PROGRESSIVE INCOME TAXES AND OPTION VALUES: THE CASE OF A FARMER WHO OWNS A FOREST

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ABSTRACT

Consider an individual having two sources of taxable income. First, a stochastic, e.g. farm income. Secondly, a controllable arising when a stock of capital is converted into taxable income, e.g. the harvesting of a stock of timber. The harvest decision is an optimal stopping problem when the individual i) maximises expected post-tax income, ii) has the option to observe a period's farm income before deciding on the harvest policy, and iii) income taxes are progressive. Two cases are considered. The case where farm income is generated by random draws from a stationary distribution, and the case where farm income is generated by a stationary, autoregressive process. In the first case, the solution is a single value of farm income for each period. In the second case, the optimal stopping rule is a set of two values of farm income.

Keywords: Optimal harvesting, optimal stopping, real options, stochastic non-timber income.

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INTRODUCTION

This paper studies the case of an individual having two sources of taxable income and facing a progressive income tax scheme. The income from the first source is stochastic and has a known distribution; in the paper this income is represented by farm income. The second type of income is a controllable income arising when a stock of capital is converted into taxable income, represented by the harvesting of a stock of timber. The problem for the individual is to decide when it is optimal to harvest the stock of timber. This is analysed as a real option pricing and optimal stopping problem.

Many studies on the economics and management of forests under uncertainty have addressed the concept of optimal stopping, e.g., Norström (1975), Lohmander (1987), and

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Brazee & Mendelsohn (1988). These studies analyse the decision to harvest a forest stand when timber prices are stochastic. Morck *et al*. (1989) use contingent claims to price the real option of harvesting a forest stand within a limited time period when prices and inventories follow geometric brownian motions. A related study is Thomson (1992). A recent study by Plantinga (1998) also focuses on prices and explicitly links the field of analysis to the theory of real options.

The optimal timing of the harvesting decision is also the object of analysis in the present study. It will, however, be assumed that timber prices are constant. The focus, instead, will be on the effect that a progressive income tax scheme has on the harvesting decision and stand value when the forest owner has a different and stochastic main income. In Scandinavia tax progression affects a large part of the population. Furthermore, many forest owners have their main income outside forestry, because their forest area is too small and/or unproductive to provide sufficient income. Among this group of small-scale forest owners, the typical main source of income is agriculture. This income is highly variable due to variations in product prices and growth conditions. These facts are the main sources of inspiration for the model presented in this study.

Only few related studies exist. For example Taylor & Novak (1992) calculate optimal cropping and storage policies with and without progressive income taxes. However, they do not discuss or attempt any quantification of option values or solve the implicit optimal stopping problem explicitly.

In the following section, a model is developed which describes the problem of maximising the post-tax value of the option to harvest a forest stand, when income tax is progressive and the forest owner has a stochastic income from farming, which he can observe before making the harvesting decision. The problem is solved and analysed as an optimal stopping problem for two different specifications of the stochastic behaviour of the farm income. Numerical analyses of the problem are provided. The results and their interpretations are discussed and the paper is concluded with a discussion of qualifications and possible extensions of the model.

THE MODEL

Consider the case where a farmer earns a taxable income x_t each year $t < T$. The income is stochastic, varies between $\underline{x}$ and $\bar{x}$, and has the probability density function $f(x_t)$, $f(\underline{x} < x_t < \bar{x}) > 0$ and $F(\bar{x}) = 1$. The income is subject to tax paid according to a progressive tax scale constructed in the following way: when taxable income is below a level I, the income tax rate is s_L and all taxable income is subject to this tax. When income is above I, the marginal tax rate is s_H, i.e., the part of x_t below I is subject to the tax s_L whereas the part above I is subject to the higher tax s_H.

The farmer also owns a small forest. For simplicity, it is assumed that it consists of a single stand which cannot be partially harvested. For small forests and forest stands this assumption is reasonable, because harvesting often implies substantial fixed costs. The stumpage value K is subject to income tax only when the stand is harvested. For simplicity, the value of the land in its next use is assumed to be zero. Finally, it is assumed that there is no property taxes on K.

The expected post-tax value $E(V(K, x_t))$ of harvesting the stand before x_t has been observed is given by the expression:

$$
\begin{aligned}
E\big(V(K,x_t)\big) = & \int_{\underline{x}}^{I-K} K(1-s_L) f(x_t)\,dx_t \\
& + \int_{I-K}^{I} \Big[(I-x_t)(1-s_L) + \big(K-(I-x_t)\big)(1-s_H)\Big] f(x_t)\,dx_t \\
& + \int_{I}^{\bar{x}} K(1-s_H) f(x_t)\,dx_t. \qquad (1)
\end{aligned}
$$

It is a basic and important assumption that the farmer has the opportunity to observe x_t before deciding whether he wants to harvest the stand this year or wait at least one more year. The farmer essentially holds an option to harvest the stand. Assuming that the farmer is risk neutral and wishes to maximise the post-tax value of his stand or in other words the price, P, of his harvesting option, then for any given x_t and for $t \le T$ the price is given by:

$$P_t\left(K,x_t\middle|x_t\right)=\max\left\{V\left(K,x_t\right);(1+\rho)^{-1}P_{t+1}\left(K,x_{t+1}\middle|x_t\right)\right\},\qquad(2)$$

where ρ is the discount rate and P_{t+1} is the expected post-tax value of postponing the harvesting decision at least one period. In option pricing theory, $V(K, x_t)$ is called the stopping value and the expected value of holding the option in the next period, $(1 + \rho)^{-1}P_{t+1}(K, x_{t+1})$, the continuation value (cf. Dixit & Pindyck, 1994). Note that $(1+\rho)^{-1}P_{t+1} = E(V(K, x_T))$ at $t = T-1$.

The problem for the farmer is to know for what values of farm income, x_t, stopping is optimal. This is the optimal stopping rule that separates the continuation region from the stopping region, i.e., the value of x_t where $V(K, x_t) = (1 + \rho)^{-1}P_{t+1}(K, x_{t+1} \mid x_t)$, and the value that maximizes the price $P_t(K, x_t)$ of the harvesting option before x_t has been observed. Finally, define the option value, $O.V._t$, as $O.V._t \equiv P_t\left(K,x_t\right)-E\left(V\left(K,x_t\right)\right)$. That is, the option value is the difference between the option price and the expected value of stopping, and hence it is the expected value of the option to wait and observe x_t before making the decision to harvest or not. It is well-known that the option value is non-negative.

ANALYTICAL RESULTS

In the case of no autocorrelation in the process generating x_t, the continuation value, P_{t+1}, does not depend on the present period's outcome, and a single optimal stopping rule exists for each $t \leq T$[1].

Proposition 1: Assume x_t exists such that $V(K,x_t) < (1+\rho)^{-1} P_{t+1}(K)$. Then the optimal stopping rule, x_t^*, maximises the post-tax value of the option to harvest the stand:

$$\begin{aligned}P_t(K)=\max_{x_t^*}\Bigg\{&\int_{\underline{x}}^{I-K}K\left(1-s_L\right)f\left(x_t\right)dx_t\\&+\int_{I-K}^{x_t^*}\left[\left(I-x_t\right)\left(1-s_L\right)+\left(K-\left(I-x_t\right)\right)\left(1-s_H\right)\right]f\left(x_t\right)dx_t\\&+(1+\rho)^{-1}\int_{x_t^*}^{\bar{x}}P_{t+1}(K)f\left(x_t\right)dx_t\Bigg\}.\end{aligned}\qquad(3)$$

[1] The proof of existence for the optimal stopping rule is not repeated here, but see, e.g., Forboseh (1994).

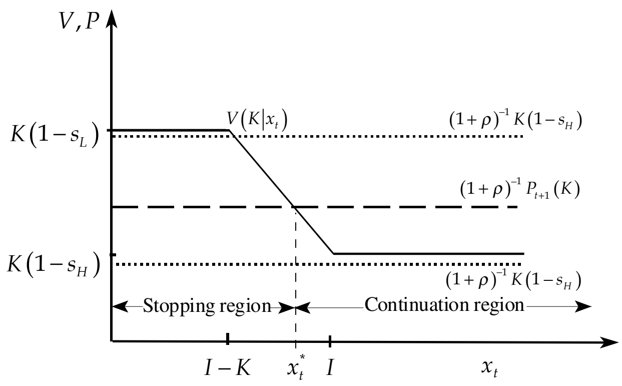

FIGURE 1. THE CASE OF NO AUTOCORRELATION.

An illustration of the solution in the case of no autocorrelation in farm income x_t. Broken lines represent the limits of the value of continuation.

Furthermore, x_t^* is smaller than I, is unique, and given by:

$$x_t^* = \frac{(1+\rho)^{-1} P_{t+1} - \left(I(s_H - s_L) + K(1-s_H)\right)}{s_L - s_H} \tag{4}$$

Proof: To prove that $x_t^* < I$ must be true, assume it is not. This would imply that $V(K, x_t \,|\, x_t = I) \geq (1+\rho)^{-1}P_{t+1}(K)$, but this would be true for all $x_t \geq I$. Furthermore, it would be true that $V(K, x_t \,|\, x_t \leq I) > (1+\rho)^{-1}P_{t+1}(K)$. Stopping would be optimal for any x_t, but this contradicts the assumption made. Differentiating (3) w.r.t. x_t^* the first-order condition for optimality can be rearranged to obtain (4). Uniqueness is established by observing that $\partial^2 P_t(K) \big/ \left(\partial x_t^*\right)^2 = (s_L - s_H) f\left(x_t^*\right) < 0$.

A graphical illustration of the problem and its solution is given in Figure 1. The optimal stopping rule x_t^* partitions the set of farm income values, x, into a stopping and a continuation region. Note that the value of continuation can never exceed $(1+\rho)^{-1}K(1-s_L)$, in turn implying both a lower limit for x_t^*, cf. (4), and that it can never be lower than $(1+\rho)^{-1}K(1-s_H)$.

A few comparative statics results merit discussion. Using (3) and the fact that the first-order condition is satisfied for the optimal program, one can show that:

$$\frac{dP_t}{dP_{t+1}} = \frac{\partial P_t}{\partial P_{t+1}} + \frac{\partial P_t}{\partial x_t^*}\frac{\partial x_t^*}{\partial P_{t+1}} = \frac{\partial P_t}{\partial P_{t+1}} = (1+\rho)^{-1}\int_{x_t^*}^{\bar{x}} f(x)dx > 0. \qquad (5)$$

It is easy to show that this result implies that dP_t/dP_{t+n} > 0 for all $n \geq 0$. Anything increasing the value of harvesting the stand in future, $t + i$, $i = 1, 2,...$, increases the continuation value at time t, and hence lowers the optimal stopping rule x_t^*, cf. (4). In other words, the more profitable continuation is expected to be, the less likely stopping becomes. These observations also indicate the effect of a growth in K. If K grows, the value of continuation, P_{t+1}, increases and, hence, the optimal stopping rule, x_t^*, decreases. With growth the value of continuation theoretically could exceed $K_t\,(1-s_L)$. In that case, continuation is optimal for all x_t^* and the stand will never be harvested.

Next the case of autocorrelated farm income is considered. Several factors affect the level of farm income in any given year. Some of these, e.g. the weather, may be very well described as independent realizations over time. Others, like prices, often show some degree of serial dependence. This may cause farm income to show serial correlation too. In fact, when analysing the development of taxable income during the years 1961 through 1992 for several size groupings, the hypothesis of no serial correlation could in general be rejected. Serial correlation in farm income makes the value of continuation dependent on the present period's farm income. Clearly, several different autoregressive processes may be relevant, but for simplicity it is assumed that farm income is generated by a weakly stationaryAR(1) process.

Proposition 2: Assume that farm income is generated by a stationary AR(1) process with the representation: $x_{t+1} = \alpha + \beta x_t + \varepsilon_t$, where $0 < \beta < 1$, and ε is a symmetrically, independently and identically distributed error term, $N(0, \sigma_\varepsilon^2)$. Furthermore, assume that x_t exists such that $V(K, x_t \mid x_t) < (1+\rho)^{-1} P_{t+1}(K \mid x_t)$. Then the optimal stopping rule that maximises the option price is a set of *at most* two values, x_t^L and x_t^U, of farm income. Hence, the maximized option price is:

$$P_t(K)^* = \max_{x_t^L, x_t^U}\left\{\int_{\underline{x}}^{x_t^L} V(K) f(x_t|x_{t-1})dx_t + \right.$$

$$\left.\int_{x_t^U}^{\bar{x}} V(K) f(x_t|x_{t-1})dx_t + (1+\rho)^{-1}\int_{x_t^L}^{x_t^U} P_{t+1}(K|x_t) f(x_t|x_{t-1})dx_t\right\}. \quad (6)$$

The lower value x_t^L is given by:

$$x_t^L = \frac{(1+\rho)^{-1} P_{t+1}(K|x_t^L) - (I(s_H - s_L) + K(1-s_H))}{s_L - s_H} \quad (7)$$

and $x_t^L \leq I$. The upper value x_t^U satisfies:

$$K(1-s_H) = (1+\rho)^{-1} P(K|x_t^U) \quad (8)$$

and $x_t^U \leq I$.

Proof:[2] At $t = T-1$ it is known that $P_T = E_{T-1}(V(K, x_T))$, then using Equation (1) and integration by parts:

$$P_T = K(1-s_H) - (s_L - s_H)\int_{I-K}^{I} F(x_T)dx_T. \quad (9)$$

Differentiating Equation (14) with respect to x_{T-1} results in:

$$\frac{\partial P_T}{\partial x_{T-1}} = (s_L - s_H)\frac{\beta}{\sigma}\left[F(I) - F(I-K)\right]. \quad (10)$$

Thus it is evident that $0 \geq \partial P_T/\partial x_{T-1} \geq s_L - s_H$ and, hence, there can be at most two intersections between $(1+\rho)^{-1}P_T$ and $V(K\,|\,x_{T-1})$. To show that this holds for all t, use (2) recursively. To show that (7) and (8) constitute the optimal stopping rule, rewrite (6) by inserting the relevant parts of the function $V(K, x_t)$. Differentiating with respect to x_t^L and x_t^U and rearranging shows that (7) and (8) fulfil the first-order necessary conditions for an optimum.

The solution is illustrated in Figure 2. The optimal stopping rules x_t^L and x_t^U are the income levels where the value

2 Details of the proof are given in the Appendix.

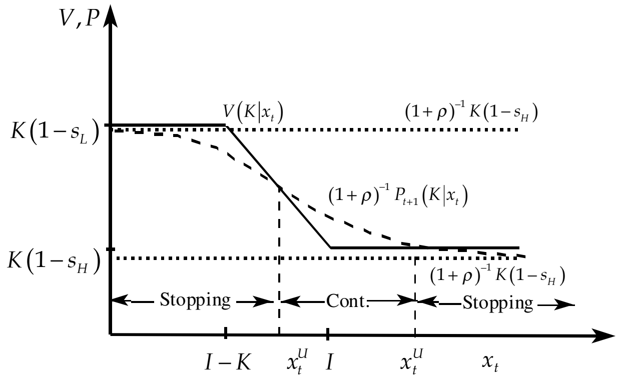

FIGURE 2. THE CASE OF AUTOCORRELATION.

An illustration of the solution to the optimal stopping problem when the stochastic process governing farm income x_t exhibits autocorrelation. Broken lines represent the limits of the value of continuation.

of stopping, that is, harvesting now and getting $V(K\,|\,x_t)$, equals the expected value of continuation $(1+\rho)^{-1}\,P_{t+1}(K\,|\,x_t)$, i.e. the expected value of waiting at least one period before harvesting. P_{t+1} is decreasing in x_t because — with positive autocorrelation — an increase in current farm income implies an increase in future income levels and, hence, a higher marginal tax rate. This decreases the value of continuing *not* to harvest the forest stand.

The relevance of the analytical conclusions made depends largely upon their numerical significance. Therefore, numerical sensitivity analysis will be used to investigate the behaviour of the optimal stopping rule and the relative importance of the option value.

NUMERICAL ANALYSIS

The numerical sensitivity analysis is conducted for the case of no autocorrelation in farm income. The presented qualitative results concerning the price (value) of the option to harvest the stand, $P_t(K)$, are in general valid for the case of autocorrelation too. Using (7) and (8), it is possible to determine the effects on the stopping rules.

The numerical analysis proceeds from a base case scenario with parameters which are relevant in Denmark. The base case parameters of the distribution function are ob-

TABLE 1. THE NUMERICAL ANALYSIS— PARAMETERS AND VALUES.

Parameter	Value	Parameter	Value
μ, DKK	260,000	I,DKK	240,000
σ	52,000	K,DKK	50,000
s_L	0.5	ρ	0.03
s_H	0,62	$\overline{x}, \underline{x}$	$\mu \mp 4 \times \sigma$

tained by rounding off the estimated mean, μ, and variance, σ^2, from a time series of taxable income for Danish farms in the size group 30–49.9 ha, assuming a farm size of 40 ha and no autocorrelation (Statistics Denmark 1965–1995). This farm size is chosen because it is close to the average farm size in Denmark, and because the average yearly income fluctuates around a level where changes in marginal income tax occur.

The remaining parameters are given values relevant to the Danish farmers. All parameters are reported in Table 1. A pre-tax value of the standing timber K = 50,000 Danish Kroner (hereafter DKK) represents a minor, mature stand of Norway spruce (*Picea abies*, L. (Karst.)) containing approximately 250 m^3 and covering an area of 0.5 – 1.0 ha depending on the age and site class.

Preliminary analysis revealed that the effect of the finite time horizon declined rapidly and was essentially absent 10 periods before the final period. Therefore only results for period T – 20 are presented. That is, these results are valid for all periods except the last 5 to 10.

The optimal stopping rule may increase or decrease as K changes, see Figure 3. The relative increase in the value of the stand is shown in Figure 3. It is seen to be strictly decreasing in K. Perhaps the most interesting question is: how does the optimal stopping rule and stand value react to changing progressivity, i.e., the difference $s_H - s_L$? To analyse this question, the degree of progressivity in the tax scheme is increased (decreased) by changing the high tax, s_H. The effect of such changes on the optimal stopping rule is shown in Figure 4, where the relative importance of the option value is shown too.

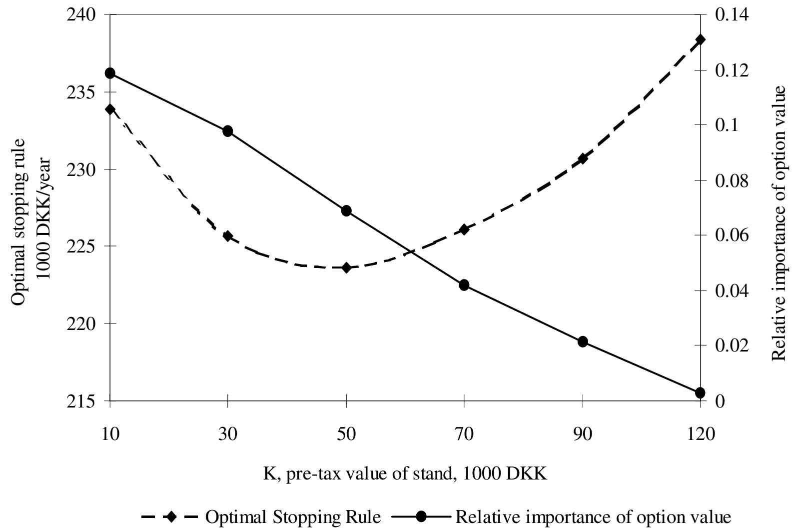

FIGURE 3. THE RESULTS' SENSITIVITY TO CHANGES IN K.

The optimal stopping rule, x_t^*, and the relative importance of the option value for different levels of K, the pre-tax value of the stand, in period T−20. The relative importance of the option value is defined as $[P_t(K)-E(V(K,x_t))]/E(V(K,x_t))$.

Another important parameter related to the tax scheme is the point I where the marginal income tax jumps from s_L to s_H. From Figure 5 it can be seen that the optimal stop-

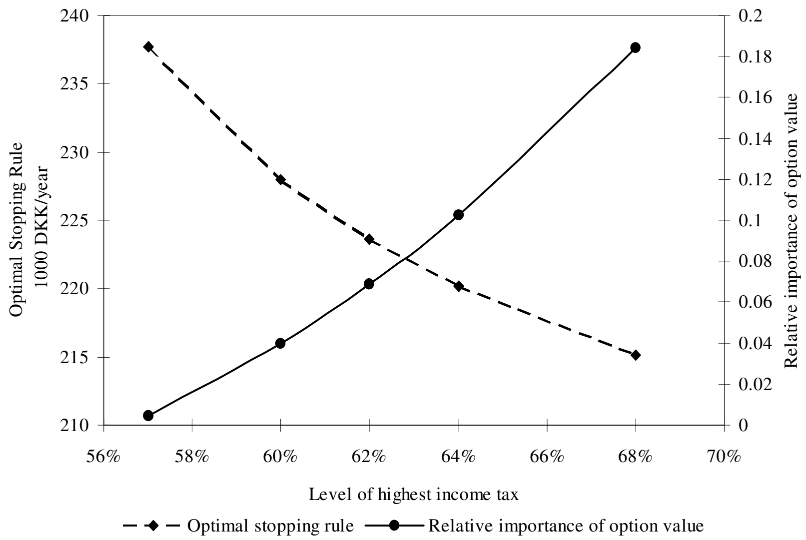

FIGURE 4. THE RESULTS' SENSITIVITY TO CHANGES IN s_H.

The optimal stopping rule and the relative importance of the option value for different levels of progressivity in the tax scheme. Values in period T − 20.

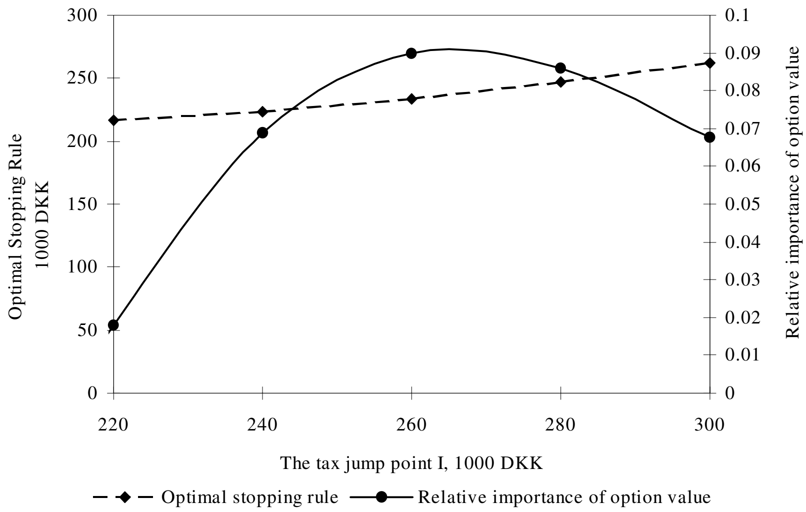

FIGURE 5. THE RESULTS' SENSITIVITY TO CHANGES IN JUMP-POINT I.
The optimal stopping rule and the relative importance of the option value for different values of the point I, where the tax jumps from s_L *to* s_H*. Values in period T - 20.*

ping rule decreases as *I* increases. The relative importance of the option value also changes with *I*, but in a more complicated pattern.

Finally, the case of a growth in the pre-tax value *K* of the stand is analysed. As an example, the growth in value of a Norway spruce stand of site class 1 according to Møller (1933) is approximated with an exponentially decreasing function. If it is not possible to observe farm income before deciding whether to harvest or not, harvesting the stand at age 64 is the optimal policy. Prior to this age, the Marshallian value of the stand is the discounted expected post-tax value of the stand if harvested at age 64. After the age 64, it will always be optimal to harvest immediately; according to the Faustmann principle the stand is overmature. Hence, at ages above 64, the Marshallian value of the stand is given by $E(V(K_t, x_t))$. Figure 6 shows the value of a stand priced according to these principles and the value of the same stand priced as an option with the embedded optimal stopping rule. The relative increase in the value of the stand, i.e., the importance of the option value, is shown in Figure 6 too.

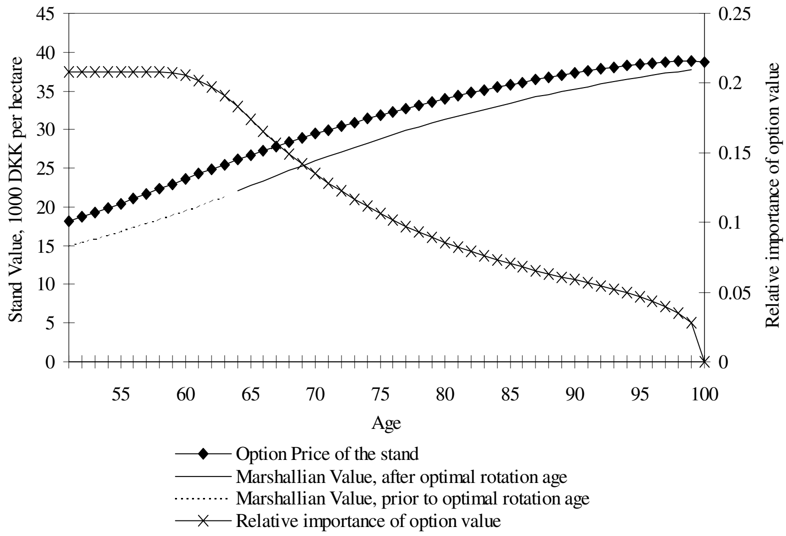

FIGURE 6. THE CASE OF GROWTH IN K.

The value of a growing stand when priced as the option to harvest the stand according to the optimal stopping rule. Compare with the traditional Marshallian net present value of the stand. Left axis shows the value of the stand. The relative increase in the value of the stand is measured on the right axis.

DISCUSSION

The analytical solution to the case where farm income is simple random draws from a known distribution has many similarities to the solutions to the problems treated by Lohmander (1987) and Brazee & Mendelsohn (1988). With autocorrelation the solution becomes more complicated due to the nonlinearity of the post-tax value function V, and the dependence of the continuation value $(1+\rho)^{-1}P_{t+1}(K)$ on the current level of farm income, x_t. Compare Figure 1 with Figure 2. The dynamics with respect to many features of the problem are, however, not different from the case of no autocorrelation, which is why the numerical analysis only concerns the latter case.

The results of the numerical analysis are presented in Figures 3–6. Increasing the pre-tax value K of the forest stand causes a decline in the relative importance of the option value (cf. Figure 3) due to the decreasing probability of getting a significant part of K subject to the lower tax only. The option price itself, and hence the continuation value, is of course affected positively by the increase in K. The optimal stopping rule is affected negatively by changes in K through the increase in the continuation value, but positively through the increase in K directly, cf. (4). Thus, the total effect on x_t^* depends on the level of K. For small K the increase in the continuation value outweighs the effect of the increase in K and hence the cost of waiting. As K increases the relative importance of these components shifts, and for large K the cost of waiting cannot be outweighed by the probability of a relatively small reduction in tax burden. This causes x_t^* to approach I, and as it reaches I stopping becomes optimal for all values of x, cf. Figure 1.

Having focused on the option value arising in the present model, a central question is how much progressivity affects the importance of the option value. Different degrees of progressivity have been modelled by changing the higher tax, s_H, and the effects on the relative increase in asset value and the optimal stopping rule are depicted in Figure 4. Increasing the upper marginal tax clearly decreases the value of harvesting now and hence the capital cost of waiting, but the value of continuation is also decreased. The overall effect of these dynamics on x_t^* in the base scenario is a decrease when s_H increases. The value of the potential future reductions in tax burden is increased, and, hence, it becomes optimal to wait for a wider range of x-values. This is also reflected in the increasing relative importance of the option value, which may become as high as 20%, with a jump from 50% to 68%. The more progressive the tax scale is, the larger is the value of waiting and observing the next period's x_t before deciding whether to harvest or not.

Another parameter of interest is the jump point, I. The relative increase in the asset value increases with increasing I until I is approximately DKK 260,000. Above this level the probability of being in a state where a significant part of K will be subject to s_H if harvested becomes so small that the continuing increase in $E(V(K, x_t))$ causes the relative importance to decline. Increasing I causes the continuation value to increase, but also affects the cost of waiting; hence, the overall effect on x_t^* is an increase.

Finally, the assumption of no growth is relaxed and a positive but declining growth rate allowed in the pre-tax value K_t. In the previous example, the option embedded in holding the stand was in practice the option to postpone the harvest decision, even though the growth rate in $E(V(K, x_t))$ is zero and hence below the interest cost ρ. In the case of growth, the option to harvest when income tax levels are attractive in principle embeds two different kinds of options. At an age of 64, the expected relative growth in the post-tax value of the stand equals the interest cost of the current expected post-tax value of the stand. Hence, this is the optimal rotation age if the farmer can not observe x_t before deciding whether to harvest or not. In the stochastic dynamic optimization problem the stand may be harvested before or after the age prescribed by classical capital budgeting or Faustmann principles. The effect on the valuation of a forest stand can be quite dramatic as illustrated in Figure 6.

As all models the present one has its limitations and simplifications. The present study has used a 'single stand' model as the object of analysis. However, the analytic results obtained may change if the model included several stands, because selling one stand in a year where $x_t < I$ could affect the taxation of subsequent harvests. In other words, it may not be valid to solve the optimal stopping problem for each stand independently; separability does not necessarily apply.

As a simplification, land value, potentially the value of future forest generations, is set to zero in this study. However, usually land value is positive. If a land value, L, is added to the value of stopping and the value of continuation in (2) and (3), the optimal stopping rule will also increase, and hence stopping becomes more likely. The reason is that the cost of waiting increases as more capital is 'tied up' on the land. If one wants to determine a land value reflecting the value of future forest generations managed according to the optimal stopping rule derived, an iterative procedure is needed which takes into account the stochastic rotation age.

As pointed out in the introduction, most studies on forest management decisions under uncertainty have considered stochastic prices on timber, i.e., stochastic K, as the stochastic variable of choice. The present model can be extended to allow K to be stochastic too. Again several possible specifications of the stochastic behaviour of K are possible, including the random draws, the stationary autoregressive, process and different types of brownian motions. The optimal stopping rule will consist of a set of timber prices and farm income levels.

CONCLUSION

The problem of determining the optimal stopping rule was solved in two cases; a case where farm income is modelled as random draws from a known distribution, and a case where farm income is modelled as a stationary mean reverting AR(1)-process. A numerical sensitivity analysis of the model has been provided for the first case.

The numerical analyses also gave some insight into the magnitude of the option value and hence the increase in the value of an asset, e.g., a forest stand. For the parameters in the base scenario the result showed an increase of more than 20% in the value of a Norway spruce stand when priced as an option to harvest the stand, as compared to its value when priced with traditional expected net present value techniques. This figure of course depends much on the premises of the model. As shown and discussed the degree of progressivity strongly affects the relative importance of the option value. The jump from 50 to 62% in marginal income tax rate at an income level of DKK 240,000 may seem dramatic, but in Denmark the marginal income tax jumps 6% at app. DKK 130,000 and 12% at DKK 240,000. Thus, the scheme used is not nearly as progressive as the Danish tax scheme. Clearly, the relative importance of the option value is larger under the Danish tax scheme than under the tax scheme used in the present model. In countries where income taxes are much less progressive, the problem described here may have much less relevance or would at least require different levels of income and variance in income to be relevant.

Another relevant extension is to allow partial harvesting of the timber stock. In forest ecosystems characterised by unevenaged, multispecies stands, this may be the only relevant specification. Clearly, if extraction from such systems can be done without fixed costs and/or returns to scale in harvesting, then it will always be optimal to extract an amount of timber worth at least $I - x_t$ from the forest. Whether timber above this amount should be extracted now or later will depend on the value of waiting. It must be decided not only for what values of farm income harvesting should be postponed, but also how much timber to postpone harvesting of. The problem has an important analogue outside the resource management field. In many countries capital gains on capital assets like stocks are only subject to income tax when sold. Thus, the emerging class of small private stockholders with stochastic income from other sources, may find it optimal to sell parts of the stocks in years when the marginal income tax is favourable. If transactions cost are significant, the problem faced by the individual may be more like the problem analysed in this study.

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REFERENCES

Brazee, R. & Mendelsohn, R., 1988. Timber Harvesting with Fluctuating Prices. *Forest Science*, 32: 359–372.

Dixit, A. K. & Pindyck, R.S., 1994. *Investment under Uncertainty* (Princeton: Princeton University Press).

Forboseh, P.F., 1994: *Price-Adaptive Strategies for Multi-Product Forest Management with and without Yield Constraints.* Ph.D.-dissertation, Michigan Technological University, Michigan, USA.

Lohmander, P., 1987: *The Economics of Forest Management Under Risk*. Department of Forest Economics, Swedish University of Agricultural Sciences, Umeå, Report 79.

Morck, R., Schwartz, E. & Stangeland, D., 1989: The Valuation of Forestry Resources under Stochastic Prices and Inventories. *Journal of Financial and Quantitative Analysis*, 24: 473–487.

Møller, C. M., 1933. Boniteringstabeller og Bonitetsvise tilvækstoversigter for Bøg, Eg og Rødgran i Danmark. [Site tables and Growth tables by Site for beech, oak and Norway spruce in Denmark.] *Dansk Skovforenings Tidsskrift*, 18: 457–627.

Norstrøm, C. J., 1975. A stochastic model for the growth period decision in forestry. *Swedish Journal of Economics*, 77: 329–337.

Plantinga, A., 1998. Optimal Harvesting Policies with Stationary and Non-Stationary Prices: An Option Value Analysis. *Forest Science*, 44, 192–202.

Statistics Denmark, *Landbrugsstatistisk Årsbog 1965–1995*, [Yearbook of Agricultural Statistics, issues 1965–1995].

Taylor C. R. & Novak, F.S., 1992. Optimal Flex Cropping and Storage of Spring Wheat under Progressive Income Tax. *Canadian Journal of Agricultural Economics*, 40: 369–384.

Thomson, T. A., 1992. Optimal Forest Rotation When Stumpage Prices Follow a Diffusion Process. *Land Economics*, 68: 329–342.

APPENDIX

To prove proposition 2 formally it must be shown that:

$$\frac{dP_{t+1}}{dx_t} \geq (s_L - s_H) \tag{A1}$$

Using the fact that $P_T = E_{T-1}(V(K, x_T))$ at $t = T - 1$, then, from Equation (1):

$$\begin{aligned} P_T\left(V\left(K, x_T\right)\right) = {} & K\left(s_H - s_L\right)F\left(x_T \leq I - K\right) + K\left(1 - s_H\right) \\ & + I\left(s_H - s_L\right)\left[F\left(x_T \leq I\right) - F\left(x_T \leq I - K\right)\right] \\ & + \left(s_L - s_H\right)\int_{I-K}^{I} x_T f\left(x_T\right)dx_T . \end{aligned} \tag{A2}$$

Integrating the last term by parts provides:

$$\begin{aligned} \left(s_L - s_H\right)\int_{I-K}^{I} x_T f\left(x_T\right)dx_T = {} & -I\left(s_H - s_L\right)\left[F\left(I\right) - F\left(I - K\right)\right] \\ & -K\left(s_H - s_L\right)F\left(I - K\right) - \left(s_L - s_H\right)\int_{I-K}^{I} F\left(x_T\right)dx_T , \end{aligned} \tag{A3}$$

and substituting back into the expression for P_T:

$$P_T = K\left(1 - s_H\right) - \left(s_L - s_H\right)\int F\left(x_T\right)dx_T . \tag{A4}$$

Differentiating Equation (A4) with respect to x_{T-1} results in:

$$\frac{\partial P_T}{\partial x_{T-1}} = -(s_L - s_H)$$

$$\int_{I-K}^{I}\left[\int_{-\infty}^{x_T} \frac{-\beta}{\sigma^2}(\alpha+\beta x_{T-1}-x_T)\frac{1}{\sqrt{(2\pi)}\sigma}\exp^{\left(-\frac{(x_T-\alpha-\beta x_{T-1})^2}{2\sigma^2}\right)}dx_T\right]dx_T$$

$$= (s_L - s_H)\frac{\beta}{\sigma^2}\int_{I-K}^{I}\left[E(x_T)F(x_T) - \int_{-\infty}^{x_T} x_T f(x_T)dx_T\right]dx_T$$

$$= (s_L - s_H)\frac{\beta}{\sigma^2}\int_{I-K}^{I} E(x_T)F(x_T) - \left(-\sigma f(x_T) + E(x_T)F(x_T)\right)dx_T$$

$$= (s_L - s_H)\frac{\beta}{\sigma}\left[F(I) - F(I-K)\right]. \tag{A5}$$

Thus it is evident that:

$$0 \geq \frac{\partial P_T}{\partial x_{T-1}} \geq s_L - s_H$$

and hence there can be at most two intersections between P_T and $V(K\,|\,x_{T-1})$. Using (2) recursively, this can be shown to be true for all $t < T$.

To obtain (A5) the following standard result from the standard normal distribution has been used. Let:

$$\tilde{x}_T = \frac{x_T - E(x_T)}{\sigma}$$

Substitute this into the expectation in the integral of the third line in (A5):

$$\int_{-\infty}^{x_T} x_T f(x_T)dx_T = \int_{-\infty}^{\tilde{x}_T}\left(\sigma\tilde{x}_T + E(x_T)\right) f\left(\sigma\tilde{x}_T + E(x_T)\right)\partial\tilde{x}_T$$

$$= \int_{-\infty}^{\tilde{x}_T} \sigma\tilde{x}_T \frac{1}{\sqrt{2\pi}} e^{-\frac{\tilde{x}_T^2}{2}} d\tilde{x}_T + E(x_T)\Phi(\tilde{x}_T) \tag{A6}$$

Evaluating the integral and taking appropriate limits gives:

$$\int_{-\infty}^{\tilde{x}_T} \sigma\tilde{x}_T \frac{1}{\sqrt{2\pi}} e^{-\frac{\tilde{x}_T^2}{2}} d\tilde{x}_T = \left[-\sigma\frac{1}{\sqrt{2\pi}} e^{-\frac{\tilde{x}_T^2}{2}}\right]_{-\infty}^{\tilde{x}_T} = -\sigma\varphi(\tilde{x}_T)$$

upon substituting back into (A6) and using the relations between $\varphi(\tilde{x}_T)$ and $f(x_T)$ the result is:

$$-\sigma f(x_T) + E(x_T)F(x_T) \tag{A7}$$

Use this in (A5) to get from the third line to the fourth. Another application of this result in forest economics is found in Forboseh (1994).