Online Appendix

Deciphering Private Equity Incentive Contracting and Fund Leverage Choice

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Appendix A

The Case for Incorporating the Costs of Financial Distress

Models of private equity (PE) valuation and capital structure have largely sidestepped financial distress costs to focus on other frictions. The lack of attention on financial distress costs in PE may in part follow from some of the earlier findings of Andrade and Kaplan (1998). They show that high debt levels, and not operating inefficiencies, are the sources of distress in their leveraged buyout (LBO) sample. They also find that financial distress is resolved with fewer losses on average relative to a non-treated sample, and that financial distress costs are nearly non-existent for LBO transactions that do not experience a negative shock. Yet, in the end, the authors estimate the costs of financial distress to be 10% to 20%.

These results were generated from a sample of buyout funds in the 1980s and 1990s. During that period, buyout funds performed very well on average, with high alphas helping offset the usual costs of financial distress. But excess returns have declined in recent years (e.g., Gupta and Van Nieuwerburgh, 2021; Phalippou, 2021), implying greater likelihoods of default, as well as loss given default.

In the commercial real estate sector, publicly listed firms and non-institutional PE market alternatives exist to compete directly with institutional private equity real estate (PERE). This implies diminished marginal operating, governance, and financial engineering gains attributable to PERE, resulting in relatively lower alphas (Gupta and Van Nieuwerburgh, 2017; Pagliari, 2020; Riddiough and Li. 2023). Furthermore, in their analysis of insurance company loans backed by income-producing collateral, Brown et al. (2006) document distress costs in commercial real estate lending on the order of 20% to 30% above and beyond losses attributable to the asset's internal transfer value at the time of foreclosure. These costs include addressing deferred maintenance and realizing fire-sale discounts when disposing of the asset. For PERE value-add and opportunity funds, which focus on real estate development and repositioning opportunities, I would expect lender losses to be meaningfully higher than those found in Brown et al. (2006).[1]

Appendix B

Proofs to Propositions and Corollaries

For all of the proofs, the subscripts are suppressed wherever doing so does not introduce any ambiguity into the meaning of the variables.

Proof of Proposition 1: Starting with equation (1b), after doing some algebra and using the standard normal pdf, I have that $\frac{\partial D_0}{\partial B} = e^{-rT}\left[N[d_2] - \frac{1}{\sigma\sqrt{2\pi T}}e^{-\frac{1}{2}d_2^2}\right] + (1-k)V_0 e^{\alpha T}\left[\frac{1}{B\sigma\sqrt{T}\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2}\right]$. From (1b'), $d_1^2 = d_2^2 + 2d_2\sigma\sqrt{T} + \sigma^2 T$. After substituting d_1^2 into the prior equation, using the definition of d_2^2 from (1b'), and after completing the squares in the exponents, I get $\frac{\partial D_0}{\partial B} = e^{-rT}\Big[N[d_2] - \frac{1}{\sigma\sqrt{T}}n(d_2)\Big] + (1-k)e^{-rT}\frac{1}{\sigma\sqrt{T}}n(d_2) = e^{-rT}\left[N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}\right]$.

To prove the existence and uniqueness of a finite B_k^* when $k > 0$ that satisfies $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}} = 0$, I note that $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}$ is everywhere continuous, $N[d_2] - \frac{k}{\sigma\sqrt{T}}n(d_2) = 1$ for $B = 0$, and that $N[d_2] - \frac{k}{\sigma\sqrt{T}}n(d_2) \to 0$ as $B \to \infty$. Now I claim that $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}} \to 0$ from below (i.e., the quantity is negative when B is large). For this to be true, $\frac{k}{\sigma\sqrt{T}} > \frac{N[d_2]}{n(d_2)}$ for any $k > 0$ as B gets large. Applying L'Hospital's rule to the right-hand side of the inequality shows that it goes to zero in the limit, confirming that $N[d_2] - \frac{k}{\sigma\sqrt{T}}n(d_2) \to 0$ from below. Next, I take the derivative of $N[d_2] - \frac{k}{\sigma\sqrt{T}}n(d_2)$ with respect to B, which results in $n(d_2)\frac{\partial d_2}{\partial B}\left[1 + d_2\frac{k}{\sigma\sqrt{T}}\right]$. The terms outside the bracket together are negative. The term inside the bracket is initially positive when B is small, and then eventually turns negative for some sufficiently large B. This implies that the slope of $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}$ is initially negative as a function of B, but then turns positive for some unique sufficiently large B, and then stays positive thereafter. This is all that is needed for the existence and uniqueness of B_k^*, since, for the above collection of facts to be true, it must be the case that there is a single crossing

[1] I note that tax shield effects in PE appear to be less important than standard corporate financial analysis might suggest. For example, Jenkinson and Stucke (2011) find that incremental tax shield benefits to the issuance of buyout debt largely accrue to preexisting shareholders through the acquisition share price. In addition, not all PE debt is issued at the target firm (Op-Co) level, with increasing debt in recent years being issued at the fund or sponsor level, presumably with no tax shield pass-through benefits. Similarly, in PERE, there is no taxation at the property-firm level, which negates double taxation on equity. Even listed commercial real estate firms, REITs, are not taxed at the firm level. Furthermore, the vast majority of LP equity investors in PERE (as well as certain other forms of PE) are tax-exempt institutions, such as pension funds, endowments, and sovereign wealth funds.

at zero in which there is one and only one B for which $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}} = 0$. Finally, given these facts, in the range of $B \in [0, B_k^*)$ it immediately follows that $\frac{\partial D_0}{\partial B} > 0$ and $\frac{\partial^2 D_0}{\partial B^2} < 0$ when $k > 0$. QED

Proof of Proposition 2: From equation (2b), the FOC is: $\frac{\partial \Phi^V}{\partial B} = \rho\left[V_0 e^{(\mu+\alpha)T} n(h_1)\frac{\partial h_1}{\partial B} - N[h_2]\frac{\partial \chi_0}{\partial B} - \chi_0 n(h_2)\frac{\partial h_2}{\partial B}\right] = 0$. Recalling that $\chi_0 = B + (V_0 - D_0)e^{\psi T}$, it follows that $\frac{\partial \chi_0}{\partial B} = 1 - e^{(\psi-r)T}\left[N[d_2] - \frac{k}{\sigma\sqrt{T}} n(d_2)\right]$, $\psi \geq r$. Subbing this into the FOC and utilizing results from Proposition 1, as well as well-known comparative static relations for call options with respect to B (the exercise price), the FOC simplifies to $\frac{\partial \Phi^V}{\partial B} = -\rho N[h_2]\left[1 - e^{(\psi-r)T}\left[N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}\right]\right] = 0$. For $\rho > 0$, the FOC reduces to equating the terms inside the brackets to zero. Existence and uniqueness follow from the logic spelled out in the proof to Proposition 1. The lack of dependence on ρ is based on inspection of the FOC above. Finally, when $\psi < r$, inspection of $\frac{\partial \Phi^V}{\partial B}$ reveals that an internal optimum does not exist. Further inspection reveals that Φ^V is universally decreasing in B within the feasible range, implying $B^* = 0$. QED

Proof of Corollary 1 to Proposition 2: $\psi = r - \frac{1}{T} ln\left[N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}\right]$ follows directly from the FOC derived for Proposition 2. I will refer to this relation repeatedly when deriving the comparative static results. The comparative static $\frac{\partial B^*}{\partial \psi} > 0$ follows from $\frac{\partial \psi}{\partial B} > 0$ in the above functional relation, since $N[d_2] - \frac{k}{\sigma\sqrt{T}} n(d_2) > 0$ and the derivative of this quantity is negative, per the proof of Proposition 1. I use the fact that $\frac{\partial B^*}{\partial \psi} > 0$ and implicit differentiation to generate the other stated comparative static relations. In the case of $\frac{\partial B^*}{\partial k}$, inspection of the above relation reveals that $\frac{\partial \psi}{\partial k} > 0$, implying that $\frac{\partial B^*}{\partial k} < 0$. In the case of $\frac{\partial B^*}{\partial \alpha}$, $\frac{\partial \psi}{\partial \alpha}$ is seen to be negative, implying that $\frac{\partial B^*}{\partial \alpha} > 0$. For the cases of $\frac{\partial B^*}{\partial \sigma}$, $\frac{\partial B^*}{\partial T}$, and $\frac{\partial B^*}{\partial r}$, after quite a bit of tedious algebra, I am unable to sign $\frac{\partial \psi}{\partial \sigma}$, $\frac{\partial \psi}{\partial T}$, and $\frac{\partial \psi}{\partial r}$. This in turn implies that I am unable to sign $\frac{\partial B^*}{\partial \sigma}$, $\frac{\partial B^*}{\partial T}$, and $\frac{\partial B^*}{\partial r}$. QED

Proof of Proposition 3: As a first step, it is useful to write out $\frac{\partial \Phi_{CU}^V}{\partial B}$. This quantity can be expressed as $\frac{\partial \Phi_{CU}^V}{\partial B} = \xi[N(m_2) - N(h_2)]\left[1 - e^{(\psi-r)T}\left[N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}}\right]\right] - \rho N(m_2)$. Note that the bracketed term to the far left is always negative in the relevant range for B, as is the last term. When the larger bracketed term (that also contains the choke condition term) is positive, then $\frac{\partial \Phi_{CU}^V}{\partial B} < 0$. This is always true when $\psi \leq r$, and holds for all $B \geq 0$ and follows because $N[d_2] - n(d_2)\frac{k}{\sigma\sqrt{T}} = 1$ at $B = 0$, which then decreases in the feasible range $B \in [0, B_k^*]$. In this case, $B_{CU}^* = 0$ as claimed. In the case of $\psi > r$ and $1 - e^{(\psi-r)T} \geq \frac{\rho N[m_2]}{\xi[N[m_2] - N[h_2]]}$ at $B = 0$, inspection of $\frac{\partial \Phi_{CU}^V}{\partial B}$ above reveals that $\frac{\partial \Phi_{CU}^V}{\partial B} < 0$ for all $B \geq 0$, implying that $B_{CU}^* = 0$ in this case as well. Lastly, in the case of $\psi > r$ and $1 - e^{(\psi-r)T} <$

$\frac{\rho N[m_2]}{\xi[N[m_2]-N[h_2]]}$ at $B = 0$, inspection of $\frac{\partial \Phi_{CU}^{V}}{\partial B}$ above shows that the larger bracketed term is negative and that $\frac{\partial \Phi_{CU}^{V}}{\partial B} > 0$ at $B = 0$. Now, because the right-hand side of the FOC expressed in equation (10b) is always negative, and because the left-hand side of (11b) is increasing in B, there will exist a $B_{CU}^{*} < B^{*}$ that satisfies the FOC written in (10b). QED

Appendix C

Alternative Model of Private Equity Fundraising

In consideration of the alternative fundraising model, I fix $E_0 = \bar{E}_0$ and ask how large V_0 should be given that debt finances all fund acquisitions in excess of $\bar{E}_0$. Consequently, the PE fundraising game now has two independent stages, with equity fundraising coming in the first stage and optimal fund size determination based on debt financing in the second stage.

This approach to fundraising complicates debt valuation, because debt is now self-referencing. That is, $\breve{D}_0 = e^{-rT}BN\left[\breve{d}_2\right] + (1-k)\breve{V}_0 e^{\alpha T}N\left[-\breve{d}_1\right]$ as before, with $\breve{d}_1 = \frac{ln\left[\breve{V}_0/_B\right]+\left((r+\alpha)+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$, $\breve{d}_2 = \breve{d}_1 - \sigma\sqrt{T}$. But now $\breve{V}_0 = \bar{E}_0 + D_0$ and $\breve{D}_0 = D_0$ are imposed as constraints. I note that $\breve{D}_0$ is well-behaved (continuous and increasing) as it depends on *B,* for $B \in [0, B_k^*)$.

Taking the total derivative of $\breve{D}_0$ with respect to *B*, I obtain $\frac{d\breve{D}_0}{dB} = \frac{\partial\breve{D}_0}{\partial B} + \frac{\partial\breve{D}_0}{\partial\breve{V}_0}\frac{\partial\breve{V}_0}{\partial B}$. Since $\breve{V}_0 = \bar{E}_0 + D_0$ and $\breve{D}_0 = D_0$, $\frac{d\breve{D}_0}{dB} = \frac{\partial\breve{V}_0}{\partial B}$ and the total derivative can be rewritten as $\frac{d\breve{D}_0}{dB} = \frac{\partial D_0}{\partial B}\left[1 + \frac{\partial\breve{D}_0}{\partial\breve{V}_0}\right]$. And since $\breve{V}_0 = \bar{E}_0 + D_0$, it is clear that $\frac{\partial\breve{D}_0}{\partial V_0} > 0$ in the relevant range for *B*. In particular, $\frac{\partial\breve{D}_0}{\partial V_0} = (1-k)N\left[-\breve{d}_1\right] + n\left(\breve{d}_1\right)\frac{k}{\sigma\sqrt{T}} > 0$. Thus, not only is $\frac{d\breve{D}_0}{dB}$ positive, but $\frac{d\breve{D}_0}{dB} > \frac{\partial D_0}{\partial B}$ for $B \in [0, B_k^*)$.

With this result I am now in a position to consider the GP's problem of optimizing fund size as it depends on *B.* As before, the problem is stated as: $\underset{B}{Max}\ \Phi^V(B;\psi,\rho) = \rho\left[\breve{V}_0 e^{(\mu+\alpha)T}N\left[\breve{h}_1\right] - \chi_0 N\left[\breve{h}_2\right]\right]$, $\breve{h}_1 = \frac{ln\left[\breve{V}_0/_{\chi_0}\right]+\left((\mu+\alpha)+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$, $\breve{h}_2 = \breve{h}_1 - \sigma\sqrt{T}$. Further, as above, $\breve{V}_0$ is a function of *B,* with the previously imposed constraints applying. Evaluating incentive compatibility results in the following relation: $\frac{1}{\frac{d\breve{D}_0}{dB}} = \frac{e^{(\mu+\alpha)T}N[\breve{h}_1]}{N[\breve{h}_2]} > e^{(\mu+\alpha)T} > e^{\psi T}$ when $\mu + \alpha > \psi$. A unique interior solution therefore exists when $r < \mu + \alpha$, which is always the case as long as $\mu > r$ and $\alpha \geq 0$. Since $\frac{d\breve{D}_0}{dB} > \frac{\partial D_0}{\partial B}$ for $B \in [0, B_k^*)$, the marginal cost of debt in this case (left-hand side of the above relation) is less than the marginal cost of debt with the baseline fundraising model. This implies that the optimal *B* is larger in this fundraising model whenever $\mu + \alpha > \psi$. And since $\frac{d\breve{D}_0}{dB} > \frac{\partial D_0}{\partial B}$, the optimal $\breve{D}_0$ will also be larger than D_0. Lastly, note that $\breve{D}_0 = 0$ when $D_0 = 0$, implying that the marginal cost of debt increases without bound as D_0 approaches zero. This in turn implies that the optimal $\breve{D}_0$ is finite and therefore fund size is finite.

Appendix D

Parameter Value Selection

Total Asset Acquisition Cost and Management Fees: Proper accounting for management fees is more complex than typically characterized. Many management fees, such as acquisition, monitoring, and development fees, scale directly to total fund size as opposed to invested capital. Moreover, management fees are paid over the entire life of the fund, and not simply over the cash-based duration of fund investment. For example, although fund durations may commonly be on the order of five to seven years, management fees are incurred continually over a typical fund life of 10 years. Consequently, I assume 1.50% in total management fees per annum (as displayed in column (1) of Table 1), taken over a 10-year fund life, to result in 15.0% total management fees as a percentage of invested capital. Then I rescale these fees based on representative fund leverage ratios, resulting in total fixed management fees equal to 5.0% of the fund's asset acquisition cost. In all cases, I assume that the fund's asset acquisition cost is $V_0 = 100$.

Incentive Fee Contract Variables: For baseline model estimation purposes, in this sub-section I specify a carried interest share of 20.0% and a carried interest hurdle rate of 9.0%, with no catch-up fee provision term. Catch-up fees are considered separately, after the completion of this initial calibration exercise.

Fund Assets' Unlevered Equilibrium Rate of Return: Reference to monthly reports issued by Green Street suggests unlevered returns of around 6.0%. Real Estate Research Corporation (RERC) also produces estimates of unlevered discount rates for major property types. As of the fourth quarter of 2020, their estimates range from 5.0% to 8.5% for A+ to A quality property.

Value-add and opportunity PERE funds typically acquire assets that require repositioning, and oftentimes significant development or redevelopment. This increases asset risk relative to otherwise equivalent income-producing assets. According to RERC, unlevered discount rates for B and C quality assets range from approximately 7.0% to 12.0%. In referencing Pagliari's (2020) analysis of value-add and opportunity funds, he pegs unlevered expected asset returns at approximately 10.0%. I am not aware of any other direct evidence on the topic, so I use $\mu = .10$ as my base-case value.

Fund Assets' Unlevered Standard Deviation of Return: Empirical estimates of PERE fund return volatility exist in industry publications (e.g., *CEM Benchmarking 2020*). But such estimates are generally made on a portfolio or index of levered funds on a net-of-fee basis. With that in mind, fund volatility estimates are typically in the 17.5% to 25.0% range. Pagliari (2020) generates direct volatility estimates on value-add and opportunity PERE funds in the 15.0% to 20.0% range. Based on my analysis of Preqin data, I find the standard deviation of IRRs realized on 399 value-add and opportunity PERE funds to be 16% to 17%. Altogether, these data points result in $\sigma = .175$, which I use as my base-case value.

I note that the base-case fund volatility is significantly below asset volatilities estimated or assumed in venture capital and buyout funds (e.g., Cochrane, 2005; Metrick and Yasuda, 2010; Sorensen, Wang, and Yang, 2014). But, given the highly specific idiosyncratic risks associated with these investments, they generally have a low correlation structure that significantly reduces variation in payoffs at the fund level. Resulting variation in fund-level payoffs is therefore much more in line with my PERE-based fund payoff volatility estimate.[2] Furthermore, PERE funds often specialize by property type and geographical area or region, resulting in a strong correlation structure relative to that observed in buyout and venture capital funds.

Alpha: Alpha is estimated on a gross-of-fee basis. There are no direct estimates of gross-of-fee value-add and opportunity PERE fund alphas that I know of. As noted earlier, there are good reasons to believe that operational, governance, and financial engineering benefits to PERE are limited relative to the benefits available in buyout or venture capital funds. Limited benefits are closely related to the existence of a viable and liquid parallel public market for the ownership of commercial real estate. That said, the repositioning and redevelopment of assets held by PERE funds do offer opportunities to add value relative to holding run-of-the-mill income-producing property.

Recent studies document PERE fund alphas on a net-of-fee basis. Gupta and Van Nieuwerburgh (2021) estimate that closed-end PERE funds lose 17 cents on average for every dollar invested. Given fund durations of five to seven years on average, this equates to approximately a 2.0% to 3.0% negative net-of-fee alpha. Applying a standard mean-variance framework, Bollinger and Pagliari (2019) and Pagliari (2020) generate similar net-of-fee alpha estimates for PERE value-add and opportunity funds. Risk adjustments are made in both studies and do not explicitly account for liquidity differences between PERE funds and the liquid benchmark indices.

This leads me to choose a base case $\alpha = .02$. This estimate seems reasonable based on net-of-fee performance ranging from 0.0% to −3.0 %, with fee drag in the 3.5%-4.0% range.

Fund Duration and Debt Term: I obtain *Preqin* data on 78 PERE value-add and opportunity funds that have liquidated and for which I have a full set of cash flows. In these data, the average fund life is 10.5 years. Durations and weighted average fund lives are significantly shorter, however. These data generate an average fund duration of 4.7 years [as measured by the method suggested in Phalippou and Gottschalg (2009)] and an average weighted average life of just under 5.0 years. Other data I have seen indicate weighted average PERE fund lives in the four- to eight-year range. Given these data points, I take $T = 6$ years as my base-case value.

Risk-free Rate of Return: The U.S. Treasury rate is often referenced as the risk-free rate. Previous work suggests that the risk-free rate exceeds the Treasury rate due to a convenience yield (Krishnamurthy and Vissing-Jorgensen, 2012). Thus, I specify the risk-free rate as $r = .02$. This relatively low rate applies to the 2004-2020 time period that I use for my empirical analysis.

Costs of Financial Distress: Financial distress costs were previously pegged at $k = .30$ (see Appendix A).

[2] For example, Sorensen et al. (2014) assume 60% volatility at the deal level, with 20% pairwise correlations and 15 deals in a fund. This reduces volatility to 25% at the fund level.

Appendix E

LP Return Targeting and the Baseline Contract

To start, I make three initial observations. First, incentive fees are strictly decreasing in λ. Second, incentive fees are strictly increasing in ρ, with the interest carry share acting as a linear scaling factor. Third, based on these first two facts, the GP is able to extract fees to the point where the target return constraint binds. Specifically, the GP sets $\bar{\rho}$ such that:

$$\bar{\rho} = \frac{\mathcal{E}_T - e^{\lambda^* T}[E_0 + \Phi^F]}{V_0 e^{(\mu+\alpha)T} N[h_1] - \chi_0 N[h_2]}. \tag{E.1}$$

Two additional observations are in order. First, for $\psi < \lambda^*$, the target return constraint in (7a) cannot be satisfied when $\rho = 1$. That is, at least some profit sharing by the GP through interest carry must occur. Second, GPs will exit the market when they cannot meet the return target constraint with a positive carried interest share. That is, the GPs will not contract for $\rho \leq 0$. As a result, based on these structural considerations, the $0 < \rho < 1$ constraint in (7a) will always be satisfied and hence can be ignored.

To solve (7) subject to (7a), I form the Lagrangian, with the resulting Karush-Kuhn-Tucker (KKT) conditions:

$$\mathcal{L}(B,\rho) = -\Phi^V(\rho, B; \psi) + \mu_1\big(\lambda^* - \lambda^N(B,\rho)\big) + \mu_2(D_0(B) - \bar{D}_0), \tag{E.2}$$

where $\mu_1, \mu_2 \geq 0$ denote Lagrange multipliers. With this I can now state the major result of this subsection.

Proposition E1 (The Constrained Baseline Contract When LPs Target Returns): *When maximizing incentive fees, the target return constraint always binds, with $\mu_1 = T[\mathcal{E}_T - \Phi^V] > 0$. As a result, the GP sets the carried interest share according to equation (E.1) in order to just meet the target return constraint. In general, depending on parameter values, the target fund leverage constraint may or may not be binding. When the fund leverage constraint binds, $\mu_2 = \frac{1}{\frac{\partial D_0}{\partial B}}\left[\frac{\partial \mathcal{E}_T}{\partial B} - \frac{\partial E_0}{\partial B}\left(\frac{\mathcal{E}_T - \Phi^V}{E_0 + \Phi^F}\right)\right] > 0$. Target return and fund leverage constraints simultaneously bind if and only if $\frac{E_0 + \Phi^F}{\mathcal{E}_T} < e^{-\lambda^* T} < \frac{\frac{\partial E_0}{\partial B}}{\frac{\partial \mathcal{E}_T}{\partial B}}$. This inequality relation is independent of ψ.*

Proof: Given the Lagrangian stated in (E.2), after examining the necessary FOCs I find that, regardless of whether the fund leverage constraint is binding, $\mu_1 = \frac{\frac{-\partial \Phi^V}{\partial \rho}}{\frac{-\partial \Phi^V}{\partial \rho}\left(\frac{1}{T}\right)\left(\frac{1}{\mathcal{E}_T - \Phi^V}\right)} = T[\mathcal{E}_T - \Phi^V] > 0$. This implies that the fund target return constraint always binds. As a result, one finds $\bar{\rho} > 0$ as defined in equation (E.1), where positive $\bar{\rho}$ is necessary to satisfy GP participation. With μ_1, in the case of a binding fund leverage constraint, after some algebra I obtain $\mu_2 = \frac{1}{\frac{\partial D_0}{\partial B}}\left[\frac{\partial \mathcal{E}_T}{\partial B} - \left(\frac{\mathcal{E}_T - \Phi^V}{E_0 + \Phi^F}\right)\frac{\partial E_0}{\partial B}\right]$. The term inside the bracket must be verified to be positive for $\mu_2 > 0$. When $\mu_2 > 0$, it must be that $\frac{\frac{\partial E_0}{\partial B}}{\frac{\partial \mathcal{E}_T}{\partial B}} > \frac{E_0+\Phi^F}{\mathcal{E}_T-\Phi^V}$. Recalling equation (6), this inequality implies that $\frac{\frac{\partial E_0}{\partial B}}{\frac{\partial \mathcal{E}_T}{\partial B}} > e^{-\lambda^* T}$. Also, GP participation vis-à-vis equation (E.1) implies that $\mathcal{E}_T - [E_0 + \Phi^F]e^{\lambda^* T} > 0$, which in turn implies that $e^{-\lambda^* T} > \frac{E_0+\Phi^F}{\mathcal{E}_T}$. Altogether I have that $\frac{E_0+\Phi^F}{\mathcal{E}_T} < e^{-\lambda^* T} < \frac{\frac{\partial E_0}{\partial B}}{\frac{\partial \mathcal{E}_T}{\partial B}}$ when $\mu_2 > 0$ and GP participation is satisfied as stated in the proposition. Furthermore, $\frac{E_0+\Phi^F}{\mathcal{E}_T} < e^{-\lambda^* T} < \frac{\frac{\partial E_0}{\partial B}}{\frac{\partial \mathcal{E}_T}{\partial B}}$ implies $\mu_2 > 0$. All terms are independent of ψ, as claimed. QED

In general, the target fund leverage constraint may or may not be binding, depending on the parameter values. It will not be binding when $\mu_2 = 0$ for some B such that $D_0(B) < \bar{D}_0$. Given the calibrated parameter values I use in my analysis, I find that the target fund leverage constraint always binds, implying that $\mu_2 > 0$ for $\bar{B}$ such that $D_0(\bar{B}) = \bar{D}_0$. Thus the LP essentially breaks even from a risk-return perspective. Interestingly, μ_2, the shadow value associated with relaxing the fund leverage constraint, equals the marginal cost of debt, $\frac{1}{\frac{\partial D_0}{\partial B}}$, multiplied by the (partially-adjusted) increase in LP return associated with a marginal dollar of debt, $\left[\frac{\partial \mathcal{E}_T}{\partial B} - \frac{\partial E_0}{\partial B}\left(\frac{\mathcal{E}_T-\Phi^V}{E_0+\Phi^F}\right)\right]$.

One can simply use the inequality relation, $\frac{E_0+\Phi^F}{\mathcal{E}_T} < e^{-\lambda^* T} < \frac{\frac{\partial E_0}{\partial B}}{\frac{\partial \mathcal{E}_T}{\partial B}}$, to verify that the GP participates in the fund with $\bar{\rho} > 0$ and with the fund leverage constraint binding at $D_0(\bar{B}) = \bar{D}_0$. The participation constraint, $\frac{E_0+\Phi^F}{\mathcal{E}_T} < e^{-\lambda^* T}$, requires finding an $\alpha^{Min} > 0$ such that $\bar{\rho} = 0$. Then for any $\alpha > \alpha^{Min}$, the GP participates and sets $\bar{\rho}$ according to equation (E.1), with a net-of-fee return to the LP of $\lambda = \lambda^*$. For the fund leverage constraint to bind, it must be that $e^{-\lambda^* T} < \frac{\frac{\partial E_0}{\partial B}}{\frac{\partial \mathcal{E}_T}{\partial B}}$, which follows from the second KKT condition. In all cases, the inequality relations do not depend on the carry hurdle rate, ψ.

Appendix F

LP Return Targeting and the Catch-Up Provision

To solve equation (11), I form the Lagrangian, with KKT conditions as follows:

$$\mathcal{L}(B,\xi) = -\Phi^V(\rho,B;\psi) + \mu_1\big(\lambda^* - \lambda^N(B,\rho)\big) + \mu_2(D_0(B) - \overline{D}_0) + \mu_3(\xi - 1) \qquad \text{(F.1)}$$

and where $\mu_1, \mu_2, \mu_3 \geq 0$ denote Lagrange multipliers.

Prior to stating solutions to the constrained optimization problem, it is also useful to recall the following relation:

$$\Phi^V_{CU} = \mathcal{E}_T - e^{\lambda^* T}[E_0 + \Phi^F]. \qquad \text{(F.2)}$$

This is simply a rearrangement of equation (6), along with constraining λ^N to equal λ^*. I use this relation to locate solutions when analyzing KKT conditions according to equation (F.1).

With this, I am now in a position to state the constrained efficient solution to the catch-up fee contracting problem.

Proposition F1 (The Constrained Catch-Up Fee Contracting Problem): *For empirically-based parameter value ranges established previously, there are three solution ranges to consider. To bracket the solution ranges, let $\dot{\alpha}$ be that $\alpha > 0$ with $\lambda = \lambda^*$ and $D_0 = \overline{D}_0$ given $\xi = \rho$, and let $\ddot{\alpha}$ be that $\alpha > \dot{\alpha}$ at which $\lambda = \lambda^*$ and $D_0 = \overline{D}_0$ given $\xi = 1.0$. The contracting solution ranges are as follows: 1) when $\alpha < \dot{\alpha}$, no catch-up provision is included the incentive contract; 2) when $\dot{\alpha} \leq \alpha < \ddot{\alpha}$, both the target return and fund leverage constraints bind, with ξ chosen to satisfy equation (E.2); 3) when $\alpha \geq \ddot{\alpha}$, full catch-up $\xi = 1.0$ is implemented, along with $\lambda = \lambda^*$. $D_0(B) < \overline{D}_0$ is chosen to satisfy equation (F.2).*

Proof: The first step is to determine $\frac{\partial \Phi^V_{CU}}{\partial \xi}$ and $\frac{\partial \lambda}{\partial \xi}$. After differentiating Φ^V_{CU} with respect to ξ and doing some algebra, I obtain $\frac{\partial \Phi^V_{CU}}{\partial \xi} = \big[V_0 e^{(\mu+\alpha)T} N[h_1] - \chi_0 N[h_2]\big] - \big[V_0 e^{(\mu+\alpha)T} N[m_1] - \chi_0 N[mh_2]\big]$, which can be re-expressed as $\int_{\chi_0}^{\mathrm{X}_0} (V_T - \chi_0) f(V_T) dV_T > 0$ given that $\chi_0 < \mathrm{X}_0$. As for $\frac{\partial \lambda}{\partial \xi}$, differentiate lambda as defined in equation (6) with respect to ξ, noting that only Φ^V_{CU} depends on ξ. Thus I have that $\frac{\partial \Phi^V_{CU}}{\partial \xi} = V_0 e^{(\mu+\alpha)T}\big[N[h_1] - N[m_1]\big] - \chi_0\big[N[h_2] - N[m_2]\big] > 0$ and $\frac{\partial \lambda}{\partial \xi} = \frac{\frac{-\partial \Phi^V_{CU}}{\partial \xi}}{T(\mathcal{E}_T - \Phi^V_{CU})} < 0$.

Now, for the parameter values considered herein, LP returns increase in fund leverage at and below the fund leverage constraint. Altogether this implies that there exists a minimum $\dot{\alpha} > 0$ at which $\xi = \rho$ with $\lambda = \lambda^*$ and $D_0 = \overline{D}_0$. Because LP return increases in α, any $\alpha \leq \dot{\alpha}$ implies that the catch-up fee provision cannot be implemented with $\xi > \rho$ while simultaneously satisfying the target return and fund leverage constraints. This establishes the lower range of α's.

Next, I identify an $\ddot{\alpha} > \dot{\alpha}$ such that $\xi = 1.0$ with $\lambda = \lambda^*$ and $D_0 = \overline{D}_0$. This establishes an upper bound for the middle range of α's. I now claim that for $\dot{\alpha} < \alpha < \ddot{\alpha}$, the target return and fund leverage constraints bind with ξ chosen to satisfy equation (F.2). This claim requires that μ_1, $\mu_2 > 0$ for there to be a constrained optimum. After examining FOCs that follow from equation (F.1), I find $\mu_1 = T[\mathcal{E}_T - \Phi_{CU}^V] > 0$ and $\mu_2 = \frac{1}{\frac{\partial D_0}{\partial B}}\left[\frac{\partial \mathcal{E}_T}{\partial B} - \left(\frac{\mathcal{E}_T - \Phi_{CU}^V}{E_0 + \Phi^F}\right)\frac{\partial E_0}{\partial B}\right]$. These are precisely the same conditions that are required to hold in the constrained baseline contract problem characterized in Proposition 3, where the only difference is that I am optimizing the expanded contract with respect to the catch-up rate, ξ, instead of the baseline contract with respect to the carry share percentage, ρ. μ_1 is seen to be always positive, while the bracketed term of μ_2 must be verified as positive given the parameter set in question.

Lastly, $\alpha \geq \ddot{\alpha}$ defines the higher range of αs. In this case, I claim that the target return and full catch rate constraints bind. Here D_0 is positive but less than $\overline{D}_0$, with D_0 determined by (F.2). For this claim to hold, I must verify that μ_1, $\mu_3 > 0$. Solving the constrained optimization problem in (F.1) generates that $\mu_1 = \frac{\frac{-\partial \Phi_{CU}^V}{\partial B}}{\frac{\partial \lambda^N}{\partial B}}$ and $\mu_3 = \frac{\partial \Phi_{CU}^V}{\partial \xi}\left[1 - \frac{\mu_1}{T[\mathcal{E}_T - \Phi_{CU}^V]}\right]$. These KKT conditions are verified to hold for empirically supported parameter ranges applied herein. QED

For $\alpha \leq \dot{\alpha}$, there is no $\xi > \rho$ that can simultaneously satisfy the target return and fund leverage constraints for the given ψ and ρ, while also resulting in GP incentive fees that are positive in expectation. Implicit in this result is that, for empirically-based parameter values established previously, and because $\alpha > 0$, LP returns are increasing in fund leverage at the fund leverage constraint. This causes the low-skill LP to lever the fund up to the $D_0 = \overline{D}_0$ constraint in a (failed) attempt to meet the LP's return target.

In all cases for which the catch-up fee provision is implementable, constrained optimal GP incentive fees are determined as a two-part tariff. For $\dot{\alpha} \leq \alpha < \ddot{\alpha}$, the GP optimizes incentive fees by setting $D_0 = \overline{D}_0$, and then using equation (F.2) to find the catch-up rate, ξ, to meet the LP's return target. The GP optimizes over the catch-up rate rather than fund leverage, since the LP's return target is more sensitive to reductions in fund leverage than it is to increases in the catch-up rate. In this case, the Lagrange multipliers are of the exact same form as in the baseline contracting case, except now incentive fees paid to the GP incorporate the fee-increasing effects of the catch-up provision. This augmented contract, which reduces fund leveraging incentives of the GP, has the effect of reducing shadow costs at both the return target and fund leverage constraint. For $\alpha \geq \ddot{\alpha}$, the catch-up rate binds at $\xi = 1.0$. Here the GP finds $D_0(B) < \overline{D}_0$ that satisfies equation (F.2). Because fund leverage moves inversely with α, the GP reduces leverage to increase fees. This decreases the LP's returns, with the GP reducing fund leverage to the point at which $\lambda^N = \lambda^*$.