# Online Appendix
# The Family Firm Ownership Puzzle

## Appendix 1: Model Implied Family Ownership

This appendix presents simulation results of the share of family wealth invested in the family firm in models with and without ambiguity. The implied family ownership is computed for different levels of ambiguity about the diversified portfolio relative to the family firm, and for three cases where the worst expected return of the family firm is the same, less, and more than that of the diversified portfolio. For comparison, we also present the implied ownership of a single stock in the model without ambiguity.

**Table A.1.**
**Calibration of Stock Ownership**

Model implied share of wealth invested in the single stock in models with and without ambiguity[32]. These estimates are computed in the benchmark model in Section 2. In the model without ambiguity, the true values of annual expected return ($\bar{r}_{2,0}$) and variance ($\sigma^2_{2,0}$) of the diversified portfolio are assumed to be 9% and 0.037, respectively. The true value of annual return variance ($\sigma^2_{1,0}$) of the single stock is assumed to be 24.2%, the correlation coefficient between diversified portfolio and single stock is assumed to be 0.123 (see Elton and Gruber, 1977). In the model with uncertainty, the worst-case values of annual expected return ($\bar{r}_{2,min}$) and variance ($\sigma^2_{2,max}$) of the diversified portfolio are assumed to be the same as the true values in the model without ambiguity.

| *Panel A: Model with Uncertainty* | | | |
|---|---|---|---|
| Relative Ambiguity of Risk ($\sigma^2_{2,max}/\sigma^2_{1,max}$) | Relative Ambiguity of Expected Returns $\bar{r}_{1,min} = \bar{r}_{2,min}$ | $\bar{r}_{1,min} = 0.5\bar{r}_{2,min}$ | $\bar{r}_{1,min} = 1.5\bar{r}_{2,min}$ |
| Infinity | 100% | 81% | >100% |
| 4.00 | 82% | 65% | 98% |
| 2.00 | 68% | 54% | 82% |
| 1.00 | 50% | 39% | 61% |
| 0.50 | 30% | 22% | 38% |
| 0.25 | 13% | 8% | 17% |
| *Panel B: Model without Uncertainty about Risk* | | | |
| Relative Risk ($\sigma^2_{2,0}/\sigma^2_{1,0}$) | Relative Ambiguity of Expected Returns $\bar{r}_{1,min} = \bar{r}_{2,min}$ | $\bar{r}_{1,min} = 0.5\bar{r}_{2,min}$ | $\bar{r}_{1,min} = 1.5\bar{r}_{2,min}$ |
| 0.15 | 3% | 0% | 6% |

32 The risk aversion and wealth level are calibrated such that 30% of wealth is invested in the risky portfolio and 70% invested in the safe asset, in Markowitz (1952) framework that contains a safe asset with return of 2% and a risky asset with expected return of 9%.

## Appendix 2: Family investment decision with Treasury Bills

In this appendix, we consider the optimal portfolio choice problem faced by the family owners who choose to invest in three assets, the family's firm with return $r_1$, a diversified portfolio with return $r_2$ and Treasury Bills with return $r_3$. We use Treasury Bills as a proxy for the risk-free asset with nonzero variance[33].

**Table A.2.**
**Summary Statistics of Return on Treasury Bonds and Market Index**

The returns are deflated using the inflation rate computed from CPI. The returns on treasuries, CPI, and the CRSP value-weighted market index are from the CRSP dataset available on WRDS.

| | *Market Index* | *Treasury Bonds/Bills* | | | | |
|---|---|---|---|---|---|---|
| | | *30 year* | *10 year* | *5 year* | *2 year* | *30 day* |
| *Annual Return (%) 1942-2013* | 9.15 | 2.11 | 1.86 | 1.80 | 1.31 | 0.15 |
| *Annual Volatility (%) 1942-2013* | 18.50 | 13.51 | 9.71 | 7.10 | 5.20 | 3.55 |
| *Annual Return (%) 1995-2013* | 9.56 | 6.10 | 4.38 | 3.64 | 2.16 | 0.41 |
| *Annual Volatility (%) 1995-2013* | 20.19 | 17.61 | 8.95 | 5.93 | 3.59 | 2.11 |

Table A.2 shows that the average return and standard deviation of 30-day Treasury Bills are significantly lower than that of the market portfolio. The standard deviation of the return on Treasury Bills is significantly different from zero. The standard deviations of longer-term Treasury Bonds are much higher.

We assume that returns on all three assets, $r_1$, $r_2$ and $r_3$ are independent and normally distributed. To focus on the investment choice between the family firm and the market portfolio, we assume no ambiguity regarding the return on Treasury Bills. That is, we assume the true value of the mean and variance of returns on the family firm and the market portfolio, $r_1$ and $r_2$, are unknown to investors, while the true value of the mean and variance of returns on Treasury Bills are unique and known to all investors. The true value of the mean and variance of the return on asset $i$, $r_i$, is denoted as $(\bar{r}_{i,0}, \sigma_{i,0})$ for $i$ =1, 2, 3. Investors perceive that the mean and variance of $r_1$

[33] 30-day Treasury Bills is commonly used as proxy of the risk free asset in asset pricing and optimal portfolio choice literature.

and $r_2$ belong to a set of possible values $\Theta_1 = \{(r_{1,n}, \sigma_{1,m}), n = 1,2, \dots, N, m = 1,2, \dots, M\}$, and $\Theta_2 = \{(r_{2,n}, \sigma_{2,m}), n = 1,2, \dots, N, m = 1,2, \dots, M\}$, which contain the true values of mean and standard deviation $(\bar{r}_{i,0}, \sigma_{i,0})$, for $i$ = 1, 2. That is, the true values $\bar{r}_{i,0}$ and $\sigma_{i,0}$ lie between $[\bar{r}_{i,min}, \bar{r}_{i,max}]$ and $[\sigma_{i,min}, \sigma_{i,max}]$, respectively, for $i$ = 1, 2. We further assume that the mean return on Treasury Bills is smaller than the minimum mean returns on family firm and market portfolio, and the variance of return on Treasury Bills is smaller than the minimum variances of returns on family firm and market portfolio, that is,

$$\begin{gathered} \bar{r}_{1,min} > \bar{r}_3,\ \bar{r}_{2,min} > \bar{r}_3 \\ \sigma_{1,min} > \sigma_3,\ \sigma_{2,min} > \sigma_3 \end{gathered} \tag{A.1}$$

As in our benchmark model, we assume that family owners have private information regarding the fundamentals of the family firm, which allows them to reduce the ambiguity about the fundamentals of their own firms over time and through experience. However, the family owner cannot obtain material, non-public information about firms in the broad portfolio as they do not manage these other firms. In our setup, the reduction in ambiguity is captured by the shrinkage in the range of all possible means and variances on the return on the family firm.

*A.1.1 The family's investment decision*

For each time period $t$, the decision problem of the family owner with CARA utility can be formulated as

$$\max_{\alpha_1,\alpha_2} \min_{\theta_1 \in \Theta_1, \theta_2 \in \Theta_2} E_t[-exp(\gamma W_{t+1})]$$

$$s.t. W_{t+1} = W_t[\alpha_1 r_{1,t+1} + \alpha_2 r_{2,t+1} + (1 - \alpha_1 - \alpha_2) r_{3,t+1}]$$

(A.2)

where $W_t$ is the wealth of the family at time $t$, $r_{1,t+1}$, $r_{2,t+1}$ and $r_{3,t+1}$ are the returns on the family firm, market portfolio, and Treasury Bills at time $t$+1, respectively. $\alpha_1$ and $\alpha_2$ are the shares of wealth invested in the family firm and market portfolio, respectively. We allow for short sale in all the assets, and do not restrict $\alpha_1$ and $\alpha_2$ to be positive and less than one.

It is straightforward to show that the decision problem of the family owner can be transformed as,

$$\max_{\alpha_1,\alpha_2} \min_{\bar{r}_{1,n_1}, \bar{r}_{2,n_2}, \sigma_{1,m_1}, \sigma_{2,m_2}} \{[\alpha_1(\bar{r}_{1,n_1} - \bar{r}_3) + \alpha_2(\bar{r}_{2,n_2} - \bar{r}_3) + \bar{r}_3]$$
$$- \frac{\gamma W_t}{2}[\alpha_1^2 \sigma_{1,m_1}^2 + \alpha_2^2 \sigma_{2,m_2}^2 + (1 - \alpha_1 - \alpha_2)^2 \sigma_3^2]\} \tag{A.3}$$

Let's first examine the minimization problem of (A.3). Since the objective function in (A.3) is monotonically decreasing in $\sigma_1^2$ and $\sigma_2^2$, the maximum possible value of variance is always chosen, that is

$$\sigma_1^* = \sigma_{1,max} \equiv \max_{m=1,2,..,M}\{\sigma_{1,m}\}$$
$$\sigma_2^* = \sigma_{2,max} \equiv \max_{m=1,2,..,M}\{\sigma_{2,m}\}$$

As long as the family owner is long in the assets, then the minimum possible mean returns are chosen for this asset. If the family shorts the asset, then the maximum possible mean return of this asset is chosen, that is,

$$(\bar{r}_1^*, \bar{r}_2^*) = \begin{cases} (\bar{r}_{1,min}, \bar{r}_{2,max}), & if\ \alpha_1 > 0\ and\ \alpha_2 < 0 \\ (\bar{r}_{1,min}, \bar{r}_{2,min}), & if\ \alpha_1 > 0\ and\ \alpha_2 > 0 \\ (\bar{r}_{1,max}, \bar{r}_{2,min}), & if\ \alpha_1 < 0\ and\ \alpha_2 > 0 \end{cases}$$

Assumption (A.1) implies that it is never optimal for the family owners to short both family firm and market portfolio at the same time. Given the solution to the minimization problem (A.3), the optimal share invested in the family business and market portfolio can be then characterized as follows:

*Case 1:* If the following condition is satisfied,

$$(\bar{r}_{1,min} - \bar{r}_{2,max})\sigma_3^2 > (\bar{r}_{2,max} - \bar{r}_3)\sigma_{1,max}^2 + \gamma W_t \sigma_3^2 \sigma_{1,max}^2$$

then the family owners go long on the family stock and short the diversified portfolio

$$\alpha_1^* = \frac{(\bar{r}_{1,min} - \bar{r}_3)\sigma_{2,max}^2 + (\bar{r}_{1,min} - \bar{r}_{2,max})\sigma_3^2 + \gamma W_t \sigma_{2,max}^2 \sigma_3^2}{\gamma W_t(\sigma_{1,max}^2\sigma_{2,max}^2 + (\sigma_{1,max}^2 + \sigma_{2,max}^2)\sigma_3^2)} > 0$$

$$\alpha_2^* = \frac{(\bar{r}_{2,max} - \bar{r}_3)\sigma_{1,max}^2 + (\bar{r}_{2,max} - \bar{r}_{1,min})\sigma_3^2 + \gamma W_t \sigma_{1,max}^2 \sigma_3^2}{\gamma W_t(\sigma_{1,max}^2\sigma_{2,max}^2 + (\sigma_{1,max}^2 + \sigma_{2,max}^2)\sigma_3^2)} < 0$$

$$1 - \alpha_1^* - \alpha_2^* = \frac{-(\bar{r}_{1,min} - \bar{r}_3)\sigma_{2,max}^2 - (\bar{r}_{2,max} - \bar{r}_3)\sigma_{1,max}^2 + \gamma W_t \sigma_{1,max}^2 \sigma_{2,max}^2}{\gamma W_t(\sigma_{1,max}^2\sigma_{2,max}^2 + (\sigma_{1,max}^2 + \sigma_{2,ma}^2)\sigma_3^2)} \gtrless 0$$

In this case, family owner's private information allows them to significantly reduce the ambiguity of their family firm relative to the other firms, and the minimum expected return of family firm is much larger than the maximum possible expected return of the diversified portfolio (in excess of the return on Treasury Bills), and the family chooses to invest as much as possible in their firm. In this case, the share invested in the family firm decreases with the risk aversion and family wealth. However, we argue this is an extreme case and does not apply to the marginal family owner who may exit, as the family would never exit in this case. Furthermore, we argue this is a case rare in reality, as it is extreme for the family to think the minimum risk-adjusted expected return of the family firm is much larger than the maximum possible expected return of the diversified portfolio.

*Case 2:* If the following condition is satisfied,

$$-(\bar{r}_{1,min}-\bar{r}_3)\sigma_{2,max}^2-\gamma W_t\sigma_3^2\sigma_{2,max}^2<(\bar{r}_{1,min}-\bar{r}_{2,min})\sigma_3^2<(\bar{r}_{2,min}-\bar{r}_3)\sigma_{1,max}^2+\gamma W_t\sigma_3^2\sigma_{1,max}^2,$$

then the family owners invest positive shares in both the family stock and the diversified portfolio

$$\alpha_1^*=\frac{(\bar{r}_{1,min}-\bar{r}_3)\sigma_{2,max}^2+(\bar{r}_{1,min}-\bar{r}_{2,min})\sigma_3^2+\gamma W_t\sigma_{2,max}^2\sigma_3^2}{\gamma W_t(\sigma_{1,max}^2\sigma_{2,max}^2+(\sigma_{1,max}^2+\sigma_{2,max}^2)\sigma_3^2)}>0$$

$$\alpha_2^*=\frac{(\bar{r}_{2,min}-\bar{r}_3)\sigma_{1,max}^2+(\bar{r}_{2,min}-\bar{r}_{1,min})\sigma_3^2+\gamma W_t\sigma_{1,max}^2\sigma_3^2}{\gamma W_t(\sigma_{1,max}^2\sigma_{2,max}^2+(\sigma_{1,max}^2+\sigma_{2,max}^2)\sigma_3^2)}>0$$

$$1-\alpha_1^*-\alpha_2^*=\frac{-(\bar{r}_{1,min}-\bar{r}_3)\sigma_{2,max}^2-(\bar{r}_{2,min}-\bar{r}_3)\sigma_{1,max}^2+\gamma W_t\sigma_{1,max}^2\sigma_{2,max}^2}{\gamma W_t(\sigma_{1,max}^2\sigma_{2,max}^2+(\sigma_{1,max}^2+\sigma_{2,max}^2)\sigma_3^2)}\gtrless 0$$

In this case, the family owner still has less ambiguity about the expected return on their firm relative to other firms. However, the private information advantage is not as large as in Case 1, so the family firm invests positive wealth shares in both the family firm and the diversified portfolio.

*Case 3:* If the following condition is satisfied,

$$(\bar{r}_{1,max}-\bar{r}_{2,min})\sigma_3^2<-(\bar{r}_{1,max}-\bar{r}_3)\sigma_{2,max}^2-\gamma W_t\sigma_3^2\sigma_{2,max}^2$$

then the family owners short the family stock and long the diversified portfolio

$$\alpha_1^*=\frac{(\bar{r}_{1,max}-\bar{r}_3)\sigma_{2,max}^2+(\bar{r}_{1,max}-\bar{r}_{2,min})\sigma_3^2+\gamma W_t\sigma_{2,max}^2\sigma_3^2}{\gamma W_t(\sigma_{1,max}^2\sigma_{2,max}^2+(\sigma_{1,max}^2+\sigma_{2,max}^2)\sigma_3^2)}<0$$

$$\alpha_2^*=\frac{(\bar{r}_{2,min}-\bar{r}_3)\sigma_{1,max}^2+(\bar{r}_{2,min}-\bar{r}_{1,max})\sigma_3^2+\gamma W_t\sigma_{1,max}^2\sigma_3^2}{\gamma W_t(\sigma_{1,max}^2\sigma_{2,max}^2+(\sigma_{1,max}^2+\sigma_{2,max}^2)\sigma_3^2)}>0$$

$$1-\alpha_1^*-\alpha_2^*=\frac{-(\bar{r}_{1,max}-\bar{r}_3)\sigma_{2,max}^2-(\bar{r}_{2,min}-\bar{r}_3)\sigma_{1,max}^2+\gamma W_t\sigma_{1,max}^2\sigma_{2,max}^2}{\gamma W_t(\sigma_{1,max}^2\sigma_{2,max}^2+(\sigma_{1,max}^2+\sigma_{2,max}^2)\sigma_3^2)}\gtrless 0$$

In this scenario, the private information does not help to reduce the ambiguity of the family firm enough, so the minimum possible expected return of the family business is less than the minimum possible expected return of the diversified portfolio.

Let us again focus on Case 2, where family owners invest positive shares in the family firm and the diversified portfolio. In this case, we can rewrite the optimal investment share in the family firm as

$$\alpha_1^*=\frac{(\bar{r}_{1,min}-\bar{r}_3)/\sigma_3^2+(\bar{r}_{1,min}-\bar{r}_{2,min})/\sigma_{2,max}^2+\gamma W_t}{\gamma W_t(\sigma_{1,max}^2/\sigma_{2,max}^2+\sigma_{1,max}^2/\sigma_3^2+1)} \tag{A.4}$$

The optimal share depends on the Sharpe ratio of the family firm relative to the treasury bills and the market portfolio. When the family has less ambiguity about the variance and the expected return of the family firm, that is, the smaller are $\sigma^2_{1,max}/\sigma^2_{2,max}$ and $\sigma^2_{1,max}/\sigma^2_3$ or the larger are $(\bar{r}_{1,min} - \bar{r}_3)/\sigma^2_3$ and $(\bar{r}_{1,min} - \bar{r}_{2,min})/\sigma^2_{2,max}$, the family is more likely to continue to invest in the family firm. Thus, the concern about risk-adjusted return centers on the ambiguity of the variance rather than the level of the variance. Moreover, if the family has little information about the diversified portfolio (relative to their firm), then they would put a very high upper bound on the perceived variance of the return of this asset (extremely high $\sigma^2_{2,max}$), which implies that they would invest little in the diversified portfolio. Suppose the family is extremely averse to ambiguity, or the relative ambiguity about the risk of the diversified portfolio gets extremely large, then the family would invest nothing in the diversified portfolio, regardless of their risk aversion or wealth level.

To study the relationship between the share invested in the family firm and the characteristics of the family, such as risk aversion and wealth of the family, we found it is necessary to focus on Case 2. Taking the derivative of the optimal share in family share ($\alpha^*_1$) with respect to risk aversion ($\gamma$) and wealth ($W_t$), we have

$$\frac{\partial \alpha^*_1}{\partial(\gamma W_t)} = -\frac{(\bar{r}_{1,min}-\bar{r}_3)/\sigma^2_3+(\bar{r}_{1,min}-\bar{r}_{2,min})/\sigma^2_{2,max}}{(\gamma W_t)^2(\sigma^2_{1,max}/\sigma^2_{2,max}+\sigma^2_{1,max}/\sigma^2_3+1)} \tag{A.5}$$

If the family owners have less ambiguity regarding the family firm and think the return on their family firm is so good that the minimum return is sufficiently larger than that of the market portfolio such that the Sharpe ratio relative to the market portfolio is larger than the Sharpe ratio relative to the Treasury Bills, that is,

$$\frac{(\bar{r}_{1,min} - \bar{r}_{2,min})}{\sigma^2_{2,max}} > -\frac{(\bar{r}_{1,min} - \bar{r}_3)}{\sigma^2_3}$$

then the family will always invest a positive share in the family business and not consider exiting. However, we are more interested in the case where

$$\frac{(\bar{r}_{1,min} - \bar{r}_{2,min})}{\sigma^2_{2,max}} < -\frac{(\bar{r}_{1,min} - \bar{r}_3)}{\sigma^2_3} < 0$$

The family thinks the minimum expected return of the family business is smaller than that of the market portfolio. This is the case that captures the marginal family investors who might consider the exit decision. In this case, (A.5) is positive, so the more risk averse is the family owner, the more wealth the family has, the larger share of wealth invested in the family firm, the less likely the family to exit *ceteris paribu.* Hence, the predictions of our benchmark model still hold when the choice set of the family owners includes not only two risky assets with different levels of ambiguity but also Treasury Bills that are proxies for the relatively low-risk asset without ambiguity.