A Theory of Intra-Party Factions and Electoral Accountability: Online Appendix

Contents

A Extensions

A.1 Primary Elections

In the baseline model, the voter only participates in a general election between the incumbent and a random challenger from the opposite party. In this extension, consider a primary stage (before the general election) in which the voter also decides between the incumbent and a random challenger from the same party.

Lemma 3: For districts $y < 0$, the primary voter's reelection decision can be described as follows:

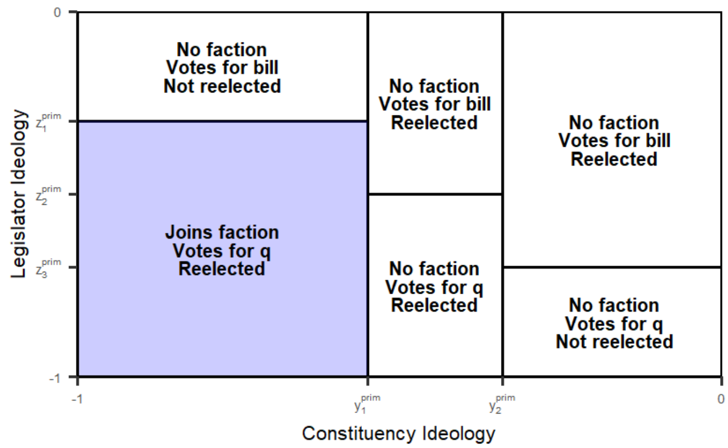
- If $\mu(z|f,v) \leq (\geq) -\frac{1}{2}$, then the voter reelects when

$$y \leq (\geq) \quad -\frac{1-3\mu(z|f,v)^2 - 3\sigma^2(z|f,v)}{3(1+2\mu(z|f,v))}$$

Incumbents who signal in a more "extreme" direction get reelected in the primary, if they represent districts that are sufficiently extreme. This result, captured in the first part of Lemma 3, is similar to the result in Lemma 1 in the sense that extreme incumbents in moderate districts get kicked out of office. The difference is that unlike in the general election, where all incumbents can expect to get reelected in sufficiently extreme constituencies, incumbents who signal as more moderate in these districts will face a tougher time in their primary. Thus, when the incumbent is perceived as less extreme in expectation, the primary voter has to be sufficiently moderate, as the condition in Lemma 3 indicates, in order to favor reelection. A fitting example here is the number of Republicans who were primaried out of their districts during the Obama Administration because they were perceived as moderate "RINOs," or Republicans In Name Only.[1] These Republicans were ultimately swapped out for more conservative replacements.

[1]The Hill, "Republicans on notice: Get ready for 2012 primary challenges," November 17, 2010

Figure A.1: Extreme Party Factions with Primaries



Note: Figure A.1 presents the equilibrium with primary elections, for $q = -\frac{1}{2}$ and $b = 0$. The region shaded in blue represents faction members. This equilibrium differs from the one presented in Proposition 1 in that those in ideologically extreme districts who signal as moderate lose reelection.

Proposition 4: When $q < b$, a larger interval of legislators join a faction and vote in favor of the status quo. When $b < q$, the same interval of legislators do so.

The fundamental change this makes to the equilibria in Propositions 1 and 2 is that in very liberal districts, since those who are perceived as moderate are thrown out of office, a larger interval of incumbents are incentivized to send the more "extreme" signal. In the case of a conservative shift in the status quo, this means that a larger continuum of incumbents choose to join a faction and vote against the status quo in liberal districts (and those that do not, lose their primary). This is described in Proposition 4 and shown visually in Figure A.1. In the case of a liberal shift in the status quo, we attain the same result, substantively speaking, as before, in the sense that the same-sized interval of legislators join a faction.

A.2 Insincere Votes and Non-Connected Coalitions

An increasingly common phenomenon in the U.S. Congress is the presence of non-connected voting coalitions where legislators at the ideological extremes vote against the majority party and legislators in the middle are in favor. The model can help explain this phenomenon. For the purpose of this extension, I retain the variant explored in the previous section that includes primary elections in addition to a general election.

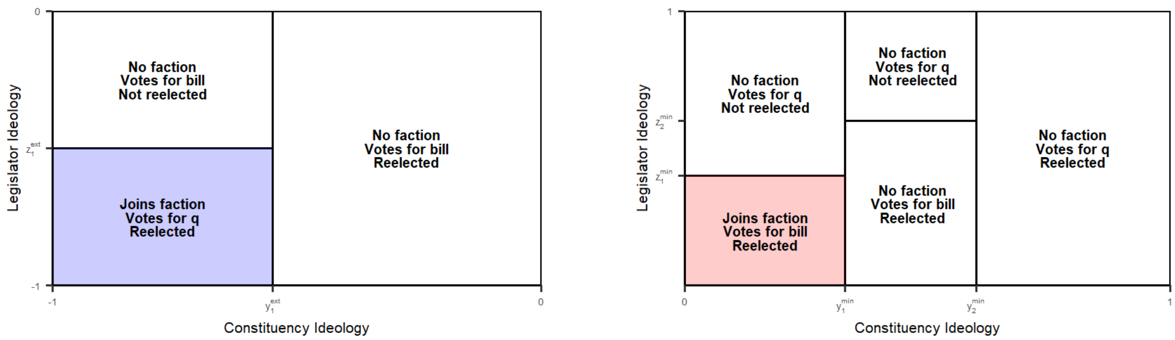
Proposition 5: If $q > b$, $\frac{b+q}{2} \geq 0$, $x_f \leq b - q - \frac{1}{2}$, $(q-b)^2 + (x_f)^2 + 2(q-b)b \geq w \geq (1+x_f)^2 + 2(q-b)(1+b) + (q-b)^2$, incumbents $[-1, z_1^{ext}]$ in districts $[-1, y_1^{ext}]$ vote for the status quo, and all other incumbents in the majority party vote for the bill.

This equilibrium is described in Proposition 5 and captured in Figure A.2. In the majority party, shown in panel (a), almost all legislators vote for the agenda because they are acting sincerely and voting with the party is on the equilibrium path (and leads to winning reelection). The exceptions are progressive incumbents in ideologically liberal constituencies, who defect from the agenda. The equilibrium behavior of these progressive Democrats is justified because reelection benefits are sufficiently high that it overwhelms the policy loss they incur by voting for the status quo over the party agenda (despite the party agenda being closer to their ideal points). In this case, voting sincerely would lead to losing their primary election; thus, incumbents take advantage of the opportunity to signal their preferences to their constituencies and vote against their party.

In the minority party, moderate Republicans join a faction to vote with the majority party. In this case, the equilibrium logic is the same as in Proposition 2, except the bill and status quo are flipped (since members of the minority party incur the cost of party discipline c when voting for the bill). Joining a faction in this case mitigates the cost of party discipline and garners reelection for incumbents. Those who are sufficiently conservative and prefer to vote for the status quo lose reelection.

Figure A.2: Non-Connected Voting Coalitions

(a) Majority Party ($q > b$) (b) Minority Party ($q > b$)



Note: Figure A.2 presents an equilibrium that features non-connected roll-call voting coalitions, for $q = \frac{1}{4}$ and $b = 0$. The region shaded in blue represents majority party faction members, and the region shaded in red reflects minority party faction members. Note that those at the "ideological ends" are voting for the status quo, whereas those in the middle vote for the bill in equilibrium.

One example of this equilibrium is the vote to pass President Biden's proposed infrastructure bill. On November 5, 2021, the House passed the bill on a 228-206 vote, with six Democrats voting against the bill (and 13 Republicans voting in support). The six Democrats included progressive lawmakers, like Rep. Alexandria Ocasio-Cortez (D-NY) and Rep. Ilhan Omar (D-MN), who voted "no" on the legislation because they wanted the larger Build Back Better plan to be paired with the infrastructure bill. These Democrats are apt examples of the equilibrium in panel (a) of Figure A.2. In panel (b), many of the Republicans who voted with the majority party were moderates who represented electorally competitive or moderate constituencies.

A.3 Endogenous Faction Platform

In this section, we relax the assumption that the faction platform is exogenous. In particular, we consider an extension where the platform is constructed as the average of the ideal points of the faction members. One can note here that legislators' ideal points are distributed according to a continuous distribution; as such, an individual legislator's decision to join a faction does not, in and of itself, affect the faction platform, since each legislator ideal point has mass zero in the distribution of legislator ideology. In this section, I derive the equilibrium platforms for Propositions 1 and 2.

Proposition 6: The equilibrium extremist factions platform is $x_f^{ext} = -1 - 2(b-q) + \sqrt{2(b-q)(2(b-q)+1+\frac{b+q}{2})}$. The equilibrium moderate factions platform is $x_f^{mod} = 2(q-b) - \sqrt{4(q-b)^2 - q^2 + b^2 + w}$.

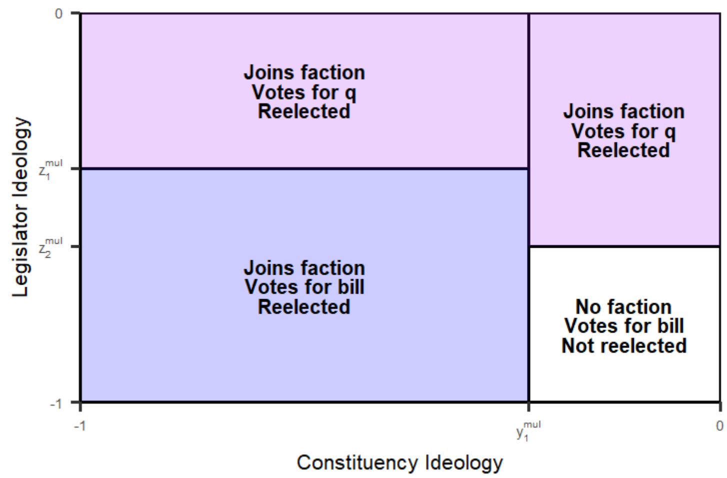
For the extremist factions platform, as the status quo q moves closer to the bill b, there are two counteracting effects. First, the faction platform becomes more conservative, since more legislators sincerely prefer the status quo over the bill. However, the distance between b and q is smaller, which means that the legislators do not place as much weight on policy. Whereas the extremist factions platform is solely a function of the bill b and the status quo q, the moderate faction platform is also a function of reelection benefits w. As reelection benefits increase, the faction platform becomes more liberal. When it comes to b and q, the same tradeoff holds for the moderate factions case as for the extremist factions case.

A.4 Multiple Factions

In this section, I relax the assumption in the model that there is only one faction in each party. Here, I explore equilibrium results with two factions in the majority party, for the case of $b < q$. The first faction's platform is denoted by x_{1f} and the second faction's platform is denoted by x_{2f}, where $x_{1f} < x_{2f}$ (i.e., faction 1 represents the more extreme faction). The two factions are shown visually in Figure A.3, with the blue faction representing faction 1 and the purple faction representing faction 2.

The first equilibrium suggests that all legislators in ideologically extreme districts join a faction, but some vote for the status quo and others vote for the party's agenda. In this

Figure A.3: Two Factions ($b < q$)



Note: Figure A.3 presents an equilibrium that features two factions, for $q = 0$ and $b = -0.5$. The region shaded in blue represents the extremist faction, and the region shaded in purple reflects moderate faction members.

case, legislators join the first faction if and only if

$$-(z-b)^2 - (z - x_{1f})^2 \geq -(z-q)^2 - (z - x_{2f})^2 \Rightarrow$$

$$\Rightarrow z \leq z_1^{mul} = \frac{b^2 - q^2 + x_{1f}^2 - x_{2f}^2}{2(b - q + x_{1f} - x_{2f})}$$

This equilibrium is justified if no legislators have an incentive to not join a faction and vote the same way. This requires $w \geq max\{(1 + x_{1f})^2, (z_1^{mul} - x_{1f})^2, (x_{2f} - z_1^{mul})^2, (x_{2f})^2\}$. We also need legislators not to deviate in the roll-call voting stage and vote for the opposite policy than the one supported by the legislator's faction. This is satisfied given the condition on w. The second equilibrium takes place in electorally competitive districts and is similar to the one illustrated in Proposition 2.

An example of this two-faction case might be the Congressional Progressive Caucus and the New Democrat Coalition. The Congressional Progressive Caucus mostly comprises liberal incumbents in ideologically extreme districts. The New Democrat Coalition comprises moderate Democrats (relative to the rest of the party), but the New Democrat Coalition contains a broader cross-section of Democrats.

A.5 Voter Welfare

In this section, we consider the voter's welfare function:

$$-E_y[E_z[(y-z)^2]]$$

Note that this objective function mimics the voter's utility function from earlier, except that we now take the expectation across districts. Accordingly, faction equilibria may be welfare-improving by more carefully delineating the ideological type of the incumbent to the voters, thereby allowing the voters to weed out more of the dissonant type in equilibrium. In the literature, this is sometimes referred to as the "selection" effect.

When comparing the equilibrium in Lemma 2 with the one in Proposition 1, extremist factions are weakly optimal for voters if they are sufficiently small (i.e., when $z_1^{ext} \leq z_1^{NF,ext}$). Otherwise, when the interval of legislators that join a faction increases, the interval of districts willing to reelect these incumbents is larger. As such, the corresponding interval of districts in which some incumbents lose reelection (i.e., the moderate or electorally competitive districts) becomes smaller. Therefore, fewer extreme incumbents are thrown out of office in equilibrium, and in expectation, voters are stuck with more "dissonant" incumbents than in the "no factions" equilibrium. This is the first part of Proposition 7.

Proposition 7 (Voter Welfare):

- The extremist factions equilibrium accrues greater voter welfare than the no factions equilibrium if and only if $z_1^{ext} \leq z_1^{NF,ext}$.

- The moderate factions equilibrium accrues greater voter welfare if and only if $z_3^{mod} \geq z_{mod}^{NF}$.

In terms of the moderate factions equilibrium, if a larger interval of incumbents in moderate districts vote with their party (and get reelected), this diminishes voter welfare in the sense that fewer extreme incumbents get thrown out of office. Since some of the dissonant (or ideologically extreme) types are pooling with the moderate types (and creating a noisier signal), voters are left with more of the ideologically extreme incumbents after the election has taken place. Thus, moderate factions are only a welfare improvement if they have the effect of reducing the interval of incumbents who vote with the party and winning reelection. This is the second part of Proposition 7.

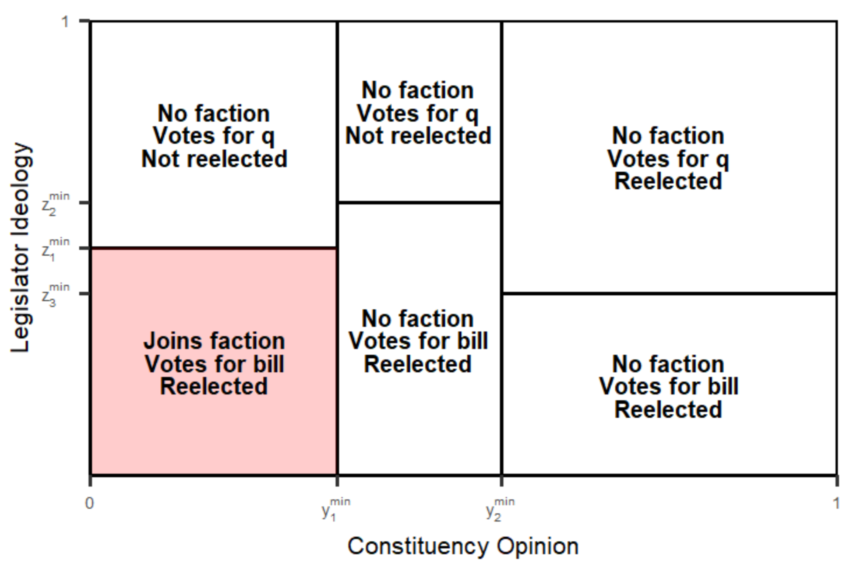
B Minority Party Factions

The agenda put forward by majority party leaders also affects the decisions of minority party members to join factions as well. In particular, the result in Proposition 8 suggests that when there is an ideological split within the minority party, there exists a faction that comprises moderate members who coalesce with the majority party to vote in favor of the proposed agenda. One should note that in the set-up of this model, minority and majority party members are assumed not to have overlapping ideal points (i.e., the most conservative Democrat is to the left of the most liberal Republican). As a result, a coalition of minority and majority party members does not exist because some minority party members share the same ideal points as some in the majority, but rather they have the same preference ranking over the bill and status quo or members of the minority may have an incentive to appear moderate to their constituencies by voting with the majority.

Proposition 8: If $q \geq b$, members $z \in [0, z_1^{min}]$ in districts $y \in [0, y_1^{min}]$ join a faction and vote for the majority party's bill, and members $z \in (z_1^{min}, 1]$ vote for the status quo.

When the bill proposed by the majority party is more liberal than the status quo, in the most moderate districts ($y \in [0, y_1^{min}]$), liberal incumbents will join a faction and vote for the party's bill, and incumbents that are sufficiently conservative will not join a faction and vote for the status quo. This is shown visually in Figure B.4. One example that fits this setting (if one flips the axes again to look at the Republican side) is during the Bush Administration, when the House was controlled by Republicans. The Blue Dog Democrats voted with Republicans on several initiatives, notably the Bush tax cuts, various national security measures like the National Commissions Act, and the Partial-Birth Abortion Act. On each of these votes, moderate Democrats representing more conservative districts sided with the majority party.

Figure B.4: Minority Party Factions



Note: Figure B.4 presents an equilibrium that features a minority party faction, for $q = 1$ and $b = 0.25$. The region shaded in red represents the minority party faction. Note that in this case, faction members vote in favor of the majority party's agenda in equilibrium.

C Additional Analysis on Roll-Call Votes

In the case studies presented in the main text, I focused on two key roll-call votes. To corroborate the analysis I provided and to link these votes to the equilibria described in the paper, I show breakdowns of the roll-call votes by legislator ideology, constituency ideology, and faction membership.

On the DHS funding measure in the first case study, one can look at Table C.1, which breaks down the vote on the DHS funding bill by membership in the House Freedom Caucus (as measured by Clarke 2020), constituency ideology (as measured by Tausanovitch and Warshaw 2013), and legislator ideology (as measured by the first-dimension DW-NOMINATE score). The first two rows show that members of the House Freedom Caucus were much more likely to vote against the DHS funding measure than other Republicans (in fact, none of them voted in favor). Moreover, the third and fourth row show that incumbents who represented very conservative districts were also more likely to vote against the bill, and the same is true for very conservative incumbents (as seen in the fifth and sixth row). Finally, virtually all conservative legislators (about 98.3 percent) who represented conservative districts voted against the bill.

Table C.1: DHS Vote by HFC Membership, Constituency Ideology, and Legislator Ideology

Group	Proportion of Ayes	Proportion of Nays
HFC	0	1
Non-HFC	0.368	0.632
Cons. Districts (>75th percentile)	0.088	0.912
Non-cons. Districts (<=75th percentile)	0.378	0.622
Cons. Legislators (>75th percentile)	0.017	0.983
Non-cons. Legislators (<=75th percentile)	0.407	0.593
Cons. Legislators in Cons. Districts	0	1
~ (Cons. Legislators in Cons. Districts)	0.338	0.662
All Members of Congress	0.31	0.69

Similarly, on the Waxman-Markey vote, which is broken down in Table C.2, members of the Blue Dog Democrats were much more likely to vote against the legislation than non-members (54 percent versus 8.3 percent). Those in the most conservative districts were also more likely to vote against the bill, and the same is true for the most conservative legislators in the Democratic Party. Finally, this proportion is even larger (about 58.1 percent) when one considers conservative Democratic legislators who represent conservative districts.

Table C.2: Waxman-Markey Vote by Blue Dog Membership, Constituency Ideology, and Legislator Ideology

Group	Proportion of Ayes	Proportion of Nays
Blue Dog	0.46	0.54
Non-BD	0.917	0.083
Cons. Districts (>75th percentile)	0.492	0.508
Non-cons. Districts	0.942	0.058
Cons. Legislators (>75th percentile)	0.531	0.469
Non-cons. Legislators	0.927	0.073
Cons. Legislators in Cons. Districts	0.419	0.581
~ (Cons. Legislators in Cons. Districts)	0.91	0.09
All Members of Congress	0.827	0.173

D Proofs

Proof of Lemma 1: The first part of Lemma 1 is proved in the body of the paper. The second part can be shown by solving the following inequality:

$$-(y-\frac{1}{2})^2-\frac{1}{12}\leq -(y-\mu(z|f,v))^2-\sigma^2(z|f,v)\Rightarrow$$

$$\Rightarrow y\leq\left(\frac{1-3(\mu(z|f,v))^2-3\sigma^2(z|f,v)}{3(1-2\mu(z|f,v))}\right)$$

For the case in which $\sigma^2(z|f,v)>\frac{1}{12}$, the same inequality above holds where $\mu(z|f,v)\leq-\frac{1}{2}$. If the variance is larger and the mean is closer to zero (than -1), then there are some cases in which the reduction in bias always outweighs the increase in the variance, particularly when $\sigma^2(z|f,v)\leq\frac{1}{3}-(\mu(z|f,v))^2$. Otherwise, one can again use the inequality above to identify the districts that would prefer to reelect the incumbent.

Proof of Lemma 2: The terms invoked in Lemma 2 are defined as follows: $y^{NF}_{mod} = \frac{1}{3} z^{NF}_{mod} \frac{z^{NF}_{mod}-1}{z^{NF}_{mod}-2}$, $y^{NF}_{ext} = \frac{1}{3} z^{NF,ext}_{1} \frac{z^{NF,ext}_{2}-1}{z^{NF,ext}_{2}-2}$, $z^{NF}_{mod} = \frac{b+q}{2} - \frac{w-c}{2(q-b)}$, $z^{NF,ext}_{2} = \frac{b+q}{2} - \frac{w}{2(b-q)}$, $z^{NF,ext}_{1} = \frac{b+q}{2} - \frac{c}{2(b-q)}$. The proofs for these equilibria are analogous to those described in the proofs for Propositions 1 and 2.

Proof of Lemma 3: Lemma 3 can be shown from the inequality

$$-(y+\frac{1}{2})^2 - \frac{1}{12} \leq -(y-\mu(z|f,v))^2 - \frac{(2(\mu(z|f,v)+1))^2}{12} \Rightarrow$$

$$\Rightarrow y \leq \frac{1-3\mu(z|f,v)^2-3\sigma^2(z|f,v)}{3(1+2\mu(z|f,v))}$$

if $\mu(z|f,v) \leq -\frac{1}{2}$ and

$$\Rightarrow y \geq \frac{1-3\mu(z|f,v)^2-3\sigma^2(z|f,v)}{3(1+2\mu(z|f,v))}$$

otherwise

Proof of Proposition 1: The full equilibrium is described below:

1. Incumbents $-1 \leq z \leq z^{ext}_1$ in districts $[-1, y^{ext}_1]$ join a faction and vote in favor of the status quo, and incumbents $z > z^{ext}_1$ do not.
2. If $c \leq \hat{c} = 2(b-q)(\frac{3b-q}{2} - x_f - \sqrt{2(b-q)(b-x_f)})$:
   - Incumbents $z \leq z^{ext}_2$ in districts $[y^{ext}_1, y^{ext}_2]$ vote in favor of the status quo, and incumbents $z > z^{ext}_2$ vote in favor of the party's bill.
   - Incumbents $z \leq z^{ext}_3$ in districts $[y^{ext}_2, 0]$ vote in favor of the status quo and do not get reelected. Incumbents $z > z^{ext}_3$ vote in favor of the party's bill and do get reelected.
3. If $c > \hat{c} = 2(b-q)(\frac{3b-q}{2} - x_f - \sqrt{2(b-q)(b-x_f)})$:
   - Incumbents $z \leq z^{ext}_3$ in districts $y \in [y^{ext}_1, 0]$ vote in favor of the status quo and do not get reelected. Incumbents $z > z^{ext}_3$ vote in favor of the party's bill and do get reelected.

First, we ascribe values to the thresholds in this equilibrium (which are also noted in Proposition 1). We have $z^{ext}_1 = x_f + q - b + \sqrt{2(b-q)(b-x_f)}$, $z^{ext}_2 = \frac{b+q}{2} - \frac{c}{2(b-q)}$, $z^{ext}_3 = \frac{b+q}{2} - \frac{w}{2(b-q)}$, $y^{ext}_1 = \frac{1}{3} z^{ext}_1 (\frac{1-z^{ext}_1}{2-z^{ext}_1})$, and $y^{ext}_2 = \frac{1}{3} z^{ext}_2 (\frac{1-z^{ext}_2}{2-z^{ext}_2})$. The upper threshold for x_f is $\underline{x}^1_f = -1 + \sqrt{(1+b)^2 - (1+q)^2}$ and if $b+q \geq 0$, it is equal to $\underline{x}_f^1 = min\{-\sqrt{b^2-q^2}, -1+\sqrt{(1+b)^2-(1+q)^2}\}$. The lower bound on w is $\tilde{w} = max\{(1+x_f)^2, (q-b+\sqrt{2(b-q)(b-x_f)})^2\}$.

The voters' posterior mean in $y \in [-1, y_1^{ext}]$ is:

$$\mu(z) = \begin{cases} \frac{1}{2}(-1+z_1^{ext}) & f=1 \text{ and } v=q \\ \frac{1}{2}(z_1^{ext}) & f=0 \text{ and } v=b \\ \frac{1}{2} & f=1 \text{ and } v=b \text{ or } f=0 \text{ and } v=q \end{cases}$$

The voters' posterior variance is:

$$\sigma^2(z) = \begin{cases} \frac{1}{12}(-1-z_1^{ext})^2 & f=1 \text{ and } v=q \\ \frac{(z_1^{ext})^2}{12} & f=0 \text{ and } v=b \\ \frac{1}{12} & f=1 \text{ and } v=b \text{ or } f=0 \text{ and } v=q \end{cases}$$

Those who join a faction prefer to do so when

$$U(z, f=1, v=q) = -(z-x_f)^2 - (z-q)^2 \geq -(z-b)^2 = U(z, f=0, v=b)$$

which is equivalent to $z \leq x_f + q - b + \sqrt{2(b-q)(b-x_f)}$.

If $c \leq 2(b-q)(\frac{3b-q}{2} - x_f - \sqrt{2(b-q)(b-x_f)})$, the voter's posterior mean in $y \in [y_1^{ext}, y_2^{ext}]$ is

$$\mu(z) = \begin{cases} \frac{1}{2}(-1+z_2^{ext}) & f=0 \text{ and } v=q \\ \frac{1}{2}z_2^{ext} & f=0 \text{ and } v=b \\ \frac{1}{2} & f=1 \end{cases}$$

The voters' posterior variance is:

$$\sigma^2(z) = \begin{cases} \frac{1}{12}(-1-z_2^{ext})^2 & f=0 \text{ and } v=q \\ \frac{1}{12}(z_2^{ext})^2 & f=0 \text{ and } v=b \\ \frac{1}{12} & f=1 \end{cases}$$

A legislator prefers to vote in favor of the status quo when

$$U(z, f = 0, v = q) = -(z - q)^2 - c \geq -(z - b)^2 = U(z, f = 0, v = b)$$

which is true for $z \leq \frac{b+q}{2} - \frac{c}{2(b-q)}$. Finally, we consider districts $y \in [y_2^{mod}, 0]$. The voter's posterior mean is

$$\mu(z) = \begin{cases} \frac{1}{2}(-1 + z_3^{mod}) & f = 0 \text{ and } v = q \\ \frac{1}{2} z_3^{mod} & f = 0 \text{ and } v = b \\ \frac{1}{2} & f = 1 \end{cases}$$

The voters' posterior variance is:

$$\sigma^2(z) = \begin{cases} \frac{1}{12}(-1 - z_3^{mod})^2 & f = 0 \text{ and } v = q \\ \frac{1}{12}(z_3^{mod})^2 & f = 0 \text{ and } v = b \\ \frac{1}{12} & f = 1 \end{cases}$$

The legislators prefer to vote for the status quo when

$$U(z, f = 0, v = q) = -(z - q)^2 \geq -(z - b)^2 + w = U(z, f = 0, v = b)$$

which is true for $z \leq \frac{b+q}{2} - \frac{w}{2(b-q)}$.

Next, we consider what happens when $c > 2(b-q)(\frac{3b-q}{2} - x_f - \sqrt{2(b-q)(b-x_f)})$. The voter's posterior mean is

$$\mu(z) = \begin{cases} \frac{1}{2}(-1 + z_3^{mod}) & f = 0 \text{ and } v = q \\ \frac{1}{2} z_3^{mod} & f = 0 \text{ and } v = b \\ \frac{1}{2} & f = 1 \end{cases}$$

The voters' posterior variance is:

$$\sigma^2(z) = \begin{cases} \frac{1}{12}(-1 - z_3^{mod})^2 & f = 0 \text{ and } v = q \\ \frac{1}{12}(z_3^{mod})^2 & f = 0 \text{ and } v = b \\ \frac{1}{12} & f = 1 \end{cases}$$

The legislators prefer to vote for the status quo when

$$U(z, f = 0, v = q) = -(z - q)^2 \geq -(z - b)^2 + w = U(z, f = 0, v = b)$$

which is true for $z \leq \frac{b+q}{2} - \frac{w}{2(b-q)}$.

Proof of Proposition 2: The full equilibrium can be described as follows:

- In districts $y \in [y_2^{mod}, 0]$, incumbents $z \in [z_3^{mod}, 0]$ join a faction and vote for the status quo. Incumbents $z < z_3^{mod}$ vote for the party's bill and lose reelection.
- If $c < \frac{q-3b}{q-b} + 2(q-b)x_f + w - 2(q-b)\sqrt{2(q-b)(x_f - b) + w}$:
  - In districts $y \in [y_1^{mod}, y_2^{mod})$, incumbents $z \geq z_2^{mod}$ vote in favor of the status quo, whereas incumbents $z < z_2^{mod}$ vote in favor of the party's bill and lose reelection
  - In districts $y \in [-1, y_1^{mod}]$, all incumbents vote in favor of the bill and get reelected.
- If $c \geq \frac{q-3b}{q-b} + 2(q-b)x_f + w - 2(q-b)\sqrt{2(q-b)(x_f - b) + w}$:
  - In districts $y \in [-1, y_2^{mod}]$, all incumbents vote in favor of the bill and get reelected.

We have $z_2^{mod} = \frac{b+q}{2} - \frac{w-c}{2(q-b)}$, $z_3^{mod} = q - b + x_f - \sqrt{2(q-b)(x_f - b) + w}$, $y_2^{mod} = \frac{1}{3} z_3^{mod} (\frac{1-z_3^{mod}}{2-z_3^{mod}})$, and $y_1^{mod} = \frac{1}{3} z_2^{mod} (\frac{1-z_2^{mod}}{2-z_2^{mod}})$. We need $x_f \geq b - \frac{w}{2(q-b)}$ and $w \geq \hat{w} = max\{q^2 - b^2 + x_f^2, x_f^2, (x_f - \frac{b+q}{2})^2\}$.

If $c < \frac{q-3b}{q-b} + 2(q-b)x_f + w - 2(q-b)\sqrt{2(q-b)(x_f - b) + w}$, the voters' posterior mean in $y \in [y_1^{ext}, y_2^{ext}]$ is:

$$\mu(z) = \begin{cases} \frac{1}{2}(-1 + z_2^{ext}) & f = 0 \text{ and } v = b \\ \frac{z_2^{ext}}{2} & f = 0 \text{ and } v = q \\ \frac{1}{2} & f = 1 \end{cases}$$

The voters' posterior variance is:

$$\sigma^2(z) = \begin{cases} \frac{1}{12}(-1 - z_2^{ext})^2 & f = 0 \text{ and } v = b \\ \frac{(z_2^{ext})^2}{12} & f = 0 \text{ and } v = q \\ \frac{1}{12} & f = 1 \end{cases}$$

Incumbents in these districts vote in favor of the bill when

$$U_{z,f=0,v=b} = -(z-b)^2 \geq -(z-q)^2 + w - c = U_{z,f=0,v=q}$$

which is equivalent to $z \leq \frac{b+q}{2} - \frac{w-c}{2(q-b)}$.

If $c \geq \frac{q-3b}{q-b} + 2(q-b)x_f + w - 2(q-b)\sqrt{2(q-b)(x_f-b)+w}$, the equilibrium in $y \in [y_1^{ext}, y_2^{ext}]$ is the same as in $[-1, y_1^{ext}]$. In districts $y \in [y_1^{ext}, 0]$, the voters' posterior mean is

$$\mu(z) = \begin{cases} \frac{1}{2}(-1+z_3^{ext}) & f=0 \text{ and } v=b \\ \frac{z_3^{ext}}{2} & f=1 \text{ and } v=q \\ \frac{1}{2} & f=0 \text{ and } v=q \text{ or } f=1 \text{ and } v=b \end{cases}$$

The voters' posterior variance is:

$$\sigma^2(z) = \begin{cases} \frac{1}{12}(-1-z_3^{ext})^2 & f=0 \text{ and } v=b \\ \frac{(z_3^{ext})^2}{12} & f=1 \text{ and } v=q \\ \frac{1}{12} & f=0 \text{ and } v=q \text{ or } f=1 \text{ and } v=b \end{cases}$$

Incumbents in these districts join a faction and vote for the status quo (instead of voting for the bill) when

$$U_{z,f=0,v=b} = -(z-b)^2 \leq -(z-q)^2 - (z-x_f)^2 + w = U_{z,f=1,v=q}$$

which suggests $z \geq q - b + x_f - \sqrt{2(q-b)(x_f-b)+w} = z_3^{mod}$.

Proof of Proposition 3: If $z_1^{ext} \geq z_1^{NF,ext}$, then the extremist factions equilibrium accrues greater party welfare if and only if

$$\alpha(z_2^{NF,ext}+1)(y_2^{ext}-y_{ext}^{NF}) - (z_1^{ext}+1)(y_2^{ext}+1) + (y_{ext}^{NF}+1)(z_1^{NF,ext}+1) + (y_2^{ext}-y_{ext}^{NF})(z_2^{NF,ext}+1) \geq 0 \Rightarrow$$

$$\Rightarrow (1+\alpha)(z_2^{NF,ext}+1)(y_2^{ext}-y_{ext}^{NF}) \geq (z_1^{ext}+1)(y_2^{ext}+1) - (y_{ext}^{NF}+1)(z_1^{NF,ext}+1) \Rightarrow$$

$$\Rightarrow \alpha \geq -1 + \frac{(z_1^{ext}+1)(y_2^{ext}+1) - (y_{ext}^{NF}+1)(z_1^{NF,ext}+1)}{(z_2^{NF,ext}+1)(y_2^{ext} - y_{ext}^{NF})} = \hat{\alpha}$$

Note that $z_1^{ext} \geq z_1^{NF,ext}$ is equivalent to $c \geq 2(b-q)(\frac{3b-q}{2} - x_f - \sqrt{2(b-q)(b-x_f)}) = \hat{c}$ or $\hat{x}_f^{ext} = \frac{b+q}{2} - \frac{c}{2(b-q)} - \sqrt{c} \leq x_f \leq \frac{b+q}{2} - \frac{c}{2(b-q)} + \sqrt{c} = \tilde{x}_f^{ext}$.

For the moderate factions equilibrium, if $z_3^{mod} \geq z_{NF}^{mod}$, then factions yield greater party welfare when $(z_3^{mod} - z_{NF}^{mod})(-y_2^{mod})(1-\alpha) \geq 0$ or $\alpha \leq 1$. If $z_3^{mod} \leq z_{NF}^{mod}$, then factions yield greater party welfare when

$$-((z_{mod}^{NF} - z_3^{mod})(-y_{mod}^{NF}) + (y_{mod}^{NF} - y_2^{mod})(-z_3^{mod})) + \alpha((z_{mod}^{NF} - z_3^{mod})(-y_{mod}^{NF}) - (y_{mod}^{NF} - y_2^{mod})(z_3^{mod}+1)) \geq 0$$

which simplifies to

$$\alpha \geq \frac{(z_{mod}^{NF} - z_3^{mod})(-y_{mod}^{NF}) + (y_{mod}^{NF} - y_2^{mod})(-z_3^{mod})}{(z_{mod}^{NF} - z_3^{mod})(-y_{mod}^{NF}) - (y_{mod}^{NF} - y_2^{mod})(z_3^{mod}+1)} = \tilde{\alpha}$$

Note that $z_3^{mod} \geq z_{NF}^{mod}$ is equivalent to $c \leq 2(q-b)(\frac{q-3b}{2} + x_f + \frac{w}{2(q-b)} - \sqrt{2(q-b)(x_f-b)+w}) = \tilde{c}$ or $\hat{x}_f^{mod} = -(\frac{w-c}{2(q-b)} - (q-b)) - \sqrt{-\frac{w-c}{2(q-b)}(b+q) + w + \frac{1}{4}(3q-b)(q-3b)} \leq x_f \leq -(\frac{w-c}{2(q-b)} - (q-b)) + \sqrt{-(\frac{w-c}{2(q-b)}(b+q) + w + \frac{1}{4}(3q-b)(q-3b)} = \tilde{x}_f^{mod}$.

Proof of Proposition 4: The threshold for the extreme party factions in the absence of primaries is $q - b + x_f + \sqrt{2(b-q)(b-x_f)}$, and the corresponding threshold with primaries is $q - b + x_f + \sqrt{2(b-q)(b-x_f)+w}$. The threshold for moderate party factions is the same, with and without primaries.

Proof of Proposition 5: Since the equilibrium on the Republican side is analogous to the results in the Minority Party Factions section, I focus here on the proof for the Democratic faction. For districts $[-1, y_1^{ext}]$, the voters' posterior mean is

$$\mu(z) = \begin{cases} \frac{-1+z_1^{ext}}{2} & f=1 \text{ and } v=q \\ \frac{z_1^{ext}}{2} & f=0 \text{ and } v=b \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

and the voters' posterior variance is

$$\sigma^2(z) = \begin{cases} \frac{1}{12}(1+z_1^{ext})^2 & f=1 \text{ and } v=q \\ \frac{(z_1^{ext})^2}{12} & f=0 \text{ and } v=b \\ \frac{1}{12} & \text{otherwise} \end{cases}$$

Incumbents join a faction when

$$w - (z - x_f)^2 - (z-q)^2 \geq -(z-b)^2 \Rightarrow$$

or $z \leq x_f + q - b + \sqrt{2(b-q)(b-x_f)+w}$.

Proof of Proposition 6: To derive the equilibrium platform for the extremist factions equilibrium, one has to rearrange the following equation:

$$\frac{1}{2}(x_f^{ext} - 1 + q - b + \sqrt{2(b-q)(b-x_f^{ext})}) = x_f^{ext} \Rightarrow$$

$$\Rightarrow x_f^{ext} = -1 - 2(b-q) + \sqrt{2(b-q)(2(b-q)+1+\frac{b+q}{2})}$$

For the moderate factions equilibrium, the factions platform satisfies the following equation:

$$\frac{1}{2}(q - b + x_f^{mod} - \sqrt{2(q-b)(x_f^{mod}-b)+w}) = x_f^{mod} \Rightarrow$$

$$\Rightarrow x_f^{mod} = 2(q-b) - \sqrt{4(q-b)^2 + b^2 - q^2 + w}$$

Proof of Proposition 7: The first part of Proposition 7 was proved earlier. We now prove the second part of the proposition. The voter welfare for a particular equilibrium can be described as

$$E_y[-(y-\mu(\cdot))^2 - \sigma^2(\cdot)]$$

The welfare associated with the "no factions" equilibrium is larger than that for Proposition 2 if

$$-\int_0^{y^{NF}_{mod}}((y-\frac{z^{NF}_{mod}}{2})^2+\frac{(z^{NF}_{mod})^2}{12})dy-(-\int_0^{y_1^{mod}}((y-\frac{z_3^{mod}}{2})^2+\frac{(z_3^{mod})^2}{12})dy)\geq 0$$

One can show that the welfare gains from the NF equilibrium are larger if $z^{NF}_{mod}\geq z_3^{mod}$.

Proof of Proposition 8: The full equilibrium can be described as follows:

- For districts $y\in[0,y_1^{min}]$, members $z>z_1^{min}$ do not join a faction and vote for the status quo. Members $z\in[0,z_1^{min}]$ do join a faction and vote the majority party's bill.
- For districts $y\in[y_1^{min},y_2^{min})$, when $c\leq 2(q-b)(\frac{3q-b}{2}-x_f-\sqrt{2(q-x_f)(q-b)+w})$, members $z\geq z_2^{min}$ vote for the status quo, and members $z<z_2^{min}$ vote for the majority party's bill. Those in the former category do not get reelected, and those in the latter category do.
- Finally, in districts $y\in[y_2^{min},1]$, when $c\leq q^2-b^2$, members $z\geq z_3^{min}$ vote for the status quo and members $z<z_3^{min}$ vote for the majority party's bill. All legislators in these districts get reelected. If $c\geq q^2-b^2$, then everyone votes for the status quo.
  - If $c>2(q-b)(\frac{3q-b}{2}-x_f-\sqrt{2(q-x_f)(q-b)+w})$, for districts $[y_1^{min},y_2^{min}]$, the voters' posterior mean and posterior variance is the same as for districts $[y_2^{min},1]$.

In this proof, $z_1^{min}=-(q-b)+x_f+\sqrt{2(q-x_f)(q-b)+w}$, $z_2^{min}=\frac{b+q}{2}+\frac{w-c}{2(q-b)}$, $z_3^{min}=\frac{b+q}{2}-\frac{c}{2(q-b)}$, $y_1^{min}=\frac{1}{3}z_1^{min}\frac{1+z_1^{min}}{2+z_1^{min}}$, and $y_2^{min}=\frac{1}{3}z_3^{min}\frac{1+z_3^{min}}{2+z_3^{min}}$. This equilibrium requires $w\geq max\{x_f^2,(x_f-\frac{b+q}{2})^2,b^2-q^2+x_f^2\}$ and $x_f\leq q+\frac{w}{2(q-b)}$.

For districts $[y_2^{min},1]$ and $c\leq q^2-b^2$, the voters' posterior mean is

$$\mu(z)=\begin{cases}\frac{1+z_3^{min}}{2} & f=0 \text{ and } v=q\\ \frac{z_3^{min}}{2} & f=0 \text{ and } v=b\\ -\frac{1}{2} & \text{otherwise}\end{cases}$$

and the voters' posterior variance is

$$\sigma^2(z)=\begin{cases}\frac{1}{12}(1-z_3^{min})^2 & f=0 \text{ and } v=q\\ \frac{(z_3^{min})^2}{12} & f=0 \text{ and } v=b\\ \frac{1}{12} & \text{otherwise}\end{cases}$$

The incumbents prefer to vote for the status quo when

$$U_{z,f=0,v=q} = -(z-q)^2 \geq -(z-b)^2 - c = U_{z,f=0,v=b}$$

which is equivalent to $z \geq \frac{b+q}{2} - \frac{c}{2(q-b)}$.

If $c \leq 2(q-b)(\frac{3q-b}{2} - x_f - \sqrt{2(q-x_f)(q-b)+w})$, for districts $[y_1^{min}, y_2^{min}]$, the voters' posterior mean is

$$\mu(z) = \begin{cases} \frac{1+z_2^{min}}{2} & f=0 \text{ and } v=q \\ \frac{z_2^{min}}{2} & f=0 \text{ and } v=b \\ -\frac{1}{2} & \text{otherwise} \end{cases}$$

and the voters' posterior variance is

$$\sigma^2(z) = \begin{cases} \frac{1}{12}(1-z_2^{min})^2 & f=0 \text{ and } v=q \\ \frac{(z_2^{min})^2}{12} & f=0 \text{ and } v=b \\ \frac{1}{12} & \text{otherwise} \end{cases}$$

Incumbents prefer to vote for the status quo when

$$U_{z,f=0,v=q} = -(z-q)^2 \geq -(z-b)^2 + w - c = U_{z,f=0,v=b}$$

which is equivalent to $z \geq \frac{w-c}{2(q-b)} + \frac{b+q}{2}$.

For districts $[0, y_1^{min}]$, the voters' posterior mean is

$$\mu(z) = \begin{cases} \frac{1+z_1^{min}}{2} & f=0 \text{ and } v=q \\ \frac{z_1^{min}}{2} & f=1 \text{ and } v=b \\ -\frac{1}{2} & \text{otherwise} \end{cases}$$

and the voters' posterior variance is

$$\sigma^2(z) = \begin{cases} \frac{1}{12}(1-z_1^{min})^2 & f=0 \text{ and } v=q \\ \frac{(z_2^{min})^1}{12} & f=1 \text{ and } v=b \\ \frac{1}{12} & \text{otherwise} \end{cases}$$

Incumbents prefer to vote for the status quo when

$$U_{z,f=0,v=q} = -(z-q)^2 \geq -(z-x_f)^2 - (z-b)^2 + w = U_{z,f=1,v=b}$$

which is equivalent to $z \geq -(q-b) + x_f + \sqrt{2(q-x_f)(q-b)+w}$.

If $c > 2(q-b)(\frac{3q-b}{2} - x_f - \sqrt{2(q-x_f)(q-b)+w})$, for districts $[y_1^{min}, y_2^{min}]$, the voters' posterior mean and posterior variance is the same as for districts $[y_2^{min}, 1]$.