Online Appendix
Incumbent Performance and Electoral Control: A Comment

B Comparative Statics

Consider now the general case in which the incumbent might come back to power after being ousted. More specifically, as in Ferejohn (1986), assume that, when a politician is out of office, she returns to power with probability $\lambda \in [0, 1]$ whenever the new incumbent is ousted.

The following proposition provides a partial analog to Ferejohn's Propositions 4 and 5.

Proposition D (Comparative Statics)**.**

(i) Assume either $\lambda = 0$ *or* $\phi(a) \equiv a$*. Then the voter's payoff* U *from the optimal retrospective rule is increasing in* W*. In the latter case, it is exactly proportional to* W*.*

(ii) The voter's maximized payoff U *is higher if* $\lambda = 0$ *than if* λ *takes any positive value in a neighborhood of* 0*.*

Proof. For part (i), suppose first that $\lambda = 0$. Then, for any rule $\mathbf{K}$, it is clear that the officeholder's value function V_t at any t is increasing in W, as her payoff is increasing in W for any fixed strategy she may follow. By (2), it follows that $\theta_t^*(\mathbf{K})$ is a decreasing function of W for all t, and hence $U(\mathbf{K})$ is increasing in W from any rule (in particular the optimal one).

Suppose now instead that $\phi(a) \equiv a$. In this case, the problem faced by the officeholder given a pair $(W, \mathbf{K})$ is homothetic in W and $\mathbf{K}$: if $(W, \mathbf{K})$ is multiplied by $\alpha > 0$, the officeholder's set of payoffs attainable by different strategies is also multiplied by α (as multiplying all effort choices by α achieves exactly this, and the process is reversible); thus the optimal payoffs in the continuation at each t, V_t, are also multiplied by α, and the best-response efforts $a_t(\theta)$ are multiplied by α for all θ, with the thresholds $\theta_t^*(\mathbf{K})$ remaining fixed. It follows that the voter's attainable payoffs are also proportional to W: if W is multiplied by α, she can multiply her equilibrium payoff by α by also scaling the rule **K** by the same factor.

For part (ii), let us rewrite the officeholder's Bellman equation (15) for the general case of $\lambda \geq 0$. Her utility when in and out of office, respectively, is

$$V_t^I = W + \int_{\theta_t^*}^{m} \left[\delta V_{t+1}^I - \phi\left(\frac{K_t}{\theta}\right) \right] f(\theta) d\theta + F(\theta_t^*) \delta V_{t+1}^O, \tag{22}$$

$$\begin{aligned} V_t^O &= F(\theta_t^*)(\lambda \delta V_{t+1}^I + (1-\lambda)\delta V_{t+1}^O) + (1 - F(\theta_t^*))\delta V_{t+1}^O = \\ &= \delta V_{t+1}^O + F(\theta_t^*)\lambda\delta(V_{t+1}^I - V_{t+1}^O), \end{aligned} \tag{23}$$

where $F(\theta_t^*)$ is the probability that the new incumbent forfeits her position, giving the officeholder a chance λ to return. Denoting $\Delta V_t = V_t^I - V_t^O$, and combining (22) and (23),

$$\begin{aligned} \Delta V_t &= W - \int_{\theta_t^*}^{m} \phi\left(\frac{K_t}{\theta}\right) f(\theta) d\theta + (1 - F(\theta_t^*))\delta V_{t+1}^I + F(\theta_t^*)\delta V_{t+1}^O - \delta V_{t+1}^O - F(\theta_t^*)\lambda\delta\Delta V_{t+1} \\ &= W - \int_{\theta_t^*}^{m} \phi\left(\frac{K_t}{\theta}\right) f(\theta) d\theta + (1 - (1+\lambda)F(\theta_t^*))\delta\Delta V_{t+1}. \end{aligned} \tag{24}$$

Assume that $\lambda \in [0, \frac{1-\delta}{\delta}]$. This assumption guarantees that $\Delta V_t \geq 0$ for all t.[15] Indeed, $\phi\left(\frac{K_t}{\theta}\right) \leq \delta \Delta V_{t+1}$ for all $\theta \geq \theta_t^*$ by (2), so (24) implies

$$\begin{aligned} \Delta V_t &\geq W - (1 - F(\theta_t^*))\delta\Delta V_{t+1} + (1 - (1+\lambda)F(\theta_t^*))\delta\Delta V_{t+1} = \\ &= W - \lambda F(\theta_t^*)\delta\Delta V_{t+1} \geq W - \lambda\delta\frac{W}{1-\delta} \geq 0. \end{aligned}$$

Denote by $V_t^I(\mathbf{K}, \lambda)$, $V_t^O(\mathbf{K}, \lambda)$, $\Delta V_t(\mathbf{K}, \lambda)$ the officeholder's payoffs as a function of the

[15] For high values of λ, it is in principle possible to have $\Delta V_t < 0$, so that the officeholder in period $t-1$ might prefer *not* to be reelected. Intuitively, this could happen if K_t is very high, $K_{t'}$ is low for $t' > t$, and λ is high. Then being in power in period t likely means being out of power forever after, whereas being out of power in period t allows the officeholder to come to power in period $t+1$ and stay there for a long time.

parameters $\mathbf{K}$, λ. We will now argue that a higher λ weakens the officeholder's incentives for effort:

Lemma 2. *If* $\lambda_0 \leq \frac{1-\delta}{\delta}$, $\Delta V_t(\boldsymbol{K}, \lambda_0) \leq \Delta V_t(\boldsymbol{K}, 0)$ *for all* t, $\boldsymbol{K}$.

Proof. We begin by showing that, if $\lambda_0 \leq \frac{1-\delta}{\delta}$ and $\Delta V_{t+1}(\mathbf{K}, \lambda_0) \leq \Delta V_{t+1}(\mathbf{K}, 0)$, then $\Delta V_t(\mathbf{K}, \lambda_0) \leq \Delta V_t(\mathbf{K}, 0)$.

Let $\tilde{V}_t^I(\mathbf{K}, 0)$ be the officeholder's continuation payoff in period t, in the case $\lambda = 0$, if she followed the optimal strategy for $\lambda = \lambda_0$ (i.e., choosing $\theta_t^* = \theta_t^*(\mathbf{K}, \lambda_0)$). Of course, $\tilde{V}_t^I(\mathbf{K}, 0) \leq V_t^I(\mathbf{K}, 0) = \Delta V_t(\mathbf{K}, 0)$ because this is in general suboptimal. By (24),

$$\begin{aligned}
\tilde{V}_t^I(\mathbf{K}, 0) &= W - \int_{\theta_t^*(\mathbf{K},\lambda_0)}^{m} \phi\left(\frac{K_t}{\theta}\right) f(\theta) d\theta + (1 - F(\theta_t^*(\mathbf{K}, \lambda_0)))\delta \Delta V_{t+1}(\mathbf{K}, 0) \geq \\
&\geq W - \int_{\theta_t^*(\mathbf{K},\lambda_0)}^{m} \phi\left(\frac{K_t}{\theta}\right) f(\theta) d\theta + (1 - F(\theta_t^*(\mathbf{K}, \lambda_0)))\delta \Delta V_{t+1}(\mathbf{K}, \lambda_0) \geq \\
&\geq W - \int_{\theta_t^*(\mathbf{K},\lambda_0)}^{m} \phi\left(\frac{K_t}{\theta}\right) f(\theta) d\theta + (1 - (1+\lambda_0) F(\theta_t^*(\mathbf{K}, \lambda_0)))\delta \Delta V_{t+1}(\mathbf{K}, \lambda_0) = \\
&= \Delta V_t(\mathbf{K}, \lambda_0),
\end{aligned}$$

where we have used that $\Delta V_{t+1}(\mathbf{K}, \lambda_0) \leq \Delta V_{t+1}(\mathbf{K}, 0)$ by assumption and that $\Delta V_{t+1}(\mathbf{K}, \lambda_0) \geq 0$ because λ_0 is low enough. More generally, the same argument shows that, if $0 < \Delta V_{t+1}(\mathbf{K}, \lambda_0) - \Delta V_{t+1}(\mathbf{K}, 0) = M$, then $\Delta V_t(\mathbf{K}, \lambda_0) - \Delta V_t(\mathbf{K}, 0) \leq \delta M$. Since $\Delta V_{t'} \leq \frac{W}{1-\delta}$ in all cases, we can conclude that either $\Delta V_t(\mathbf{K}, \lambda_0) \leq \Delta V_t(\mathbf{K}, 0)$ (if the same inequality holds for any $t' > t$) or, if not, then

$$\Delta V_t(\mathbf{K}, \lambda_0) - \Delta V_t(\mathbf{K}, 0) \leq \delta^{t'-t}(V_{t'}(\mathbf{K}, \lambda_0) - \Delta V_{t'}(\mathbf{K}, 0)) \leq \delta^{t'-t}\frac{W}{1-\delta}$$

for arbitrarily high t', which also implies the $\Delta V_t(\mathbf{K}, \lambda_0) \leq \Delta V_t(\mathbf{K}, 0)$. □

The result now follows immediately from Lemma 2: if ΔV_t is lower at all t for $\lambda \in (0, \frac{1-\delta}{\delta})$ than for $\lambda = 0$, then any fixed rule $\mathbf{K}$ extracts less effort from the officeholder in the former case, and so the voter's payoff must also be lower when comparing the respective optimal rules. □

C Teaching Guide for Proposition C

Ferejohn's model is often the first formal model of accountability taught to graduate students in political science. With that in mind, this Section provides a guide to proving the main claims of Proposition C that should make the analysis digestible for first or second-year graduate students.

(i) Note that the voter's welfare given a retrospective rule **K** can be written as $U(\mathbf{K}) = \sum_{t=0}^{\infty} \delta^t K_t (1 - F(\theta_t^*(\mathbf{K})))$. (Equation 3')

Note that, under the assumptions of Proposition C, this simplifies to $U(\mathbf{K}) = \sum_{t=0}^{\infty} \delta^t K_t (1 - \theta_t^*(\mathbf{K}))$.

(ii) Differentiate the expression for $U(\mathbf{K})$ with respect to K_t for each t to obtain the relevant FOC for each performance threshold:

$$0 = \frac{\partial U}{\partial K_t} = \delta^t(1 - \theta_t^*(\mathbf{K})) - \delta^t K_t \frac{\partial \theta_t^*(\mathbf{K})}{\partial K_t} - \sum_{s=0}^{t-1} \delta^s K_s \frac{\partial \theta_s^*(\mathbf{K})}{\partial K_t}.$$

(iii) Note that Equation 2 reduces to $\theta_t^* = \frac{K_t}{\delta V_{t+1}^I}$. Show that this implies $\frac{\partial \theta_t^*(\mathbf{K})}{\partial K_t} = \frac{\theta_t^*(\mathbf{K})}{K_t}$.

(iii') Note that Equation 8 reduces to

$$V_t^I = W + \int_{\theta_t^*}^1 \left[\delta V_{t+1}^I - \frac{K_t}{\theta} \right] d\theta = W + \delta V_{t+1}^I (1 - \theta_t^*) + K_t \ln(\theta_t^*).$$

Show that this, combined with Equation 2, implies

$$\frac{\partial \theta_t^*(\mathbf{K})}{\partial K_{t+1}} = -\frac{K_t}{\delta V_{t+1}^{I2}} \ln(\theta_{t+1}^*) = -\theta_t^{*2} \frac{\delta}{K_t} \ln(\theta_{t+1}^*).$$

More generally, for $s < t$, repeated application of Equation 8 yields $\frac{\partial V^I_{s+1}}{\partial V^I_t} = \delta^{t-s-1}(1-\theta^*_{s+1}) \times \ldots \times (1-\theta^*_{t-1})$, so

$$\frac{\partial \theta^*_s(\mathbf{K})}{\partial K_t} = -\theta^{*2}_s \frac{\delta}{K_s} \frac{\partial V^I_{s+1}}{\partial K_t} = -\theta^{*2}_s \frac{\delta}{K_s} \frac{\partial V^I_{s+1}}{\partial V^I_t} \ln(\theta^*_t) = -\theta^{*2}_s \frac{\delta^{t-s}}{K_s} \ln(\theta^*_t) \prod_{l=s+1}^{t-1} (1-\theta^*_l).$$

(iv) Combine (ii), (iii) and (iii') to obtain

$$\begin{aligned} 0 = \frac{\partial U}{\partial K_t} =& \delta^t(1-\theta^*_t) - \delta^t \theta^*_t + \sum_{s=0}^{t-1} \delta^t \ln(\theta^*_t) \theta^{*2}_s \prod_{l=s+1}^{t-1} (1-\theta^*_l) \\ \iff 0 =& 1 - 2\theta^*_t + \sum_{s=0}^{t-1} \ln(\theta^*_t) \theta^{*2}_s \prod_{l=s+1}^{t-1} (1-\theta^*_l) \end{aligned}$$

for all t.

(v) Write down the equations obtained for the first few values of t:

$$\begin{aligned} &0 = 1 - 2\theta^*_0 \\ &0 = 1 - 2\theta^*_1 + \ln(\theta^*_1)\theta^{*2}_0 \\ &0 = 1 - 2\theta^*_2 + \ln(\theta^*_2)(\theta^{*2}_0(1-\theta^*_1) + \theta^{*2}_1) \\ &0 = 1 - 2\theta^*_3 + \ln(\theta^*_3)(\theta^{*2}_0(1-\theta^*_1)(1-\theta^*_2) + \theta^{*2}_1(1-\theta^*_2) + \theta^{*2}_2) \\ &\ldots \end{aligned}$$

Rewrite the system by defining $A_t = \sum_{s=0}^{t-1} \theta^{*2}_s \prod_{l=s}^{t-1}(1-\theta^*_l)$ to obtain Equations (13)–(14) as shown in the proof of Proposition C:

$$
\begin{aligned}
&A_0 = 0 && 0 = 1 - 2\theta_0^* \\
&A_1 = \theta_0^{*2} + A_0(1-\theta_0^*) = \theta_0^{*2} && 0 = 1 - 2\theta_1^* + \ln(\theta_1^*)A_1 \\
&A_2 = \theta_1^{*2} + A_1(1-\theta_1^*) && 0 = 1 - 2\theta_2^* + \ln(\theta_2^*)A_2 \\
&A_3 = \theta_2^{*2} + A_2(1-\theta_2^*) && 0 = 1 - 2\theta_3^* + \ln(\theta_3^*)A_3 \\
&\quad \ldots
\end{aligned}
$$

Convince yourself that this recursive system pins down θ_t^* and A_t for all t. Moreover, defining T implicitly by $0 = 1 - 2T(x) + x\ln(T(x))$ and defining S by $S(x) = (1 - T(x))x + T(x)^2$, convince yourself that $\theta_t^* = T(A_t)$ for all t, and $A_{t+1} = S(A_t)$ for all t.

The steps up to this point cover the preliminary results before Proposition C as well as the first main step of the proof of this proposition ("pinning down θ_t^*"). The next step is to show that the sequence $(\theta_t^*)_{t\geq 0}$ is decreasing in t.

(vi) To prove this result analytically, show by using the definitions of S and T that (a) S is a stricty increasing function; (b) T is a strictly decreasing function; and (c) $A_1 > A_0$. Deduce that $A_{t+1} > A_t$ for all t and hence $\theta_{t+1}^* < \theta_t^*$ for all t.

You may like to check the result numerically. Here are two ways. First, using the recursive system from (v), you may solve numerically for as many elements of the sequence $(\theta_t^*)_{t\geq 0}$ as desired, and check that the sequence is decreasing. Second, you may plot the functions S and T to verify that they are increasing and decreasing, respectively. Both are simple coding exercises.

(vii) Take the limit of Equations (13)–(14) to characterize θ_∞^*.

(viii) The analytical proof that $(K_t)_{t\geq 0}$ is decreasing is involved. The interested reader may follow the argument given in Proposition C.

However, the result is easy to check numerically. Indeed, you only need to follow the logic from Equation (15) up to Equation (18). If you have solved for $(\theta_t^*)_{t\geq 0}$ numerically on a computer, you can use Equation (18) to then compute K_t for as many values of t as you like and check that the sequence is decreasing.

The only complication is that the formula for K_t involves values of θ_s^* for all $s \geq t$. Of course, you can only calculate a finite number of values of θ_s^*. However, if you are using a value of δ not too close to 1, you can approximate K_t arbitrarily well by replacing tail values of θ_s^* for $s >> t$ with θ_∞^*.