follows that $t' = \beta(t)$. When $t > 1 - \frac{1}{2(1+\kappa)}$, the left hand side of (6) is strictly decreasing in $t' \in \left[1 - \frac{1}{2(1+\kappa)}, t\right]$, which implies that it is strictly bounded above $1 - (1+\kappa)(1 - \mathbb{E}[\theta])$ for all $t' \in \left[1 - \frac{1}{2(1+\kappa)}, t\right]$. As a result, $\beta(t) < 1 - \frac{1}{2(1+\kappa)}$. When $t \leq 1 - \frac{1}{2(1+\kappa)}$, $\beta(t) < t \leq 1 - \frac{1}{2(1+\kappa)}$.

By the implicit function theorem,

$$\beta'(t) = -\frac{\frac{1}{2}}{\frac{1}{2} - (1+\kappa)(1 - \beta(t))} \frac{f(t)}{f(\beta(t))} > 0,$$

where the second inequality is due to $\beta(t) < 1 - \frac{1}{2(1+\kappa)}$, so that $\frac{1}{2} - (1+\kappa)(1 - \alpha(t)) < 0$. ■

## Proofs of Propositions

*Propositions 1 and 5*

According to Lemma 3, there are three possible optimal distributions of support:

1. $p_0^*(\theta) = \mathbb{1}(\theta < x)$ and $p_1^*(\theta) = 0$ for all $\theta$, which generates the expected payoff of $F(x)$;
2. $p_0^*(\theta) = \mathbb{1}(\theta > t)$ for some $t \in \left(\frac{\kappa}{1+\kappa}, x\right)$ and $p_1^*(\theta) = \mathbb{1}(\theta \leq \alpha(t))$, which generates the expected payoff of

$$V_\alpha(t) := F(t) + \int_0^{\alpha(t)} (\lambda(1-\theta) - 1)\, dF(\theta);$$

3. $p_0^*(\theta) = \mathbb{1}(\theta > t)$ for some $t \in \left(\frac{\kappa}{1+\kappa}, x\right)$ and $p_1^*(\theta) = \mathbb{1}(\beta(t) < \theta \leq t)$, which generates the expected payoff of

$$V_\beta(t) := F(t) + \int_{\beta(t)}^{t} (\lambda(1-\theta) - 1)\, dF(\theta).$$

We compare the expected payoffs under three above options through claims 1, 2, and 3.

**Claim 1.** *For any $t \in \left(\frac{\kappa}{1+\kappa}, x\right)$, $V_\alpha(t) > V_\beta(t)$ if and only if $\lambda < 2(1+\kappa)$.*

***Proof.*** Fix $t \in \left(\frac{\kappa}{1+\kappa}, x\right)$. Then,

$$V_\beta(t) - V_\alpha(t) = \int_0^{\alpha(t)} (1 - \lambda(1-\theta))\, dF(\theta) - \int_{\beta(t)}^{t} (1 - \lambda(1-\theta))\, dF(\theta) =: h(t, \lambda).$$

We prove the lemma by showing that $h(t,\lambda)$ is strictly increasing in $\lambda$ and that $h\left(x, 2(1+\kappa)\right)=0$. First, note that

$$\frac{\partial}{\partial\lambda}h(t,\lambda)=\int_{\beta(t)}^{t}(1-\theta)dF(\theta)-\int_{0}^{\alpha(t)}(1-\theta)dF(\theta).$$

By the definition of $\beta(t)$ in (6) and that of $\alpha(t)$ in (5),

$$\begin{aligned}\int_{\beta(t)}^{t}\left(\frac{1}{2}-(1+\kappa)(1-\theta)\right)dF(\theta)&=\int_{0}^{t}\left(1-(1+\kappa)(1-\theta)\right)dF(\theta)\\&=\int_{0}^{\alpha(t)}\left(\frac{1}{2}-(1+\kappa)(1-\theta)\right)dF(\theta),\end{aligned}\tag{7}$$

so that

$$(1+\kappa)\left(\int_{\beta(t)}^{t}(1-\theta)\mathrm{d}F(\theta)-\int_{0}^{\alpha(t)}(1-\theta)\mathrm{d}F(\theta)\right)=\frac{1}{2}\left(F(t)-F\left(\beta(t)\right)-F\left(\alpha(t)\right)\right).$$

This implies that

$$\frac{\partial}{\partial\lambda}h(t,\lambda)=\frac{F(t)-F\left(\beta(t)\right)-F\left(\alpha(t)\right)}{2(1+\kappa)}.$$

Moreover, because

$$\int_{\beta(t)}^{t}\left(\frac{1}{2}-(1+\kappa)(1-\theta)\right)dF(\theta)=\left(F(t)-F\left(\beta(t)\right)\right)\mathbb{E}\left[\frac{1}{2}-(1+\kappa)(1-\theta)|\beta(t)<\theta\leq t\right]$$

and

$$\int_{0}^{\alpha(t)}\left(\frac{1}{2}-(1+\kappa)(1-\theta)\right)dF(\theta)=F\left(\alpha(t)\right)\mathbb{E}\left[\frac{1}{2}-(1+\kappa)(1-\theta)|\theta\leq\alpha(t)\right],$$

(7) can be rewritten as

$$\begin{aligned}\left(F(t)-F\left(\beta(t)\right)\right)\mathbb{E}\left[\frac{1}{2}-(1+\kappa)(1-\theta)|\beta(t)<\theta\leq t\right]&=\int_{0}^{t}\left(1-(1+\kappa)(1-\theta)\right)dF(\theta)\\&=F\left(\alpha(t)\right)\mathbb{E}\left[\frac{1}{2}-(1+\kappa)(1-\theta)|\theta\leq\alpha(t)\right].\end{aligned}$$

Due to $t<x$, $\int_0^t\left(1-(1+\kappa)(1-\theta)\right)dF(\theta)<0$, so that

$$\mathbb{E}\left[\frac{1}{2}-(1+\kappa)(1-\theta)|\theta\leq\alpha(t)\right]<\mathbb{E}\left[\frac{1}{2}-(1+\kappa)(1-\theta)|\beta(t)<\theta\leq t\right]<0,$$

where the first inequality is due to $\alpha(t)<t$. It follows that

$$\frac{F(t)-F\left(\beta(t)\right)}{F\left(\alpha(t)\right)}=\frac{-\mathbb{E}\left[\frac{1}{2}-(1+\kappa)(1-\theta)|\theta\leq\alpha(t)\right]}{-\mathbb{E}\left[\frac{1}{2}-(1+\kappa)(1-\theta)|\beta(t)<\theta\leq t\right]}>1.$$

Hence,

$$F(t) - F(\beta(t)) > F(\alpha(t)) \tag{8}$$

which implies that $\frac{\partial}{\partial\lambda}h(t,\lambda) = \frac{F(t)-F(\beta(t))-F(\alpha(t))}{2(1+\kappa)} > 0$.

At last, again due to (7),

$$\begin{aligned} h(t, 2(1+\kappa)) &= 2\left(\int_0^{\alpha(t)}\left(\frac{1}{2} - (1+\kappa)(1-\theta)\right)dF(\theta) - \int_{\beta(t)}^{t}\left(\frac{1}{2} - (1+\kappa)(1-\theta)\right)dF(\theta)\right) \\ &= 2\left(\int_0^{t}(1-(1+\kappa)(1-\theta))\,dF(\theta) - \int_0^{t}(1-(1+\kappa)(1-\theta))\,dF(\theta)\right) \\ &= 0. \end{aligned}$$

Therefore, $h(t,\lambda) < 0$ and so $V_\alpha(t) > V_\beta(t)$ if $\lambda < 2(1+\kappa)$ while $h(t,\lambda) > 0$ and so $V_\alpha(t) > V_\beta(t)$ if $\lambda > 2(1+\kappa)$. ■

**Claim 2.** *Suppose $\lambda < 2(1+\kappa)$. There exists a unique optimal distribution of internal support $\boldsymbol{p}^*$, for which*

1. *if*

$$\lambda \leq \lambda_\alpha^*(\kappa) := 1 + \frac{1}{2}\frac{2(1+\kappa)-1}{1-(1+\kappa)(1-x)},$$

   $p_0^*(\theta) = \mathbb{1}(\theta > x)$ *and* $p_1^*(\theta) = 0$ *for all* $\theta$*;*

2. *if* $\lambda_\alpha^*(\kappa) < \lambda < 2(1+\kappa)$, $p_0^*(\theta) = \mathbb{1}(\theta > \overline{y})$ *and* $p_1^*(\theta) = \mathbb{1}(\theta \leq \underline{y})$*, where* $\underline{y} < \overline{y} < x$*.*

*Moreover,* $\lambda_\alpha^*(\kappa) < 2(1+\kappa)$ *if and only if* $\kappa > \kappa^*$*, where* $\kappa^* \in \left(0, \frac{\mathbb{E}[\theta]}{1-\mathbb{E}[\theta]}\right)$*.*

***Proof.*** We first prove the last argument. Note that the definition of $\lambda_\alpha^*(\kappa)$ implies that $\lambda_\alpha^*(\kappa) < 2(1+\kappa)$ if and only if $x > 1 - \frac{1}{2(1+\kappa)}$. By the definition of $x$ in (4), $\mathbb{E}[\theta|\theta < x] = \frac{\kappa}{1+\kappa}$, which by the implicit function theorem implies that

$$\frac{\partial x}{\partial\kappa} = \frac{1}{(1+\kappa)^2}\frac{1}{\frac{d}{dx}\mathbb{E}[\theta|\theta < x]}.$$

Because $F$ has a log-concave density, $\frac{d}{dx}\mathbb{E}[\theta|\theta < x] \leq 1$, so that

$$\frac{\partial x}{\partial\kappa} \geq \frac{1}{(1+\kappa)^2}.$$

In turn,

$$\frac{\partial}{\partial\kappa}\left(x - \left(1 - \frac{1}{2(1+\kappa)}\right)\right) = \frac{\partial x}{\partial\kappa} - \frac{1}{2}\frac{1}{(1+\kappa)^2} \geq \frac{1}{2}\frac{1}{(1+\kappa)^2} > 0.$$

Then, because

$$\lim_{\kappa\to 0}\left(x-\left(1-\frac{1}{2(1+\kappa)}\right)\right)=-\frac{1}{2}<0$$
$$\lim_{\kappa\to\frac{\mathbb{E}[\theta]}{1-\mathbb{E}[\theta]}}\left(x-\left(1-\frac{1}{2(1+\kappa)}\right)\right)=\frac{1-\mathbb{E}[\theta]}{2}>0,$$

there exists a unique $\kappa^*\in\left(0,\frac{\mathbb{E}[\theta]}{1-\mathbb{E}[\theta]}\right)$ such that $x>1-\frac{1}{2(1+\kappa)}$, so that $\lambda^*_\alpha(\kappa)<2(1+\kappa)$, if and only if $\kappa>\kappa^*$.

Because $\lambda<2(1+\kappa)$, $V_\alpha(t)>V_\beta(t)$ for all $t\in\left(\frac{\kappa}{1+\kappa},x\right)$. Note that

$$\begin{aligned}V'_\alpha(t)&=f(t)+(\lambda(1-\alpha(t))-1)\,f(\alpha(t))\,\alpha'(t)\\&=f(t)\left(1-(\lambda(1-\alpha(t))-1)\frac{1-(1+\kappa)(1-t)}{(1+\kappa)(1-\alpha(t))-\frac{1}{2}}\right),\end{aligned}$$

which has the same sign with

$$\begin{aligned}W_\alpha(t)&:=(1+\kappa)(1-\alpha(t))-\frac{1}{2}-(\lambda(1-\alpha(t))-1)(1-(1+\kappa)(1-t))\\&=\frac{1}{2}-(1+\kappa)(1-t)-(1-\alpha(t))(\lambda(1-(1+\kappa)(1-t))-(1+\kappa)).\end{aligned}$$

Because $\lambda<2(1+\kappa)$,

$$\begin{aligned}W_\alpha(t)&>\frac{1}{2}-(1+\kappa)(1-t)-(1-\alpha(t))(2(1+\kappa)(1-(1+\kappa)(1-t))-(1+\kappa))\\&=2\left(\frac{1}{2}-(1+\kappa)(1-\alpha(t))\right)\left(\frac{1}{2}-(1+\kappa)(1-t)\right).\end{aligned}$$

For any $t\le 1-\frac{1}{2(1+\kappa)}$,

$$\frac{1}{2}-(1+\kappa)(1-\alpha(t))<\frac{1}{2}-(1+\kappa)(1-t)\le 0,$$

so that

$$W_\alpha(t)>2\left(\frac{1}{2}-(1+\kappa)(1-\alpha(t))\right)\left(\frac{1}{2}-(1+\kappa)(1-t)\right)\ge 0.$$

First, suppose $\lambda\le\lambda^*_\alpha(\kappa)$. Then,

$$\begin{aligned}&\lambda(1-(1+\kappa)(1-t))-(1+\kappa)\\&\le\lambda^*_\alpha(\kappa)(1-(1+\kappa)(1-t))-(1+\kappa)\\&=\left(1+\frac{1}{2}\frac{2(1+\kappa)-1}{1-(1+\kappa)(1-x)}\right)(1-(1+\kappa)(1-t))-(1+\kappa)\\&<\left(1+\frac{1}{2}\frac{2(1+\kappa)-1}{1-(1+\kappa)(1-t)}\right)(1-(1+\kappa)(1-t))-(1+\kappa)\\&=\frac{1}{2}-(1+\kappa)(1-t).\end{aligned}$$

In turn, for any $t > 1 - \frac{1}{2(1+\kappa)}$,

$$\begin{aligned} W_\alpha(t) &> \frac{1}{2} - (1+\kappa)(1-t) - (1-\alpha(t))\left(\frac{1}{2} - (1+\kappa)(1-t)\right) \\ &= \left(\frac{1}{2} - (1+\kappa)(1-t)\right)\alpha(t) \\ &> 0. \end{aligned}$$

Therefore, $V'_\alpha(t) > 0$ for any $t \in \left(\frac{\kappa}{1+\kappa}, x\right)$. This implies that

$$F(x) = V_\alpha(x) > V_\alpha(t) = \max\{V_\alpha(t), V_\beta(t)\}$$

for all $t \in \left(\frac{\kappa}{1+\kappa}, x\right)$. As a result, $\boldsymbol{p}^*$ such that $p_0^*(\theta) = \mathbb{1}(\theta > x)$ and $p_1^*(\theta) = 0$ for all $\theta$ is optimal.

Second, suppose $\lambda^*_\alpha(\kappa) < \lambda \leq 2(1+\kappa)$. This condition necessitates $\kappa > \kappa^*$, so that $x > 1 - \frac{1}{2(1+\kappa)}$. Remember that $W_\alpha(t) > 0$ for all $t \leq 1 - \frac{1}{2(1+\kappa)}$. Because $\lambda < 2(1+\kappa)$, $\frac{\kappa}{1+\kappa} + \frac{1}{\lambda} > 1 - \frac{1}{2(1+\kappa)}$. Then, for any $1 - \frac{1}{2(1+\kappa)} < t \leq \frac{\kappa}{1+\kappa} + \frac{1}{\lambda}$,

$$\begin{aligned} \frac{1}{2} - (1+\kappa)(1-t) > 0 \\ \lambda\left(1 - (1+\kappa)(1-t)\right) - (1+\kappa) \leq 0, \end{aligned}$$

so that

$$\begin{aligned} W_\alpha(t) &= \frac{1}{2} - (1+\kappa)(1-t) - (1-\alpha(t))\left(\lambda\left(1-(1+\kappa)(1-t)\right) - (1+\kappa)\right) \\ &\geq \frac{1}{2} - (1+\kappa)(1-t) \\ &> 0. \end{aligned}$$

But for $t > \frac{\kappa}{1+\kappa} + \frac{1}{\lambda}$,

$$\begin{aligned} W'_\alpha(t) &= (1+\kappa)\left(1 - (1-\alpha(t))\lambda\right) + \left(\lambda\left(1-(1+\kappa)(1-t)\right) - (1+\kappa)\right)\alpha'(t) \\ &< (1+\kappa)\left(1 - (1-\alpha(t))\lambda\right) \\ &< (1+\kappa)\left(1 - \frac{\lambda}{1+\kappa}\right), \end{aligned}$$

where the last inequality is due to $\alpha(t) < \frac{\kappa}{1+\kappa}$. Note that

$$\lambda > \lambda^*_\alpha(\kappa) = 1 + \frac{1}{2}\frac{2\kappa+1}{1-(1+\kappa)(1-x)} > 1+\kappa.$$

Hence,

$$W'_\alpha(t) < (1+\kappa)\left(1 - \frac{\lambda}{1+\kappa}\right) < 0$$

holds for all $t > \frac{\kappa}{1+\kappa} + \frac{1}{\lambda}$. This implies that $V_\alpha(t)$ is strictly quasi-concave in $t \in \left[\frac{\kappa}{1+\kappa} + \frac{1}{\lambda}, x\right]$. Moreover, $W_\alpha\left(\frac{\kappa}{1+\kappa} + \frac{1}{\lambda}\right) > 0$ and because $\alpha(x) = 0$,

$$\begin{aligned} W_\alpha(x) &= \frac{1}{2} - (1+\kappa)(1-x) - (\lambda(1-(1+\kappa)(1-x)) - (1+\kappa)) \\ &= -(1-(1+\kappa)(1-x))\left(\lambda - \left(1 + \frac{1}{2}\frac{2(1+\kappa)-1}{1-(1+\kappa)(1-x)}\right)\right) \\ &= -(1-(1+\kappa)(1-x))(\lambda - \lambda_\alpha^*(\kappa)) < 0. \end{aligned}$$

Therefore, there exists a unique $\overline{y} \in \left(\frac{\kappa}{1+\kappa} + \frac{1}{\lambda}, x\right)$ that satisfies $W_\alpha(\overline{y}) = 0$ and, thus, maximizes $V_\alpha(t)$. Let $\underline{y} := \alpha(\overline{y})$. As a result, $\boldsymbol{p}^*$ such that $p_0^*(\theta) = \mathbb{1}(\theta > \overline{y})$ and $p_1^*(\theta) = \mathbb{1}(\theta \leq \underline{y})$ is optimal. ■

**Claim 3.** *Suppose $\lambda \geq 2(1+\kappa)$. There exists a unique optimal distribution of internal support $\boldsymbol{p}^*$, for which*

1. *if $2(1+\kappa) \leq \lambda \leq \lambda^*(\kappa)$, where*

$$\begin{aligned} \lambda^*(\kappa) &:= \min\left\{\lambda_\alpha^*(\kappa), \lambda_\beta^*(\kappa)\right\} \\ \lambda_\beta^*(\kappa) &:= \frac{1}{2(1-(1+\kappa)(1-x))(1-x)} \end{aligned}$$

   *then $p_0^*(\theta) = \mathbb{1}(\theta > x)$ and $p_1^*(\theta) = 0$ for all $\theta$;*

2. *if $\lambda > \lambda^*(\kappa)$, $p_0^*(\theta) = \mathbb{1}(\theta > \overline{z})$ and $p_1^*(\theta) = \mathbb{1}(\underline{z} < \theta \leq \overline{z})$, where $\underline{z} < \overline{z} < x$.*

*Moreover, $\lambda_\alpha^*(x) < \lambda_\beta^*(x)$ if and only if $\kappa > \kappa^*$.*

***Proof.*** We first prove the last statement. Note that $\lambda_\beta^*(\kappa) - \lambda_\alpha^*(\kappa)$ has the same sign as

$$\frac{1}{1-x} - (2(1-(1+\kappa)(1-x)) + 2(1+\kappa) - 1) = \frac{1}{1-x} - (1 + 2(1+\kappa)x),$$

which in turn has the same sign with

$$1 - (1-x) - 2(1+\kappa)x(1-x) = 2x\left(\frac{1}{2} - (1+\kappa)(1-x)\right).$$

The above expression is positive if and only if $x > 1 - \frac{1}{2(1+\kappa)}$ or, equivalently, $\kappa > \kappa^*$.

Because $\lambda \geq 2(1+\kappa)$, $V_\beta(t) \geq V_\alpha(t)$ for all $t \in \left(\frac{\kappa}{1+\kappa}, x\right)$. Note that

$$\begin{aligned} V_\beta'(t) &= \lambda(1-t)f(t) - (\lambda(1-\beta(t)) - 1)f(\beta(t))\beta'(t) \\ &= f(t)\left(\lambda(1-t) - (\lambda(1-\beta(t)) - 1)\frac{1}{2(1+\kappa)(1-\beta(t)) - 1}\right) \\ &= \frac{\lambda f(t)}{2(1+\kappa)}\left(2(1+\kappa)(1-t) - \frac{1-\beta(t) - \frac{1}{\lambda}}{1-\beta(t) - \frac{1}{2(1+\kappa)}}\right), \end{aligned}$$

which has the same sign with

$$W_\beta(t) := 2(1+\kappa)(1-t) - \frac{1-\beta(t)-\frac{1}{\lambda}}{1-\beta(t)-\frac{1}{2(1+\kappa)}}.$$

Because

$$W_\beta'(t) = -2(1+\kappa) + \frac{\frac{1}{\lambda}-\frac{1}{2(1+\kappa)}}{\left(1-\beta(t)-\frac{1}{2(1+\kappa)}\right)^2}\beta'(t) \le -2(1+\kappa) < 0,$$

$V_\beta(t)$ is strictly quasi-concave in $t$, where the second inequality is due to $\beta'(t) > 0$ and $\lambda \ge 2(1+\kappa)$.

Now consider the first case when $2(1+\kappa) \le \lambda \le \lambda^*(\kappa)$. Note that this case necessitates $\lambda_\alpha^*(\kappa) \ge 2(1+\kappa)$, which according to Lemma 2 requires $\kappa \le \kappa^*$. Because $\kappa \le \kappa^*$, $x \le 1 - \frac{1}{2(1+\kappa)}$. Then, because $\lambda \le \lambda_\beta^*(\kappa)$ and because $\beta(x) = x$,

$$\begin{aligned} W_\beta(x) &= 2(1+\kappa)(1-x) - \frac{1-x-\frac{1}{\lambda}}{1-x-\frac{1}{2(1+\kappa)}} \\ &= \frac{1}{1-x-\frac{1}{2(1+\kappa)}}\left(2(1+\kappa)(1-x)\left(1-x-\frac{1}{2(1+\kappa)}\right) - (1-x) + \frac{1}{\lambda}\right) \\ &= \frac{1}{1-x-\frac{1}{2(1+\kappa)}}\left(\frac{1}{\lambda} - 2\left(1-(1+\kappa)(1-x)\right)(1-x)\right) \ge 0, \end{aligned}$$

where the last inequality is due to $\lambda \le \lambda_\beta^*(\kappa) = \frac{1}{2(1-(1+\kappa)(1-x))(1-x)}$. It follows that $W_\beta(t) > 0$, so that $V_\beta'(t) > 0$, for all $t \in \left(\frac{\kappa}{1+\kappa}, x\right)$. This implies that

$$F(x) = V_\beta(x) > V_\beta(t) = \max\{V_\alpha(t), V_\beta(t)\}$$

for all $t \in \left(\frac{\kappa}{1+\kappa}, x\right)$. As a result, $\boldsymbol{p}^*$ such that $p_0^*(\theta) = \mathbb{1}(\theta > x)$ and $p_1^*(\theta) = 0$ for all $\theta$ is optimal.

At last, consider the second case when $\lambda > \lambda^*(\kappa)$. If $\kappa > \kappa^*$, $\lambda^*(\kappa) = \lambda_\alpha^*(\kappa)$ and $x > 1 - \frac{1}{2(1+\kappa)}$. In this case,

$$W_\beta\left(1-\frac{1}{2(1+\kappa)}\right) = 1 - \frac{1-\beta\left(1-\frac{1}{2(1+\kappa)}\right)-\frac{1}{\lambda}}{1-\beta\left(1-\frac{1}{2(1+\kappa)}\right)-\frac{1}{2(1+\kappa)}}$$

and

$$\begin{aligned} W_\beta\left(\frac{\kappa}{1+\kappa}\right) &= 2 - \frac{1-\beta\left(\frac{\kappa}{1+\kappa}\right)-\frac{1}{\lambda}}{1-\beta\left(\frac{\kappa}{1+\kappa}\right)-\frac{1}{2(1+\kappa)}} \\ &> 2 - \frac{1-\beta\left(\frac{\kappa}{1+\kappa}\right)}{1-\beta\left(\frac{\kappa}{1+\kappa}\right)-\frac{1}{2(1+\kappa)}} = \frac{1-\beta\left(\frac{\kappa}{1+\kappa}\right)-\frac{1}{1+\kappa}}{1-\beta\left(\frac{\kappa}{1+\kappa}\right)-\frac{1}{2(1+\kappa)}} > 0, \end{aligned}$$

so that there exists a unique $\overline{z} \in \left(\frac{\kappa}{1+\kappa}, 1 - \frac{1}{2(1+\kappa)}\right) \subseteq \left(\frac{\kappa}{1+\kappa}, x\right)$ that satisfies $W_\beta(\overline{z}) = 0$ and, thus, maximizes $V_\beta(t)$. If $\kappa \leq \kappa^*$, $\lambda^*(\kappa) = \lambda^*_\beta(\kappa)$ and $x \leq 1 - \frac{1}{2(1+\kappa)}$. In this case,

$$W_\beta(x) = \frac{1}{1 - x - \frac{1}{2(1+\kappa)}} \left(\frac{1}{\lambda} - 2\left(1 - (1+\kappa)(1-x)\right)(1-x)\right) < 0$$

and similarly with the previous case, $W_\beta\left(\frac{\kappa}{1+\kappa}\right) > 0$. Hence, there exists a unique $\overline{z} \in \left(\frac{\kappa}{1+\kappa}, x\right) \subseteq \left(\frac{\kappa}{1+\kappa}, 1 - \frac{1}{2(1+\kappa)}\right)$ that satisfies $W_\beta(\overline{z}) = 0$ and, thus, maximizes $V_\beta(t)$. Let $\underline{z} := \beta(\overline{z})$. As a result, $\boldsymbol{p}^*$ such that $p^*_0(\theta) = \mathbb{1}\left(\theta > \overline{z}\right)$ and $p^*_1(\theta) = \mathbb{1}\left(\underline{z} < \theta \leq \overline{z}\right)$ is optimal. ■

Propositions 1 and 5 now follow directly from the three claims:

1. If $\lambda \leq \lambda^*(\kappa)$, then $\lambda \leq \lambda^*_\alpha(\kappa)$ in the case when $\lambda < 2(1+\kappa)$ and $\lambda \leq \lambda^*_\beta(\kappa)$ in the case when $\lambda \geq 2(1+\kappa)$. In either case, by part 1 of Claim 2 and by part 1 of Claim 3, $p^*_0(\theta) = \mathbb{1}\left(\theta > x\right)$ and $p^*_1(\theta) = 0$ for all $\theta$ is optimal.

2. If $\lambda^*(\kappa) < \lambda < 2(1+\kappa)$, then it must be that $\kappa > \kappa^*$, and so $\lambda^*_\alpha(\kappa) = \lambda^*(\kappa) < \lambda < 2(1+\kappa)$. By part 2 of Claim 2, $p^*_0(\theta) = \mathbb{1}\left(\theta > \overline{y}\right)$ and $p^*_1(\theta) = \mathbb{1}\left(\theta \leq \underline{y}\right)$ is optimal.

3. Suppose $\lambda \geq 2(1+\kappa)$ and $\lambda > \lambda^*(\kappa)$. When $\kappa > \kappa^*$, these two conditions can be reduced to $\lambda \geq 2(1+\kappa)$, because $\lambda^*(\kappa) = \lambda^*_\alpha(\kappa) < 2(1+\kappa)$. When $\kappa \leq \kappa^*$, these two conditions can be reduced to $\lambda > \lambda^*(\kappa)$, because $\lambda^*(\kappa) = \lambda^*_\beta(\kappa) \geq 2(1+\kappa)$. In either case, by part 2 of Claim 3, $p^*_0(\theta) = \mathbb{1}\left(\theta > \overline{z}\right)$ and $p^*_1(\theta) = \mathbb{1}\left(\underline{z} < \theta \leq \overline{z}\right)$ is optimal.

It only remains to show that $\overline{z} < \overline{y}$. As shown in the proof of Claim 2,

$$\overline{y} > \frac{\kappa}{1+\kappa} + \frac{1}{\lambda} > 1 - \frac{1}{2(1+\kappa)}.$$

As shown in the proof of Claim 3, either

$$\overline{z} < 1 - \frac{1}{2(1+\kappa)} < x$$

in the case when $\kappa > \kappa^*$ or

$$\overline{z} < x \leq 1 - \frac{1}{2(1+\kappa)}$$

in the case when $\kappa \leq \kappa^*$. Therefore,

$$\overline{z} < 1 - \frac{1}{2(1+\kappa)} < \overline{y}.$$

*Proposition 2*

It is sufficient to prove that there exists no public information structure that induces a divisive ruling style. Assume that there exists such a public information structure $(M, \sigma)$. For each ally $i$ and each action $a$, let $M_i^a$ denote the set of messages $m_i$ for which ally $i$ would choose action $a$.

First, assume that $b(M_1^1) \cap M_2^0 \neq \varnothing$. Consider $m_2 \in b(M_1^1) \cap M_2^0$. Suppose ally 1 receives $b^{-1}(m_2) \in M_1^1$. Receiving $b^{-1}(m_2)$, ally 1 infers that ally 2 must have received $m_2 \in M_2^0$ and therefore would oppose the ruler. But given that ally 2 would oppose the ruler, ally 1 gets 1 by opposing ruler and $0 < 1$ by supporting the ruler, so that he prefers to oppose the ruler. This contradicts the fact that $b^{-1}(m_2) \in M_1^1$. Therefore, it must be true that $b(M_1^1) \cap M_2^0 = \varnothing$. Similarly, one can prove that $b(M_1^0) \cap M_2^1 = \varnothing$.

It follows that $b(M_1^1) = M_2^1$ and $b(M_1^0) = M_2^0$. But given this, either the two allies receive $(m_1, b(m_1)) \in (M_1^1, M_2^1)$ for which they both support the ruler or they receive $(m_1, b(m_1)) \in (M_1^0, M_2^0)$ for which they both oppose the ruler. As a result, $p_1(\theta) = 0$ for all $\theta$. This contradicts the induced ruling style being divisive. Therefore, it must be true that no public information structure can induce a divisive ruling style.

*Proposition 3*

First, consider unite-and-lead. Note that given $\sigma$ such that $m_1 = m_2 = \mathbb{1}(\theta \leq x)$,

$$p_0(\theta) = \mathbb{1}(\theta > x)$$
$$p_1(\theta) = 0$$

for all $\theta$. Hence, $\sigma$ induces unite-and-lead.

Second, consider divide-and-conquer. Note that given $\sigma$ such that $m_1 = \mathbb{1}(\hat{y} < \theta \leq \overline{y})$ and $m_2 = \mathbb{1}(\underline{y} < \theta \leq \overline{y}) + \mathbb{1}(\theta \leq \hat{y})$,

$$p_0(\theta) = \mathbb{1}(\theta > \overline{y})$$
$$p_1(\theta) = \mathbb{1}(\hat{y} < \theta \leq \underline{y}) + \mathbb{1}(\theta \leq \hat{y}) = \mathbb{1}(\theta \leq \underline{y})$$

for all $\theta$. Hence, $\sigma$ induces divide-and-conquer.

Third, consider divide-and-crumble. Note that given $\sigma$ such that $m_1 = \mathbb{1}(\theta \leq \hat{z})$ and $m_2 = \mathbb{1}(\theta \leq \underline{z}) + \mathbb{1}(\hat{z} < \theta \leq \overline{z})$,

$$p_0(\theta) = \mathbb{1}(\theta > \overline{z})$$
$$p_1(\theta) = \mathbb{1}(\underline{z} < \theta \leq \hat{z}) + \mathbb{1}(\hat{z} < \theta \leq \overline{z}) = \mathbb{1}(\underline{z} < \theta \leq \overline{z})$$

for all $\theta$. Hence, $\sigma$ induces divide-and-crumble.

*Proposition 4*

First, we prove (a). Because $\alpha(t)$ is strictly decreasing in $t$ and because $\overline{z} < \overline{y}$,

$$\underline{y} = \alpha(\overline{y}) < \alpha(\overline{z}),$$

so that

$$F(\underline{y}) < F\left(\alpha(\overline{z})\right).$$

As shown in (8) in the proof of Claim 1, $F(t) - F\left(\beta(t)\right) > F\left(\alpha(t)\right)$ holds for all $t \in \left(\frac{\kappa}{1+\kappa}, x\right)$. As a result,

$$F\left(\alpha(\overline{z})\right) < F(\overline{z}) - F\left(\beta(\overline{z})\right) = F(\overline{z}) - F(\underline{z}).$$

It follows that

$$F(\underline{y}) < F\left(\alpha(\overline{z})\right) < F(\overline{z}) - F(\underline{z}).$$

Hence, (a) holds.

Now we prove (b). As shown in Lemma 3, $\alpha(t) < \frac{\kappa}{1+\kappa}$ for all $t \in \left(\frac{\kappa}{1+\kappa}, x\right)$. Hence,

$$\underline{y} = \alpha(\overline{y}) < \frac{\kappa}{1+\kappa} < \overline{z}.$$

It follows that

$$\mathbb{E}[1-\theta|\underline{z} < \theta \leq \overline{z}] < \mathbb{E}[1-\theta|\theta \leq \overline{z}] < \mathbb{E}[1-\theta|\theta \leq \underline{y}],$$

so that (b) holds.

At last, we prove (c). Because $\alpha(t)$ is strictly decreasing in $t$ and because $\overline{z} < \overline{y}$, $\underline{y} = \alpha(\overline{y}) < \alpha(\overline{z})$, so that

$$\int_0^{\underline{y}}(1-\theta)dF(\theta) < \int_0^{\alpha(\overline{z})}(1-\theta)dF(\theta)$$

and

$$\int_{\underline{z}}^{\overline{z}}(1-\theta)dF(\theta) - \int_0^{\underline{y}}(1-\theta)dF(\theta) > \int_{\underline{z}}^{\overline{z}}(1-\theta)dF(\theta) - \int_0^{\alpha(\overline{z})}(1-\theta)dF(\theta).$$

Due to (7), for all $t \in \left(\frac{\kappa}{1+\kappa}, x\right)$

$$\int_{\beta(t)}^{t}\left(\frac{1}{2} - (1+\kappa)(1-\theta)\right)dF(\theta) = \int_0^{\alpha(t)}\left(\frac{1}{2} - (1+\kappa)(1-\theta)\right)dF(\theta),$$

which implies that

$$\int_{\beta(t)}^{t}(1-\theta)dF(\theta) - \int_0^{\alpha(t)}(1-\theta)dF(\theta) = \frac{F(t) - F\left(\beta(t)\right) - F\left(\alpha(t)\right)}{2(1+\kappa)}$$

holds for all $t \in \left(\frac{\kappa}{1+\kappa}, x\right)$. As a result,

$$\begin{aligned}\int_{\underline{z}}^{\overline{z}}(1-\theta)dF(\theta) - \int_0^{\alpha(\overline{z})}(1-\theta)dF(\theta) &= \int_{\beta(\overline{z})}^{\overline{z}}(1-\theta)dF(\theta) - \int_0^{\alpha(\overline{z})}(1-\theta)dF(\theta) \\ &= \frac{F(\overline{z}) - F\left(\beta(\overline{z})\right) - F\left(\alpha(\overline{z})\right)}{2(1+\kappa)}.\end{aligned}$$

As shown in the proof of (a), $F(\overline{z}) - F\left(\beta(\overline{z})\right) - F\left(\alpha(\overline{z})\right) > 0$. Hence,

$$\int_{\underline{z}}^{\overline{z}}(1-\theta)dF(\theta) - \int_0^{\underline{y}}(1-\theta)dF(\theta) > \frac{F(\overline{z}) - F\left(\beta(\overline{z})\right) - F\left(\alpha(\overline{z})\right)}{2(1+\kappa)} > 0.$$

This proves (c).